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Stochastic Dierential Equations Driven by Mixed Fractional Brownian Motion

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بسم الله الرحمن الرحيم

﴿...وَمَا تَوْفِيقِي إِلَّا باللهِ عَلَيهِ تَوَكَّلْتُ وإِلَيْهِ أُنِيبُ﴾

الآية 88 من سورة هـــــود

صدق الله العظيم

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⁕ *Dedication* ⁕

In the name of Allah, the Most Gracious, the Most Merciful
This thesis is dedicated to

- My late father (may Allah have mercy on him), who raised me with sincerity and care, and always supported me in my journey to discover and achieve my potential. He was a constant source of strength and encouragement.
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Abstract

This thesis focuses on the mixed fractional Brownian motion (mfBm), highlighting its mathematical properties and specific behaviors. This process combines the features of classical Brownian motion and fractional Brownian motion, allowing for the modeling of phenomena with both short- and long-range dependence. The study reveals properties such as increment stationarity, non-Markovian behavior, and Hölder continuity depending on the chosen parameters.

The analysis continues with the study of stochastic differential equations (SDEs) driven by by Mixed Fractional Brownian. Using tools such as the Wiener, Young, and Skorohod integrals, the work demonstrates the existence, uniqueness, and regularity of solutions under specific settings. The introduction of a stabilizing term helps extend the applicability of these SDEss, especially for Hurst parameters greater than 3/4.

Concrete applications are then presented through numerical simulations. These experiments show that Mixed EDS is particularly well-suited for financial modeling, reproducing asset price dynamics and market behavior more accurately. Practical aspects such as self-financing strategies and the no-arbitrage property are also explored.

Résumé

Ce mémoire examine le mouvement Brownien fractionnaire mixte (mfBm) en mettant en avant ses propriétés mathématiques . Ce processus combine les caractéristiques du mouvement Brownien classique et du mouvement Brownien fractionnaire, permettant de modéliser des phénomènes à dépendance courte et longue. Son étude révèle des propriétés telles que la stationnarité des incréments, la non-markovianité et une continuité Höldérienne selon les paramètres choisis.

L'analyse se poursuit par l'étude des équations différentielles stochastiques (EDS) dirigées par ce processus. En s'appuyant sur des outils tels que les intégrales de Wiener, de Young et de Skorohod, le travail démontre lexistence, l'unicité et la régularité des solutions dans des contextes spécifiques. L'introduction dun terme stabilisant permet délargir les conditions d'utilisation de ces EDS, notamment pour les indices de Hurst supérieurs à 3/4.

Des applications concrètes sont ensuite présentées à travers des simulations numériques. Ces expériences montrent que le mfBm est particulièrement adapté à la modélisation financière, en reproduisant de manière plus fidèle la dynamique des prix d'actifs et les comportements de marché. Des aspects pratiques comme les stratégies auto-finançantes et la propriété d'absence d'arbitrage sont aussi abordés.

L'objectif de ce travail est de mettre en lumière l'intérêt théorique et pratique du EDS mixte dans la modélisation stochastique. En combinant une base mathématique solide avec des applications numériques pertinentes, cette étude montre que le EDS mixte constitue une alternative puissante aux modèles classiques, ouvrant des perspectives nouvelles dans les domaines scientifiques et économiques.

Contents

Introduction

Stochastic differential equations (SDEs) serve as a foundational tool for modeling systems subject to random fluctuations. Classical SDEs driven by Wiener processes excel at capturing short-term, memoryless noise, However, many real-world phenomena-such as financial markets and climate dynamics-exhibit **long-range dependence** that traditional Markovian models fail to capture. This thesis bridges this gap by rigorously analyzing **mixed SDEs** that integrate both Wiener processes and fractional Brownian motion (fBm), thereby unifying transient randomness with persistent memory effects. Such equations take the form:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \sigma_1 \int_0^t X_s dB_s + \sigma_2 \int_0^t X_s dB_s^H,$$

where B_s is a Wiener process and B_t^H is a fractional Brownian motion (fBm) with Hurst index $H \in (1/2, 1)$.

The limitations of existing models are evident in systems where historical trends persistently influence future states. For instance, financial asset volatility often displays **long memory** (characterized by fBm with $H > 1/2$), while instantaneous market shocks are well represented by Wiener-driven jumps. Similarly, climate systems exhibit self-similar temperature anomalies over decades, a hallmark of fBm, juxtaposed with short-term weather fluctuations resembling white noise. However, current frameworks typically treat these components in isolation, neglecting their interaction. This work addresses this shortcoming by developing a unified theory for mixed SDEs, enabling the simultaneous analysis of short- and long-term stochastic behaviors.

The primary contributions of This thesis are threefold. First, we establish existence and uniqueness criteria for solutions to mixed SDEs in the space $\mathscr{S}^2([0,T])$, leveraging Picard iteration and Gronwall's inequality to accommodate the non-Markovian nature of fBm. Second, for $H \in (3/4, 1)$, we introduce a stabilizing term $\varepsilon \int_0^t \eta X_s dV_s$ to transform fBm-driven dynamics into a semimartingale framework, resolving pathwise uniqueness and enabling classical Itô calculus. Third, we prove weak convergence of stabilized solutions to the original mixed SDEs as $\varepsilon \longrightarrow 0$, quantifying rates as $\mathcal{O}(\varepsilon^{\alpha})$ for $\alpha = \min(1, 2H - 1)$. These theoretical advancements are contextualized through applications in finance and climate science. For example, a mixed Heston model incorporating fBm replicates the "volatility smile" observed in options markets, while temperature anomaly models demonstrate geometric ergodicity towards stationary distributions, aligning with empirical climate trends..

This master thesis is structured by 4 chapters.

In Chapter 1, we define and examine the properties of Brownian motion and fractional Brownian motion. This chapter establishes the foundation by looking at the basic properties and mathematical expressions of these stochastic processes. It is imperative to comprehend these fundamental ideas in order to fully appreciate the more intricate structures that are presented later in the thesis.

In Chapter 2 we delve deeply into the physics and dynamics of mixed fractional Brownian motion. This chapter examines the ways in which mfBm combines aspects of fractional Brownian motion and traditional Brownian motion, providing a thorough examination of its distinct features. We can recognize the benefits of mfBm over other models and its possible uses by comprehending its complex characteristics.

In Chapter 3, we study stochastic differential equations (SDEss) driven by mixed fractional Brownian motion, which combines both a standard Wiener process and a fractional Brownian motion (fBm). We begin by developing the necessary stochastic analysis tools, introducing three types of integrals adapted to different settings: the Wiener integral for classical Brownian motion, the Young integral suitable for Hölder-continuous paths when the Hurst parameter $H > \frac{1}{2}$, and the Skorohod integral used in anticipative frameworks.

We then examine mixed SDEss that involve both types of noise, focusing on the existence, uniqueness, and regularity of their solutions, particularly in the context of semilinear equations. Finally, we analyze the conditions under which these mixed SDEss admit unique solutions when the Hurst parameter lies in the interval $\left(\frac{3}{4}, 1\right)$, both with and without an added stabilizing term. The chapter concludes by exploring the asymptotic behavior of solutions in the presence of this stabilizing term, establishing limit results for the corresponding equations.

In Chapter 4, we explore practical applications and simulations of stochastic differential equations (SDEss) driven by mixed fractional Brownian motion (mfBm), which combines both a standard Wiener process and a fractional Brownian motion. We begin with a general simulation framework for such mixed SDEss, setting the stage for various modeling scenarios. The chapter then introduces a mixed Brownianfractional Brownian model and examines its structural properties.

Next, we investigate conditions of self-financing strategies and their implications, focusing on capital functions, Markovian strategies, and specific constraints necessary for financial modeling. We proceed with concrete financial applications, including the formulation of asset price models and a detailed numerical example. A crucial topic addressed is the arbitrage-free property of these models, where both theoretical aspects and numerical verifications are discussed.

Finally, we compare different modeling approaches and conclude with practical examples and simulations implemented in the `R` programming language, illustrating the behavior and validity of the proposed models.

This thesis attempts to give a thorough grasp of Stochastic Differential Equations Driven by Mixed Fractional Brownian Motion through these organized chapters, covering both its theoretical underpinnings and its real-world applications. This study advances the subject of stochastic processes and their applications in a variety of scientific and financial contexts by highlighting the theoretical significance and real-world applicability of mixed SDEs.

1

Generality on Fractional Brownian Motion

1.1 Brownian Motion

Brownian motion is the continuous random motion of microscopic particles when suspended in a fluid medium. Brownian motion was first observed (1827) by the Scottish botanist Robert Brown[3] (1773-1858) when studying pollen grains in water. The effect was finally explained in 1905 by Albert Einstein[6], who realized it was caused by water molecules colliding randomly with the particles. Over a century later, Brownian motion can still cause problems for scientists trying to study small biological particles in solution, because they move around too much.

1.1.1 Definition of Brownian Motion

To formally define Brownian motion, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a space on which we define the process $(B_t)_{t\geq 0}$.

Definition 1.1.1 A stochastic process $(B_t)_{t\geq 0}$ is called a ***standard Brownian motion*** if it satisfies the following conditions:

1. $B_0 = 0 \ \mathbb{P} - a.s.$

2. For all $n \geq 1$, for all times $0 = t_0 \leq t_1 \leq ... \leq t_n$, the increments $B_{t_n} - B_{t_{n-1}}, ..., B_{t_1} - B_{t_0}$ are independent random variables ("independent increments").

3. For any given times $0 \leq s \leq t$, $B_t - B_s$ has the Gaussian distribution $\mathcal{N}(0, t-s)$ with mean zero and variance $t - s$.

4. Almost surely, the function $t \to B_t$ is continuous.

Remark 1.1.1

1. *We can rewrite the second condition by : for* $s \leq t$*, the random variable* $B_t - B_s$ *is independent of* $\sigma(B_r, r \leq s)$.

2. *The natural filtration of the Brownian motion is* $\mathcal{F}_t^B = \sigma(B_s, s \leq t)$.

3. *We can define the Brownian motion without the last condition of continuous paths, because with a stochastic process satisfying the second and the third conditions, by applying the Kolmogorov's continuity theorem, there exists a modification of* $(B_t)_{t\geq 0}$ *which has continuous paths almost surely.*

Proposition 1.1.1 *The Brownian motion* $B = (B_t, t \geq 0)$ *is a centered Gaussian process with covariance :*

$$Cov(B_t, B_s) = \mathbb{E}(B_s B_t) = min(s,t) = s \wedge t, s \geq 0, t \geq 0.$$

Proof. We have that $B_t = B_t - B_0$. Thus $B_t \sim \mathcal{N}(0,t)$ by definition. Moreover, without loss of generality, we assume $s < t$. Hence, we have

$$Cov(B_t, B_s) = \mathbb{E}(B_s B_t) = \mathbb{E}(B_s(B_t - B_s) + B_s^2) = 0 + Var(B_s) = 0 + s = s, s < t,$$

hence

$$Cov(B_t, B_s) = \mathbb{E}(B_s B_t) = min(s,t) = s \wedge t, s \geq 0, t \geq 0.$$

Note that since the Brownian motion is a continuous Gaussian process, the proposition 1.1.1 characterizes uniquely the Brownian motion. ■

1.1.2 Properties of Brownian Motion

In this section, we will present some properties of Brownian motion.

Proposition 1.1.2 *[7] Let* $(B_t)_{t\geq 0}$ *be a standard Brownian motion*

1. *Self-similarity. For any* $a > 0$, $\{a^{-1/2} B_{at}\}$ *is Brownian motion.*

2. *Symmetry.* $\{-B_t, t \geq 0\}$ *is also a Brownian motion.*

3. *Time-inversion.* $\{tB_{\frac{1}{t}}, t > 0\}$ *is also a Brownian motion.*

4. *If* B_t *is a Brownian motion on* $[0,1]$*, then* $(t+1)B_{\frac{1}{t+1}} - B_1$ *is a Brownian motion on* $[0, \infty)$.

Remark 1.1.2 *Observe that a consequence of (3) is the law of large numbers for the Brownian motion, namely* $\mathbb{P}[\lim_{t\to+\infty} t^{-1} B_t = 0] = 1$.

1.1.2.1 Non-differentiability of Brownian Motion

Despite being continuous, the random nature of Brownian motion yields many interesting pathological properties. The most prominent example of this is that it is nowhere differentiable.

Lemma 1.1.1 *Almost surely*

$$\limsup_{n\to+\infty} \frac{B(n)}{\sqrt{n}} = +\infty.$$

And similarly for lim inf.

So $B(t)$ grows slower than t. But this lemma shows that its lim sup grows faster than $\sqrt{t}$.
Proof. By reverse Fatou,

$$\mathbb{P}[B(n) > c\sqrt{n}] \geq \limsup_{n\to+\infty} \mathbb{P}[B(n) > c\sqrt{n}] = \limsup_{n\to+\infty} \mathbb{P}[B(1) > c] > 0,$$

by the scaling property. Thinking of $B(n)$ as the sum of $X_n = B(n) - B(n-1)$, the event on the LHS is exchangeable and the Hewitt-Savage 0-1 law implies that it has probability 1 (where we used the positive lower bound). ■

Definition 1.1.2 ***(Upper and lower derivatives)*** *For a function f, we define the upper and lower right derivatives as*

$$D^*f(t) = \limsup_{h\downarrow 0} \frac{f(t+h) - f(t)}{h},$$

and

$$D_*f(t) = \liminf_{h\downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

We begin with an easy first result.

Theorem 1.1.1 *Fix $t \geq 0$. Then almost surely Brownian motion is not differentiable at t. Moreover, $D^*B(t) = +\infty$ and $D_*B(t) = -\infty$.*

Proof. Consider the time inversion X. Then

$$D^*X(0) \geq \limsup_{n\to+\infty} \frac{X(n^{-1}) - X(0)}{n^{-1}} = \limsup_{n\to+\infty} B(n) = +\infty,$$

by the lemma above. This proves the result at 0. Then note that $X(s) = B(t+s) - B(s)$ is a standard Brownian motion and differentiability of X at 0 is equivalent to differentiability of B at t.
In fact, we can prove something much stronger. ■

Theorem 1.1.2 *Almost surely, Brownian motion is nowhere differentiable. Furthermore, almost surely, for all t*

$$D^*B(t) = +\infty,$$

or

$$D_*B(t) = -\infty,$$

or both.

Proof. Suppose there is t_0 such that the latter does not hold. By boundedness of BM over $[0,1]$, we have

$$\sup_{h\in[0,1]} \frac{|B(t_0+h)-B(t_0)|}{h} \leq M,$$

for some $M < +\infty$. Assume t_0 is in $[(k-1)2^{-n}, k2^{-n}]$ for some k, n. Then for all $1 \leq j \leq 2^{-n} - k$, in particular, for $j = 1, 2, 3$,

$$\begin{aligned} |B((k+j)2^{-n}) - B((k+j-1)2^{-n})| &\leq |B((k+j)2^{-n}) - B(t_0)| + |B(t_0) - B((k+j-1)2^{-n})| \\ &\leq \left[M(2j+1)2^{-n}\right], \end{aligned}$$

by our assumption. Define the events

$$\Omega_{n,k} = \{|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \leq M(2j+1)2^{-n}, j = 1, 2, 3\}.$$

It suffices to show that $\bigcup_{k=1}^{2^n-3} \Omega_{n,k}$ cannot happen for infinitely many n. Indeed,

$$\mathbb{P}\left[\exists t_0 \in [0,1], \sup_{h\in[0,1]} \frac{|B(t_0+h)-B(t_0)|}{h} \leq M\right]$$

$$\leq \mathbb{P}\left[\bigcup_{k=1}^{2^n-3} \Omega_{n,k} \quad for \quad infinitely \quad many \quad n\right].$$

(Then the result follows by taking all $[k,k+1]$ intervals and all M integers). But by the independence of increments

$$\begin{aligned}
\mathbb{P}[\Omega_{n,k}] &= \prod_{j=1}^{3} \mathbb{P}[|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \leq M(2j+1)2^{-n}] \\
&\leq \mathbb{P}\left[|B(2^{-n})| \leq \frac{7M}{2^n}\right]^3 \\
&= \mathbb{P}\left[\left|\frac{1}{\sqrt{2^{-n}}} B\left(\left[\sqrt{2^{-n}}\right]^2\right)\right| \leq \frac{7M}{\sqrt{2^{-n}}.2^n}\right]^3 \\
&= \mathbb{P}\left[|B(1)| \leq \frac{7M}{\sqrt{2^n}}\right]^3 \\
&\leq \left(\frac{7M}{\sqrt{2^n}}\right)^3,
\end{aligned}$$

because the density of a standard Gaussian is bounded by 1/2. (The choice of 3 comes from summability). Hence

$$\mathbb{P}\left[\bigcup_{k=1}^{2^n-3} \Omega_{n,k}\right] \leq 2^n \left(\frac{7M}{\sqrt{2^n}}\right)^3 = (7M)^3 2^{-n/2},$$

which is summable. The result follows from BC. That is, the probability above is 0. ■

1.1.2.2 Brownian paths

Lemma 1.1.2 ***(Kolmogrov-Chentsov)***
Fix a compact interval $\mathbb{T} = [0,T] \subset \mathbb{R}_+$, and let $X = (X_t)_{t\in\mathbb{T}}$ be a centered Gaussian process. Suppose that there exists $C, \eta > 0$ such that, for all $s,t \in \mathbb{T}$,

$$\mathbb{E}[(X_t - X_s)^2] \leq C \mid t-s \mid^{\eta}. \tag{1.1}$$

*Then, for all $\alpha \in (0, \eta/2)$, there exists a modification Y of X with α-*Hölder *continuous paths. In particular, X admits a continuous modification.*

Proof. Fix $t > s$. Since X is Gaussian and centered, we have that

$$X_t - X_s \overset{Law}{=} \sqrt{\mathbb{E}[(X_t - X_s)^2]}G,$$

where $G \sim \mathcal{N}(0,1)$. We deduce from (1.1) that, for all $p \geq 1$,

$$\mathbb{E}[| X_t - X_s |^p] \leq C^{p/2}\mathbb{E}[| G |^p] \mid t-s \mid^{\eta p/2}.$$

Therefor, the general version of the classical Kolmogrov-Chentsov lemma is applied and gives the desired result. ■

Proposition 1.1.3 *A Brownian motion has its paths almost surely, locally γ-Hölder continuous for $\gamma \in [0, 1/2)$.*

Proof. Let $T > 0$, $n \in \mathbb{N}$ and $0 \leq s \leq t$. Then we have,

$$\mathbb{E}((B_t - B_s)^{2n}) = \frac{(2n)!}{2^n n!}(t-s)^n.$$

Hence, by using the Kolmogrov-Chentsov lemma 1.1.2, there exists a continuous modification $(\tilde{B}_t)_{0\leq t\leq T}$ of $(B_t)_{0\leq t\leq T}$, whose the paths are locally γ-Hölder continuous for $\forall\gamma \in [0, \frac{n-1}{2n})$. Moreover, we have

$$\mathbb{P}(\forall t \in [0,T], B_t = \tilde{B}_t) = 1,$$

because the two processes are continuous, It implies that also almost all the paths of $(B_t)_{0\leq t\leq T}$ are locally γ-Hölder continuous. ■

Proposition 1.1.4 *[7] The Brownian motion's sample paths are almost surely, nowhere differentiable.*

There is an intuitive way to understand this property of Brownian paths. Indeed, consider the increment for $h > 0$, $B_{t+h} - B_t \sim \mathcal{N}(0, h)$. Then we have that $\frac{B_{t+h}}{\sqrt{h}} \sim \mathcal{N}(0, 1)$. But the derivative is defined to be the limit, as h tends to 0, of the quantity $\frac{B_{t+h} - B_t}{h} \sim \mathcal{N}(0, \frac{1}{h})$. It is clear, now, that when we let h tends to 0, we obtain an "infinite" variance, so that there would not be a limit.

1.1.2.3 Quadratic variation and Brownian Motion

Definition 1.1.3 ***(Bounded variation)*** *A function $f : [0, t] \to \mathbb{R}$ is of bounded variation if there is $M < +\infty$ such that*

$$\sum_{j=1}^{k} |f(t_j) - f(t_{j-1})| \leq M,$$

for all $k \geq 1$ and all partitions $0 = t_0 < t_1 < \cdots < t_k = t$. Otherwise, we say that it is of bounded variation.

Functions of bounded variation are known to be differentiable. Since Brownian motion is nowhere differentiable, it must have unbounded variation. However, Brownian motion has a finite "quadratic variation".

Theorem 1.1.3 ***(Quadratic variation)*** *Suppose the sequence of partitions*

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k(n)}^{(n)} = t,$$

is nested, that is, at each step one or more partition points are added, and the mesh

$$\Delta(n) = \sup_{1 \le j \le k(n)} \{t_j^{(n)} - t_{j-1}^{(n)}\},$$

converges to 0. Then, almost surely

$$\lim_{n \to +\infty} \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2 = t.$$

Proof. By considering subsequences, it suffices to consider the case where one point is added at each step. Let

$$X_{-n} = \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2.$$

Let

$$\mathcal{G}_{-n} = \sigma(X_{-n}, X_{-n-1}, \ldots)$$

and

$$\mathcal{G}_{-\infty} = \bigcap_{k=1}^{\infty} \mathcal{G}_{-k}.$$

For more details of the proof, see([8]). ■

1.1.2.4 Markov property

Theorem 1.1.4 ***(Markov property)****[12] Let* $\{B_t : t \ge 0\}$ *is a Brownian motion started in* $x \in \mathbb{R}^d$*. Then the process* $\{B_{t+s} - B_s : t, s > 0\}$ *is a Brownian motion started at the origin and is independent of* $\{B_t : 0 \le t \le s\}$*.*

Proof. This follows directly from property the independence of increments of Brownian motion. ■

However, this is rather trivial. A preliminary means of making this property slightly stronger is establishing that Brownian motion is independent of information that exists an infinitesimal amount of time into the future.

Definition 1.1.4 *The* ***germ*** $\boldsymbol{\sigma}$***-algebra*** *is defined as* $\mathcal{F}^+(0)$*, where*

$$\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}^0(s)$$

and $\{\mathcal{F}^0 : t \ge 0\}$ *is the* σ*-algebra generated by* $\{B_t : 0 \le s \le t\}$*.*

Theorem 1.1.5 *[12] For all $s \geq 0$, the random process $\{B_{t+s} - B_s : t \geq 0\}$ is independent of $\mathcal{F}^+(s)$.*

Proof. By continuity, we can write the following for a strictly decreasing sequence $\{s_n : n \in \mathbb{N}\}$ converging to s:

$$B_{t+s} - B_s = \lim_{n \to \infty} B_{s_n + t} - B_{s_n}$$

However, the Markov property verifies that the right side of the above equation is independent of $\mathcal{F}^+(s)$. ■

Theorem 1.1.6 *Every stopping time with respect to the filtration $\{\mathcal{F}^+(t) : t \geq 0\}$ is a strict stopping time.*

Proof. First, let us establish the right-continuity of $\{\mathcal{F}^+(t) : t \geq 0\}$ To do this, we can write

$$\mathcal{F}^+(t) = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \mathcal{F}^0(t + \frac{1}{n} + \frac{1}{k}) = \bigcap_{\epsilon > 0} \mathcal{F}^+(t + \epsilon).$$

Thus,

$$\{T \leq t\} = \bigcap_{k=1}^{\infty} \{T < t + \frac{1}{k}\} \in \bigcap_{n=1}^{\infty} \mathcal{F}^+(t + \frac{1}{n}) = \mathcal{F}^+(t)$$

■

Theorem 1.1.7 ***(Strong Markov property)****[12] For every almost surely finite stopping time T, the process $\{B_{T+t} - B_T : t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}^+(T)$.*

Proof. Let T be a stopping time. We can then define

$$T_n = (m+1)2^{-n}, \text{ where } m/2^n \leq T < (m+1)/2^n.$$

Consider this as a discrete approximation that ends at the first dyadic rational adjacent to the original. Keeping in mind that this definition indicates that T_n is a stopping time, we define the following:

$$B_k(t) = B_{t+k/2^n} - B_{k/2^n} \text{ and } B_k = \{B_k(t) : t \geq 0\},$$
$$B_*(t) = B_{t+T_n} - B_{T_n} \text{ and } B_* = \{B_*(t) : t \geq 0\}.$$

Now, take $E \in \mathcal{F}^+(T_n)$ and the event $\{B_* \in A\}$. We have

$$\mathbb{P}(\{B_* \in A\} \bigcap E) = \sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\} \bigcap E \bigcap \{T_n = k/2^n\}).$$

Note, however, that $E\bigcap\{T_n = k/2^n\} \in \mathcal{F}^+(k/2^n)$, which by theorem1.1.6 is independent of $\{B_k \in A\}$. Consequently, we have

$$\mathbb{P}(\{B_* \in A\}\bigcap E) = \sum_{k=0}^{\infty} \mathbb{P}\{B_k \in A\}\mathbb{P}(E\bigcap\{T_n = k/2^n\}).$$

Now we use the Markov property we see that for all $k \in \mathbb{N}, \mathbb{P}\{B \in A\} = \mathbb{P}\{B_k \in A\}$. This yields

$$\sum_{K=0}^{\infty} \mathbb{P}\{B_k \in A\}\mathbb{P}(E\bigcap\{T_n = k/2^n\}) = \mathbb{P}\{C \in A\}\sum_{k=0}^{\infty} \mathbb{P}(E\bigcap\{T_n = k/2^n\}) = \mathbb{P}\{C \in A\}\mathbb{P}(E).$$

Consequently, B_* is independent of every E and hence independent of $\mathcal{F}^+(T_n)$. Now, recall that the sequence T_n is a uniformly decreasing sequence that converges to T, hence $\mathcal{F}^+(T_n) \subset \mathcal{F}^+(T)$ is independent of the Brownian motion $B_{s+T_n} - B_{T_n}$. Then, the random process $B_{r+T} - B_T$, defined by the increments

$$B_{s+t+T} - B_{t+T} = \lim_{n\to\infty} B_{s+t+T_n} - B_{t+T_n},$$

is independent, $N(0, s)$, and almost surely continuous. Thus, it is a Brownian motion and independent of $\mathcal{F}^+(T)$. ■

1.1.2.5 Martingal property

The standard Brownian motion and several functions of it are martingales.

Proposition 1.1.5 *[7] Let $(B_t)_{t\in\mathbb{R}_+}$ be a Brownian motion. Then the following processes are $(\mathcal{F}_t^B)$-martingales:*

1. $(B_t)_{t\in\mathbb{R}_+}$,

2. $(B_t^2 - t)_{t\in\mathbb{R}_+}$,

3. For any $u \in \mathbb{R}$, $(e^{uB(t)-\frac{u^2}{2}t})_{t\in\mathbb{R}_+}$.

1.2 Fractional Brownian Motion

The fractional Brownian motion (fBm) is a suitable generalization of standard Brownian motion, it is the most known process which is not a semi-martingale. It is the only Gaussian self similar stationary process with long-range dependance property. Due to these interesting properties it enjoyed success as a modeling tool in many field of applications including telecommunications, turbulence and finance, the demand to stochastic calculus with respect to fBm are raised. This process was introduced by Kolmogorov[9] and studied later by Mandelbrot and Van Ness[11] who provided an integral representation of fBm with respect to a standard Brownian motion over a real line time interval.

1.2.1 Definition of Fractional Brownian Motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 1.2.1 *The fractional Brownian motion (fBm) with Hurst index* $(H \in (0,1))$ *is a Gaussian process* $B^H = \{B^H_t, t \in \mathbb{R}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$*, having the properties:*

1. $B^H_0 = 0$,

2. $\mathbb{E}(B^H_t) = 0;\ t \in \mathbb{R}$,

3. $cov(B^H_t, B^H_s) = \mathbb{E}(B^H_t B^H_s) = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big);\ s, t \in \mathbb{R}.$

Remark 1.2.1 *Since* $\mathbb{E}(B^H_t - B^H_s)^2 = |t-s|^{2H}$ *and* B_H *is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem.*

Remark 1.2.2 *We have:*

- *For* $H = \frac{1}{2}$*, the fBm is the standard Brownian motion.*

- *For* $H = 1$*, we set* $B^H_t = B^1_t = t\xi$*, where* ξ *is a standard normal Random variable.*

1.2.2 Basic properties

Proposition 1.2.1 *Let* B^H *be a fractional Brownian motion of Hurst parameter* $H \in (0,1)$*. Then:*

1. *Selfsimilarity. For all* $a > 0$, $(B^H_{at}) \overset{d}{=} (a^H B^H_t)$.

2. *Stationarity of increments. For all* $h > 0$, $(B^H_{t+h} - B^H_h) \overset{d}{=} B^H_t$.

3. *Hölder continuity. For each* $0 < \varepsilon < H$ *and each* $T > 0$ *there exists a random variable* $K_{\varepsilon,T}$ *such that*

$$\mid B^H(t) - B^H(s) \mid \leq K_{\varepsilon,T} \mid t-s \mid^{H-\varepsilon}.$$

4. *Differentiability. The sample paths of fBm are nowhere differentiable.*

Proof. First, let us prove the selfsimilarity property. We have that

$$\begin{aligned}
\mathbb{E}(B^H_{at} B^H_{as}) &= \frac{1}{2}((at)^{2H} + (as)^{2H} - (a \mid t-s \mid)^{2H}) \\
&= a^{2H}\mathbb{E}(B^H_t B^H_s) \\
&= \mathbb{E}((a^H B^H_t)(a^H B^H_s)).
\end{aligned}$$

Thus, since all processes are centered and Gaussian, it implies that

$$(B^H_{at})\overset{d}{=}(a^H B^H_t).$$

Seconde, we show that it has stationary increments. Note that for all $h > 0$, we have

$$\begin{aligned}
\mathbb{E}((B^H_{t+h} - B^H_h)(B^H_{s+h} - B^H_h)) &= \mathbb{E}(B^H_{t+h}B^H_{s+h}) - \mathbb{E}(B^H_{t+h}B^H_h) - \mathbb{E}(B^H_{s+h}B^H_h) + \mathbb{E}((B^H_h)^2) \\
&= \frac{1}{2}\Big[((t+h)^{2H} + (s+h)^{2H} - \mid t-s \mid^{2H}) \\
&\quad - \Big((t+h)^{2H} + h^{2H} - t^{2H}\Big) - \Big((s+h)^{2H} + h^{2H} - s^{2H}\Big) + 2h^{2H}\Big] \\
&= \frac{1}{2}(t^{2H} + s^{2H} - \mid t-s \mid^{2H}) = \mathbb{E}(B^H_t B^H_s).
\end{aligned}$$

Therefore the fBm is of stationary increments.

For the Hölder continuity it follows from Kolmogrov-Chentsov lemma 1.1.2 and the fact that for any $\alpha > 0$, we have

$$\mathbb{E}(\mid B^H_t - B^H_s \mid^\alpha) = \mathbb{E}(\mid B^H_1 \mid^\alpha) \mid t-s \mid^{2H}.$$

Finally, lets prove the differentiability, indeed for every $t_0 \in [0, \infty]$,

$$\mathbb{P}\left(\limsup_{t\to t_0} \mid \frac{B^H_t - B^H_{t_0}}{t - t_0} \mid = \infty\right) = 1.$$

Let us denote by $\mathfrak{B}_{t,t_0} = \dfrac{B^H_t - B^H_{t_0}}{t - t_0}$, using the selfsimilarity property, we have

$$\mathfrak{B}_{t,t_0}\overset{d}{=}(t-t_0)^{H-1}B^H_1.$$

We define $\mathfrak{u}(t,\omega) = \{\sup_{0\le s\le t} \mid \dfrac{B^H_s}{s} \mid > d\}$. Then, for any sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to 0,

we have $\mathfrak{u}(t_n,\omega) \supseteq \mathfrak{u}(t_{n+1},\omega)$, thus,

$$\mathbb{P}(\lim_{n\to\infty} \mathfrak{u}(t_n)) = \lim_{n\to\infty} \mathbb{P}(\mathfrak{u}(t_n)),$$

and

$$\mathbb{P}(\mathfrak{u}(t_n)) \ge \mathbb{P}(\mid \frac{B^{(H)}_{t_n})}{t_n} \mid > d) = \mathbb{P}(\mid B^{(H)}_1 \mid > t_n^{1-H} d) \overset{n\to\infty}{\to} 1.$$

■

1.2.3 Long and Short-Range Dependence

Process with long-range dependence have many application, such as in telecommunication specially in Internet traffic problems. Basically, the notion of long-range dependence is that the variance of the sum of stationary sequence grows non-linearly with respect to n.

Definition 1.2.2 A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits long-range dependence if $\rho(n) = cov(X_k, X_{k+n})$ satisfies

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1,$$

for $\alpha \in (0,1)$ and some constant c.

Remark 1.2.3 *If a stationary sequence $(X_n)_{n\in\mathbb{N}}$ is long-range dependent, then the dependence between X_k and X_{k+1} decays slowly as n tends to infinity and $\sum_{n=1}^{\infty} \rho(n) = \infty$.*

Proposition 1.2.2 The fBm is one of the simplest processes which exhibit long-range dependency.

Proof. let us consider its increments

$$X_k = B_k^H - B_{k-1}^H, \quad X_{k+1} = B_{k+n}^H - B_{k+n-1}^H.$$

Since the fBm is centered then

$$\begin{aligned}
\rho(n) &= \mathbb{E}(X_k, X_{k+n}) = \mathbb{E}\left[(B_k^H - B_{k-1}^H)(B_{k+n}^H - B_{k+n-1}^H)\right] \\
&= \mathbb{E}\left[(B_{n+1}^H - B_n^H)B_1^H\right] = \mathbb{E}(B_{n+1}^H B_1^H) - \mathbb{E}(B_n^H B_1^H) \\
&= \frac{1}{2}\left[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right] \\
&= \frac{1}{2}n^{2H}\left[(1+\frac{1}{n})^{2H} - 2 + (1-\frac{1}{n})^{2H}\right] \\
&= \frac{n^{2H}}{2}\left[1 + \frac{2H}{n} + \frac{H(2H-1)}{n^2} - 2 + 1 - \frac{2H}{n} + \frac{H(2H-1)}{n^2} + o(\frac{1}{n^2})\right] \\
&= H(2H-1)n^{2H-2} + o(n^{2H-2}).
\end{aligned}$$

It follows that for $H > \frac{1}{2}$, we have

$$\rho(n) > 0 \;\; and \;\; \sum_n \rho(n) = \infty.$$

And for $H < \frac{1}{2}$, we have

$$\rho(n) < 0 \ \ and \ \ \sum_n \rho(n) < \infty.$$

Therefore, we say that the fBm has long-range dependence property if and only if $H > \frac{1}{2}$ and for the other case has short-range dependence. ■

1.2.4 Fractional Brownian Motion is not Markovian

Theorem 1.2.1 *Let B^H be a fractional Brownian motion of Hurst index $H \in (0,1) - \{\frac{1}{2}\}$. Then B^H is not a Markov process.*

Since the fBm is a Gaussian centered process, to prove this result we need the next lemma.

Lemma 1.2.1 If X is a Gaussian centered Markovian process, then for all $s < t < u$

$$\mathbb{E}(X_t X_s)\mathbb{E}(X_t X_u) = \mathbb{E}(X_t X_t)\mathbb{E}(X_u X_s).$$

Proof. Note that $R_{st} = cov(X_s, X_t)$. Since X is a Markov process then $\forall \ s < t < u$

$$\mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u/X_t) = \mathbb{E}(X_u) + \frac{cov(X_t, X_u)}{var(X_t)}(X_t - \mathbb{E}(X_t)).$$

Therefore,

$$\begin{cases} \mathbb{E}(X_u/X_t) = \dfrac{R_{ut}}{R_{tt}} X_t, \\ \mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u) + \theta_{uv}\theta_v^{-1}(v - \mathbb{E}(v)) \end{cases}$$

where $v = \begin{pmatrix} X_t \\ X_s \end{pmatrix}$ and $\theta_{uv} = \mathbb{E}[X_u v^t]$, $\theta_v = \mathbb{E}(v^t v)$

We have that,

$$\theta_{uv} = (R_{ut} R_{us}) \ \ and \ \ \theta_v = \begin{pmatrix} R_{tt} & R_{ts} \\ R_{st} & R_{ss} \end{pmatrix}$$

$$\theta_v^{-1} v = \frac{1}{R_{tt}R_{ss} - R_{ts}^2} \begin{pmatrix} R_{ss}X_t - R_{ts}X_s \\ R_{tt}X_s - R_{st}X_t \end{pmatrix}$$

We observe that,

$$\begin{aligned} \mathbb{E}(X_u/X_t, X_s) &= \theta_{uv}\theta_v^{-1} v \\ &= \frac{1}{R_{tt}R_{ss} - R_{ts}^2}(R_{ut}R_{ss}X_t - R_{ut}R_{ts}X_s - R_{us}R_{st}X_t + R_{us}R_{tt}X_s). \end{aligned}$$

Hence, $\mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u/X_t)$ we have

$$\frac{R_{ut}}{R_{tt}} X_t = \frac{1}{R_{tt}R_{ss} - R_{ts}^2}(R_{ut}R_{ss}X_t - R_{ut}R_{ts}X_s - R_{us}R_{st}X_t + R_{us}R_{tt}X_s).$$

Moreover,

$$\begin{aligned} X_t(R_{tt}R_{ut}R_{ss} - R_{tt}R_{ut}R_{ss} - R_{ut}R_{st}^2 + R_{tt}R_{us}R_{st}) + X_s(R_{tt}R_{ut}R_{st} - R_{tt}^2 R_{us}) &= 0 \\ R_{st}X_t(R_{tt}R_{us} - R_{ut}R_{st}) - R_{tt}X_s(R_{tt}R_{us} - R_{ut}R_{st}) &= 0. \end{aligned}$$

Or,

$$(R_{tt}R_{us} - R_{ut}R_{st})(R_{st}X_t - R_{tt}X_s) = 0,$$

then,

$$R_{tt}R_{us} - R_{ut}R_{st} = 0.$$

Which is the result. ■

Proof of theorem 1.2.1 We proceed by contradiction. Assume that B^H is a Markov process. Since it is a Gaussian process as well, by the previous lemma we have, for $s = 1 < t = 2 < u = 3$

$$\mathbb{E}(B_1^H B_2^H)\mathbb{E}(B_2^H B_3^H) = \mathbb{E}(B_2^H B_2^H)\mathbb{E}(B_1^H B_3^H).$$

So,

$$\begin{aligned} \frac{1}{4}(1 + 2^{2H} - 1)(2^{2H} + 3^{2H} - 1) &= 2^{2H}\frac{1}{2}(1 + 3^{2H} - 2^{2H}) \\ 2^{2H}(2^{2H} + 3^{2H} - 1) &= 2^{2H}[2(1 + 3^{2H} - 2^{2H})], \end{aligned}$$

by differentiating

$$\begin{aligned} 3 + 3^{2H} + 3(2^{2H}) &= 0 \\ 1 + 3^{2H-1} + 2^{2H} &= 0. \end{aligned}$$

We deduce that, $1 + 3^{2H-1} + 2^{2H} = 0$ only if $H = \frac{1}{2}$ which leads to a contradiction. ■

1.2.5 Fractional Brownian Motion is not a semi-martingale

The fact that the fBm is not a semi-martingale for $H \neq \frac{1}{2}$ has been proved by several authors. In order to verify that B^H is not a semi-martingale for $H \neq \frac{1}{2}$, it is sufficient to compute the p-variation of B^H.

Definition 1.2.3 *Let* $(X(t))_{t\in[0,T]}$ *be a stochastic process and consider a partition* $\pi = \Big\{0 = t_0 < t_1 < ...... < t_n = T\Big\}$. *Put*

$$\mathcal{S}_p(x,\pi) := \sum_{i=1}^{n} |X(t_i) - X(t_{i-1})|^p$$

The p-variation of X *over the interval* $[0,T]$ *is defined as*

$$\mathcal{V}_p(X,[0,T]) := sup_\pi \mathcal{S}_p(X,\pi),$$

where π *is a finite partition of* $[0,T]$. *The index of p-variation of a process is defined as*

$$I(X,[0,T]) := \inf\Big\{p > 0; \mathcal{V}_p(X,[0,T]) < \infty\Big\}.$$

We claim that

$$I(B^H,[0,T]) = \frac{1}{H}.$$

In fact, consider for $p > 0$,

$$Y_{n,p} = n^{pH-1} \sum_{i=1}^{n} \Big|B^H(\frac{i}{n}) - B^H(\frac{i-1}{n})\Big|^p.$$

Since B^H *has the self-similarity property, the sequence* $Y_{n,p}, n \in N$ *has the same distribution as*

$$\tilde{Y}_{n,p} = n^{-1} \sum_{i=1}^{n} \Big|B^H(i) - B^H(i-1)\Big|^p.$$

By the Ergodic theorem (see, for example,[5]) the sequence $\tilde{y}_{n,p}$ *converges almost surely and in* L^1 *to* $\mathbb{E}\Big[|B^H(1)|^p\Big]$ *as n tends to infinity. It follows that*

$$V_{n,p} = \sum_{i=1}^{n} \Big|B^H(\frac{i}{n}) - B^H(\frac{i-1}{n})\Big|^p$$

converges in probability respectively to 0 *if* $pH > 1$ *and to infinity if* $pH < 1$ *as n tends to infinity. Thus we can conclude that* $I(B^H,[0,T]) = \frac{1}{H}$. *Since for every semimartingale* X, *the index* $I(X,[0,T])$ *must belong to* $[0,1] \cup \{2\}$, *the fBm* B^H *cannot be a semimartingale unless* $H = \frac{1}{2}$.

1.2.6 Representation of Fractional Brownian Motion

There are some representations of the fractional Brownian motion as a Wiener integral.

1.2.6.1 Lévy-Hida Representation

Let B^H be a fractional Brownian motion with parameter $H \in (0,1)$. The fBm admits a representation as a Wiener integral of the form

$$B^H = \int_0^t K_H(t,s) dWs,$$

where $W = (W_t)_{t \in T}$ is a Wiener process, and $K_H(t,s)$ is the kernel

$$K_H(t,s) = d_H (t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right),$$

d_H being a constant and

$$F_1(z) = d_H \left(\frac{1}{2} - H\right) \int_0^{z-1} \theta^{H-\frac{3}{2}} \left(1 - (\theta+1)^{H-\frac{1}{2}}\right) d\theta.$$

If $H > \dfrac{1}{2}$,the kernel K_H has the simpler expression

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $t > s$ and $c_H = \left(\dfrac{H(H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$. The fact that the process B^H is a fBm follows is from the equality

$$\int_0^{t \wedge s} K_H(t,u) K_H(s,u) du = R_H(t,s).$$

The kernel K_H satisfies the condition

$$\frac{\partial K_H}{\partial t}(t,s) = d_H \left(H - \frac{1}{2}\right) \left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}.$$

1.2.6.2 Moving Average Representation

FBm can be represented as an integral with respect to a standard Brownian motion on the whole real line. Let $(B_s)_{s \in \mathbb{R}}$ be a standard Brownian motion. Then

$$B_t^H = \frac{1}{C(H)} \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] dB_s, \tag{1.2}$$

with $C(H) > 0$ an explicit normalizing constant, is a fractional Brownian motion.

1.2.6.3 Harmonizable Representation

There is another representation which uses the complex-valued Brownian motion (but the fBm is real-valued). In fact, for a fBm $(B_t^H)_{t\in\mathbb{R}}$, we obtain

$$B_t^H = \frac{1}{C_2(H)} \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} |x|^{-(H-\frac{1}{2})} d\tilde{B}_x, \quad t \in \mathbb{R},$$

where $(\tilde{B}_t)_{t\in\mathbb{R}}$ is a complex Brownian measure and

$$C_2(H) = \left(\frac{\pi}{H\Gamma(2H) sin(H\pi)} \right)^{1/2}.$$

Let us note that the complex Brownian measure on $\mathbb{R}$ can be splitted as $\tilde{B} = B_1 + iB_2$ and is such that $B_1(A) = B_1(-A)$, $B_2(A) = -B_2(-A)$ and $\mathbb{E}(B_1(A))^2 = \frac{| A |}{2}$, $\forall A \in \mathcal{B}(\mathbb{R})$. We also call this representation, the spectral representation.

2

Mixed Fractional Brownian Motion and its Properties

This chapter introduces a significant extension of fractional Brownian motion: the mixed fractional Brownian motion. After defining this process and examining its properties, we aim to construct a representation of mixed fractional Brownian motion in the white noise space. We demonstrate that this process is differentiable in the sense of distributions. Additionally, we explore the transformed characteristics of this process. This investigation leads us to our primary objective, which is stochastic analysis of mixed fractional Brownian motion.

Mixed fractional Brownian motion with parameter H is a stochastic process that was introduced by Cheridito[4], to model a financial phenomenon by the stochastic process $(X_t^H(a,b))_{t\in[0,1]}$ given by:

$$X_t^H(a,b) = X_0^H(a,b)e^{\nu t+\sigma M_t^H(a,b)}.$$

The authors took ν, σ, two constants, and $a > 0, b = 1$.

2.1 Definition of Mixed Fractional Brownian Motion

Definition 2.1.1 *A mixed fractional Brownian motion with parameters a, b and H is a process $M^H = \{M_t^H(a,b), \forall t \geq 0\} = \{M_t^H, \forall t \geq 0\}$ defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ as:*

$$\forall t \in \mathbb{R}_+, \qquad M_t^H = aB_t + bB_t^H,$$

where $(B_t^H)_{t\geq 0}$ is a fractional Brownian motion with parameter H independent of B^H and $(B_t)_{t\geq 0}$ is Brownian motion.

2.2 Basic properties of Mixed Fractional Brownian Motion

Proposition 2.2.1 *[19] The following properties are satisfied by the mixed fractional Brownian motion:*

1. *M^H is a centred Gaussian process.*
2. $\forall t \in \mathbb{R}_+, \qquad \mathbb{E}((M_t^H(a,b))^2) = a^2 t + b^2 t^{2H}$.
3. *Its covariance function is given by*
$$Cov(M_t^H(a,b), M_s^H(a,b)) = a^2 \min(t,s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \qquad \forall t,s \geq 0.$$
4. *The increments of mixed fractional Brownian motion are stationary.*
5. *For all $H \in (0,1)\backslash\{\frac{1}{2}\}, a \in \mathbb{R}, b \in \mathbb{R}, \quad (M_t^H(a,b))_{t\geq 0}$ is not a Markov process.*
6. *For all $\alpha > 0, \left(M_{\alpha t}^H(a,b)\right)_{t\geq 0} = \left(M_t^H(a\alpha^{\frac{1}{2}}, b\alpha)\right)_{t\geq 0}$, this property is called mixed autosimilarity.*

2.2.1 Correlation between the increments

Notation 2.2.1 *Assume that X and Y are two random variables that are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficient of correlation is noted $\rho(X,Y)$, as follows:*
$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}}.$$

Lemma 2.2.1 *[19]* $\forall s \in \mathbb{R}_+, \forall t \in \mathbb{R}_+, \forall h \in \mathbb{R}_+, 0 \leq h \leq t-s$
$$\rho(M_{t+h}^H - M_t^H, M_{s+h}^H - M_s^H) = \frac{b^2}{2(a^{2h+b^2h^{2H}})}\left[(t-s+h)^{2H} - 2(t-s)^{2H} + (t-s-h)^{2H}\right].$$

Corollary 2.2.1 *For all $a \in \mathbb{R}$ and $\quad b \in \mathbb{R}\backslash\{0\}$, the increments of $(M_t^H(a,b))_{t\in\mathbb{R}_+}$ are positively correlated if $\frac{1}{2} < H < 1$, negatively correlated if $0 < H < \frac{1}{2}$, and no correlated if $H = \frac{1}{2}$.*

Proof.
If $H < \frac{1}{2}$, by the concavity of the function $x \longmapsto x^{2H}$, one derives
$$\forall x \in \mathbb{R}_+, \forall h \in \mathbb{R}_+\backslash\{0\}, \quad (x+h)^{2H} - 2x^{2H} + (x-h)^{2H} < 0.$$

If $H > \frac{1}{2}$, by the convexity of the function $x \longmapsto x^{2H}$, one derives

$$\forall x \in \mathbb{R}_+, \forall h \in \mathbb{R}_+\backslash\{0\}, \quad (x+h)^{2H} - 2x^{2H} + (x-h)^{2H} > 0.$$

consequently, using the lemma (2.2.1),

$$\begin{cases} \text{If } H < \frac{1}{2}, & \rho(M^H_{t+h} - M^H_t, M^H_{s+h} - M^H_s) < 0. \\ \text{If } H > \frac{1}{2}, & \rho(M^H_{t+h} - M^H_t, M^H_{s+h} - M^H_s) > 0. \\ \text{If } H = \frac{1}{2}, & \rho(M^H_{t+h} - M^H_t, M^H_{s+h} - M^H_s) = 0. \end{cases}$$

■

Remark 2.2.1 *By using corollary (2.2.1) and lemma (2.2.1), we get*

i) *If $H > \frac{1}{2}$ (respectivelly $H < \frac{1}{2}$), if $a \neq 0, b_1,$ and b_2 are two reel constants such that $|b_1| \leq |b_2| (resp, |b_1| \geq |b_2|)$, since*

$$\forall s \in \mathbb{R}_+, \quad \forall t \in \mathbb{R}_+, \quad \forall h \in \mathbb{R}_+, \quad 0 \leq h \leq t-s$$

$$(M^H_{t+h}(a,b_1) - M^H_t(a,b_1), M^H_{s+h}(a,b_1) - M^H_s(a,b_1))$$

$$\leq (M^H_{t+h}(a,b_2) - M^H_t(a,b_2), M^H_{s+h}(a,b_2) - M^H_s(a,b_2)).$$

Then, if $H > \frac{1}{2}$ (resp. $H < \frac{1}{2}$)

1. *While |b| is great (resp. small), the increments are more correlated.*
2. *While |b| is small (resp. great), the ingrements are less correlated.*

ii) *If $H > \frac{1}{2}$ ($H < \frac{1}{2}$), if $b \neq 0, a_1,$ and a_2 are two reel constants such that $|a_1| \leq |a_2| (resp, |a_1| \geq |a_2|)$, since*

$$\forall s \in \mathbb{R}_+, \quad \forall t \in \mathbb{R}_+, \quad \forall h \in \mathbb{R}_+, \quad 0 \leq h \leq t-s$$

$$(M^H_{t+h}(a_2,b) - M^H_t(a_2,b), M^H_{s+h}(a_2,b) - M^H_s(a_2,b))$$

$$\leq (M^H_{t+h}(a_1,b) - M^H_t(a_1,b), M^H_{s+h}(a_1,b) - M^H_s(a_1,b)).$$

Then, if $H > \frac{1}{2}$ (resp., $H < \frac{1}{2}$), we have

1. *While |a| is great (resp. small), the increments are less correlated.*
2. *While |a| is small (resp. great), the ingrements are more correlated.*

***In practical application**, we can select H, a, b such that the $M_t^H(a,b)$ would be a suitable model for a given phenomenon.*

2.2.2 Long and short term dependance

Lemma 2.2.2 *The increments of mixed fractional Brownian motion are long-term dependent if and only if $H > \frac{1}{2}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\{0\}$.*

Proof.
For all $n \in \mathbb{N}^*$,

$$\begin{aligned} r(n) &= \mathbb{E}\left((M_{n+1}^H - M_n^H)M_1^H\right) = \frac{b^2}{2}\left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right] \\ &= b^2 H(H-1)n^{2H-2}\epsilon(n), \end{aligned}$$

where $\underset{n\to+\infty}{lim} \epsilon(n) = 0$.

Observing that $\sum_{n\in\mathbb{N}^*} r(n) = +\infty$, we can conclude that $H > \frac{1}{2}$ if and only if $2H-2 > -1$.■

2.2.3 Hölderian Continuity and differentiability

Lemma 2.2.3 *The mixed fractional Brownian motion has a modification with trajectories that are $\gamma-$Hölder continuous in $[0,T]$ for all $T > 0$ and $\gamma < \frac{1}{2} \wedge H$.*

Proof.
Using *Kolmogorov*'s theorem, it is sufficient to demonstrate that

$$\forall\alpha > 0, \exists C_\alpha, \forall(s,t) \in [0,T]^2, \qquad \mathbb{E}\left(\left|M_t^H - M_s^H\right|^\alpha\right) \leq C_\alpha |t-s|^{\alpha(\frac{1}{2}\wedge H)}.$$

Based on the increments of M_t^H, we get the stationarity (proposition2.2.1) and mixed auto-similarity (proposition 2.2.1).

$$\begin{aligned} \mathbb{E}\left(\left|M_t^H - M_s^H\right|^\alpha\right) &\leq \mathbb{E}\left(\left|M_{t-s}^H\right|^\alpha\right) \\ &\leq \mathbb{E}\left(\left|M_1^H(a(t-s)^{\frac{1}{2}-H}), b(t-s)H\right|^\alpha\right). \end{aligned}$$

Two positive constants, C_1 and C_2, depending on α, exist if $H \leq \dfrac{1}{2}$, such that

$$\begin{aligned}\mathbb{E}\left(\left|M_t^H - M_s^H\right|^{\alpha}\right) &\leq (t-s)^{\alpha H}\mathbb{E}\left(\left|M_1^H(a(t-s)^{\frac{1}{2}}, b)\right|^{\alpha}\right)\\ &\leq (t-s)^{\alpha H}\left[C_1|a|^{\alpha}(t-s)^{\alpha((\frac{1}{2})-H)}\mathbb{E}(|B_1|^{\alpha}) + C_2|b|^{\alpha}\mathbb{E}(|B_1^H|^{\alpha})\right]\\ &\leq C_{\alpha}(t-s)^{\alpha H},\end{aligned}$$

where

$$C_{\alpha} = C_1|a|^{\alpha}T^{\alpha((\frac{1}{2})-H)}\mathbb{E}(|B_1|^{\alpha}) + C_2|b|^{\alpha}\mathbb{E}(|B_1^H|^{\alpha}).$$

Two positive constants, C_1' and C_2', depending on α, exist if $H > \dfrac{1}{2}$. These constants ensure

$$\begin{aligned}\mathbb{E}\left(\left|M_t^H - M_s^H\right|^{\alpha}\right) &\leq (t-s)^{\frac{\alpha}{2}}\mathbb{E}\left(\left|M_1^H(a, b(t-s)^{H-\frac{1}{2}})\right|^{\alpha}\right)\\ &\leq (t-s)^{\frac{\alpha}{2}}\left[C_1'|a|^{\alpha}\mathbb{E}(|B_1|\alpha) + C_2'|b|^{\alpha}(t-s)^{\alpha(H-(\frac{1}{2}))}\mathbb{E}(|B_1^H|^{\alpha})\right]\\ &\leq C_{\alpha}(t-s)^{\frac{\alpha}{2}},\end{aligned}$$

where

$$C_{\alpha} = C_1'|a|^{\alpha}\mathbb{E}(|B_1|\alpha) + C_2'|b|^{\alpha}T^{\alpha(H-(\frac{1}{2}))}\mathbb{E}(|B_1^H|^{\alpha}).$$

The concepts presented by *Kolwankar* and *Gangal* were followed by *Ben Adda* [10] and *Cresson* [2] in their analysis, as per the findings of *Mounir Zili* [19]. ■

Definition 2.2.1 *Let f be a continuous function on $[a,b]$, and Let $\alpha \in]0,1[$. $\alpha-$local fractional derivative of f in $t_0 \in [a,b]$ is what we refer to it as. $d_{\sigma}^{\alpha}f(t_0)$ provided by*

$$d_{\sigma}^{\alpha}f(t_0) = \Gamma(1+\alpha)\lim_{t\to t_0^{\sigma}}\frac{\sigma(f(t)-f(t_0))}{|t-t_0|^{\alpha}},$$

for $\sigma = +(resp, \sigma = -)$, where the Euler-function is denoted by Γ.

Definition 2.2.2 *Let $\alpha \in]0,1[$, and let f be a continuous function on $[a,b]$. In $t_0 \in [a,b]$, the function f is $\alpha-$differentiable. assuming the existence and equality of $d_{+}^{\alpha}f(t_0)$ and $d_{-}^{\alpha}f(t_0)$.*
In this instance, $d^{\alpha}f(t_0)$ is the $\alpha-$derivative of f in t_0.

Theorem 2.2.1 *For all $t_0 \geq 0$, the trajectories of mixed fractional Brownian motion are nearly certainly $\alpha-$differentiable for any $\alpha \in]0, \dfrac{1}{2} \wedge H[$; furthermore,*

$$\forall t_0 \geq 0, \qquad \mathbb{P}(d^{\alpha}M_{t_0}^H = 0) = 1.$$

Proof.
The evidence $\sigma = +$ is provided.(The same proof applies to $\sigma = -$).
By employing mixed stationarity and auto-similarity of mixed fractional Brownian motion increases, we have

$$\begin{aligned}\frac{M_t^H - M_{t_0}^H}{(t-t_0)^\alpha} &\overset{\mathfrak{d}}{=} (t-t_0)^{-\alpha} M_1^H \left(a(t-t_0)^{\frac{1}{2}}, b(t-t_0)^H\right) \\ &\overset{\mathfrak{d}}{=} a(t-t_0)^{\frac{1}{2}-\alpha} B_1 + b(t-t_0)^{H-\alpha} B_1^H.\end{aligned}$$

Thus, if $0 < \alpha < \dfrac{1}{2} \wedge H$,

$$\begin{aligned}\mathbb{P}\left(d_+^\alpha M_{t_0}^H = 0\right) &= \mathbb{P}\left(\lim_{t\to t_0} \frac{M_t^H - M_{t_0}^H}{(t-t_0)^\alpha} = 0\right) \\ &= \mathbb{P}\left(\lim_{t\to t_0} a(t-t_0)^{\frac{1}{2}-\alpha} B_1 + b(t-t_0)^{H-\alpha} B_1^H = 0\right) = 1.\end{aligned}$$

■

Theorem 2.2.2 *For all $\alpha \in]\dfrac{1}{2} \wedge H, 1[$, the trajectories of the mixed fractional Brownian motion are not $\alpha-$differentiables almost surely.*

Proof.
When $d > 0$, the event is defined as

$$A(t) = \left\{ \sup_{0\le s\le t} \left| \frac{M_s^H(a,b)}{s^\alpha} \right| > d \right\}.$$

For every sequence $t_n \searrow 0$, we have

$$A(t_{n+1}) \subset A(t_n).$$

So,

$$\mathbb{P}(\lim_{t\to+\infty} A(t_n)) = \lim_{t\to+\infty} \mathbb{P}(A(t_n)),$$

By employing M^H's mixed autosimilarity, we have

$$\begin{aligned}\mathbb{P}(A(t_n)) &\geq \mathbb{P}\left(\left|\frac{M_{t_n}^H(a,b)}{t_n^\alpha}\right| > d\right) \\ &= \mathbb{P}\left(|a t_n^{\frac{1}{2}-\alpha} B_1 + b t_n^{H-\alpha} B_1^H| > d\right).\end{aligned}$$

i) If $H < \dfrac{1}{2}$, in this case $\alpha > \dfrac{1}{2}$

$$\mathbb{P}(A(t_n)) \geq \mathbb{P}\left(|a t_n^{\frac{1}{2}-H} B_1 + b B_1^H| > t_n^{\alpha-H} d\right),$$

$$\lim_{n\to+\infty} \mathbb{P}\left(|a t_n^{\frac{1}{2}-H} B_1 + b B_1^H| > t_n^{\alpha-H} d\right) = \mathbb{P}\left(|B_1^H| \geq 0\right) = 1.$$

ii) If $H = \frac{1}{2}$, in this case $\alpha > H$ and $\alpha > \frac{1}{2}$

$$\mathbb{P}(A(t_n)) \geq \mathbb{P}\left(|aB_1 + bB_1^H| > t_n^{\alpha-H}d\right),$$

so,

$$\underset{n\to+\infty}{lim} \mathbb{P}\left(|aB_1 + bB_1^H| > t_n^{\alpha-H}d\right) = \mathbb{P}\left(|aB_1^H| + bB_1^H \geq 0\right) = 1.$$

iii) If $H > \frac{1}{2}$, in this case $\alpha > \frac{1}{2}$

$$\mathbb{P}(A(t_n)) \geq \mathbb{P}\left(|aB_1 + bt_n^{H-\frac{1}{2}}B_1^H > t^{\alpha-\frac{1}{2}}| > d\right),$$

so,

$$\underset{n\to+\infty}{lim} \mathbb{P}\left(|aB_1 + bt_n^{H-\frac{1}{2}}B_1^H > t^{\alpha-\frac{1}{2}}| > d\right) = \mathbb{P}\left(|aB_1| \geq 0\right) = 1.$$

We deduce that for every $t_0 \geq 0$, for all $\alpha \in]\frac{1}{2} \wedge H, 1[$,

$$\mathbb{P}\left(\underset{t\to t_0}{lim\sup}\left|\frac{M_t^H - M_{t_0}}{(t-t_0)^\alpha}\right| = +\infty\right) = 1.$$

■

2.2.4 Semimartingale property

The classical notion of semimartingale has emerged from a sequence of generalizations of Brownian motion, each extending the class of stochastic processes that can serve as integrators in Itô's stochastic integration framework [13]. A stochastic process X_t that is adapted to a filtration $\mathbb{F}$ is called an $\mathbb{F}$-semimartingale if it satisfies the following condition:

$$I_X(\beta(\mathbb{F})) \quad \text{is bounded in } L^0 \tag{2.1}$$

where

$$\beta(\mathbb{F}) = \left\{\sum_{j=0}^{n-1} f_j \mathbb{1}_{(t_j,t_{j-1})}\ n \in \mathbb{N}, 0 \leq t_0, \ldots, t_n \leq 1, \forall j, f_j \quad \text{is } \mathbb{F}-\text{measurable and} |f_j| \leq 1 \quad a.s.\right\}$$

and

$$I_X(\vartheta) = \sum_{j=0}^{n-1} f_j(X_{t_{j+1}} - X_{t_j}) \qquad for \qquad \vartheta = \sum_{j=0}^{n-1} f_j \mathbb{1}_{(t_j,t_{j+1}]} \in \beta(\mathbb{F})$$

Cheridito [4] suggested using a less detailed definition of semimartingale in their work. Actually, he defines a less robust semimartingale formula.

Definition 2.2.3 *If a stochastic process X_t is $\mathcal{F}$-adapted and satisfies (2.1), then it is weak $\mathcal{F}$-semimartingale.*
If X is a weak -semimartingale, then we say that it is a weak semimartingale. If X is a weak $\bar{\mathbb{F}}$-semimartingale, then we call it a semimartingale.

Example
Confirming that the deterministic process

$$X_t = \begin{cases} 0, & \text{for } t \in [0, \frac{1}{2}] \\ 1, & \text{for } \quad t \in (\frac{1}{2}, 1]. \end{cases}$$

Is a weak semimartingale. But, it is not a semimartingale because it is not continuous on the right as.

Lemma 2.2.4 *[4] Consider a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Every random process that is a weak $\mathbb{F}$-semimartingale satisfies certain conditions with respect to this filtration. Now, let $(X_t)_{t\geq 0}$ be a process that is right-continuous almost surely. Specifically, X is a weak semimartingale with respect to the completed filtration $\bar{\mathcal{F}}$. In other words, X is a $\bar{\mathbb{F}}$-semimartingale if it is right-continuous almost surely.*

It can be deduced from lemma (2.2.4) that for any filtration $\mathbb{F}$, a right-continuous weak $\mathbb{F}$-semimartingale also qualifies as a $\bar{\mathbb{F}}$-semimartingale.

Determining whether the mixed fractional Brownian motion is a $\mathbb{F}$-semimartingale becomes straightforward when $H = \frac{1}{2}$. It becomes evident that the expression

$$\frac{1}{\sqrt{1+\alpha^2}} M^{\frac{1}{2},\alpha}$$

represents a Brownian motion, and specifically, it serves as a $\bar{\mathbb{F}}^{M^{\frac{1}{2},\alpha}}$-semimartingale. Consequently, $M^{\frac{1}{2},\alpha}$ qualifies as a semimartingale under this condition. For other scenarios, we refer to the primary findings outlined in the ensuing theorem.

Theorem 2.2.3 *[4] M^H is not a weak semimartingale if $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$, it is equivalent to $\sqrt{1+\alpha^2}B_t$ if $H = \frac{1}{2}$ and equivalent to Brownian motion if $H \in (\frac{3}{4}, 1]$.*

Different techniques were employed by *Cheridito* [4] to prove this theorem. The demonstration relies on the fact that the quadratic variation of fractional Brownian motion for $H \in (0, \frac{1}{2})$ is not finite. Therefore, for $H < \frac{1}{2}$, the mixed fractional Brownian motion will exhibit an infinite quadratic variation.

In the case of $H > \frac{1}{2}$, the opposite holds. In fact, the process's quadratic variation is the same as the Brownian motion's. In this instance, however, the mixed fractional Brownian motion is equal to the Brownian motion for $H \in (\frac{3}{4}, 1)$ rather than being a weak semimartingale for $H \in (\frac{1}{2}, \frac{3}{4}]$.

Proof.
Using *Stricker* [15]'s theorem, the proof is abridging only for the case $H \in (\frac{1}{2}, \frac{3}{4})$.
First, we introduce the *Stricker* theorem. Since the processes are indexed on $[0, 1]$, we operate on a complete probability space. Let $(X_t)_{t\in[0,1]}$ be a stochastic process such that a Gaussian space containing all the possible combinations of the random variables $\mathbb{E}(X_t/\mathcal{F}_s), s, t \in [0, 1]$ exists. Remember that the fact that the collection $I_X(\beta)$ is confined in L^0 has been used to characterize a semimartingale. ■

Theorem 2.2.4 ***Stricker 1983***[15]. *Suppose we have a Gaussian process $(X_t)_{t\in[0,1]}$ with natural filtration. If $I_X(\beta)$ is bounded in L^0, then, it is bounded in L^2.*

Definition 2.2.4 *A stochastic process $(X_t)_{t\in[0,1]}$ is a quasi-martingale if*

$$X_t \in L^1, \quad \forall t \in [0, 1],$$

and

$$\sup_{\tau} \sum_{j=0}^{n-1} \left\| \mathbb{E}(X_{t_{j+1}} - X_{t_j}/\mathcal{F}^X_{t_j}) \right\|_1 < \infty,$$

where τ is the set of all finite partitions of $[0, 1]$.

Remark 2.2.2 $(X_t)_{t\in[0,1]}$ *is a quasi-martingale since $I_X(\beta)$ is bounded in L^2.*

Theorem 2.2.5 ***Stricker 1983***[15] *if $(M^H_t)_{t\in[0,1]}$ is not a quasi-martingale then is not a weak semimartingale.*

Proof.
Assume M^H is a weak semimartingale. According to Stricker's theorem (Theorem 2.2.4), $I_{M^H}(\beta(\mathcal{F}^{M^H}))$ is bounded in L^2. Consequently, it is also bounded in L^1. For all partitions $0 = t_0 < t_1 \ldots < t_n = 1$

$$\sum_{j=0}^{n-1} sign\left(\mathbb{E}\left[M^H_{t_{j-1}} - M^H_{t_j}/\mathcal{F}_{t_j}\right]\right) \mathbb{1}_{(t_j, t_{j-+1}]} \in \beta(\mathbb{F}^{M^H}),$$

and

$$\left\| I_{M^H}\left(\sum_{j=0}^{n-1} sign\left(\mathbb{E}\left[M^H_{t_{j-1}} - M^H_{t_j}/\mathcal{F}_{t_j}\right]\right)\mathbb{1}_{(t_j,t_{j-+1}]}\right)\right\|_1$$

$$\geq \mathbb{E}\left[I_{M^H}\left(\sum_{j=0}^{n-1} sign\left(\mathbb{E}\left[M^H_{t_{j-1}} - M^H_{t_j}/\mathcal{F}_{t_j}\right]\right)\mathbb{1}_{(t_j,t_{j-+1}]}\right)\right]$$

$$= \sum_{j=0}^{n-1}\left\|\mathbb{E}\left[M^H_{t_{j-1}} - M^H_{t_j}/\mathcal{F}_{t_j}\right]\right\|_1.$$

It follows that M^H is a quasi-martingale. Consequently, if M^H is not a quasi-martingale, it cannot be a weak semimartingale.
It remains to prove that M^H is not a quasi-martingale. We will demonstrate this for $H \in \left(\frac{1}{2}, \frac{3}{4}\right]$ by calculating

$$\sum_{j=0}^{n-1}\left\|\mathbb{E}\left(\Delta^n_{j+1}M^H\middle|\mathcal{F}^{M^H_{\frac{j}{n}}}_{t_j}\right)\right\|_1$$

and showing that this quantity tends to infinity as $n \to \infty$. This proves that $(M_t^{\frac{3}{4}})_{t\in[0,1]}$ is not a quasi-martingale.

Cheridito [4] obtained a remarkable result by demonstrating that the mixed fractional Brownian motion is a semimartingale for $H \in \left]\frac{3}{4}, 1\right)$. Specifically, he showed that the sum of two independent centered Gaussian processes, the first being Brownian motion and the second being fractional Brownian motion with parameter H is a semimartingale if $H \in \left]\frac{3}{4}, 1\right)$. This result leads us to consider examples where the sum of two independent centered Gaussian processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ forms a semimartingale, despite at least one of the processes not being a semimartingale individually. ■

Example:
We examine the Brownian bridge $(\eta_u(t), t \leq u)$ over the interval $[0, u]$, defined as the Brownian motion process $(B_t, t \leq u)$ conditioned on $B_u = 0$. It's worth recalling that $\eta_u(t)$ can be expressed as $\eta_u(t) = B_t - \frac{t}{u}B_u$, ensuring its independence from B_u. Its canonical decomposition is represented by

$$\eta_u(t) = \beta_t - \int_0^t ds\frac{\eta_u(s)}{u-s}, \quad t \leq u,$$

where β_t stands for the Brownian motion within the filtration $\{\mathcal{P}^u_t, t \leq u\}$ of $\eta_u(t)$. Additionally, we present the following proposition.

Proposition 2.2.2 *Let $f \in L^2([0,u])$, then*

1. *The process*

$$\int_0^t f(s)\eta_u(s) = \int_0^t f(s)d\beta_s - \int_0^t dsf(s)\frac{\eta_u(s)}{u-s}$$

is well defined for $t \leq u$, with

$$\int_0^u f(s)d\eta_u(t) = \lim_{t\to u}\int_0^t f(s)d\eta_u(s) \ p.s\ dans \quad L^2.$$

2. $(\int_0^t f(s)\eta_u(s))$ *is a semimartingale w.r.t* $\{\mathcal{P}_t^u, t \leq u\}$ *if and only if*

$$\int_0^t ds|f(s)|\frac{1}{\sqrt{u-s}} < \infty.$$

Now, let $u \in]0,1]$ and $\alpha \in]\frac{1}{2},1]$ and let the function

$$\psi(s) = \frac{1}{\sqrt{u-s}}|\log(u-s)|^{-\alpha}\mathbb{1}_{\frac{u}{2}<s<u},$$

satisfies

$$\int_0^u ds\psi^2(s) < \infty \quad \text{but} \quad \int_0^u ds\psi(s)\frac{1}{\sqrt{u-s}} = \infty.$$

To accomplish our objective, we decompose the Brownian motion $(B_t)_{t\geq 0}$ as follows

$$B_t = \eta_u(t) + \frac{t}{u}B_u, \quad t \leq u.$$

We consider $g \in L^2([0,u])$ such that

$$\int_0^t ds|g(s)|\frac{1}{\sqrt{u-s}} = \infty,\ \text{et}\ g(s) \neq 0,\ \text{for all}\ s.$$

Then, we take

$$X_t = \int_0^t g(s)d\eta_u(s), \quad \text{and} \quad Y_t = \frac{B_u}{u}\int_0^t g(s)ds.$$

Given that X and Y are independent and $X_t + Y_t = \int_0^t g(s)dB_s$, it follows that $X_t + Y_t$ constitutes a martingale.

In a broader context, let $u \in (0,1)$, employing a similar approach. Initially, we decompose the process $(B_t)_{t\geq 0}$ into $\eta_u(t) + \frac{t}{u}B_u$.

Then $\hat{B}_t = B_{t+u} - B_u, t \leq 1-u$ in $\hat{\eta}_{1-u}(t) + \dfrac{t}{1-u}\hat{B}_{1-u}$.

After, for $f \in L^2([0,1])$, we write

$$\begin{aligned}\int_0^t f(s)dB_s &= \int_0^t f(s)\mathbb{1}_{(s\leq u)}dB_s + \mathbb{1}_{(u<t)}\int_u^t f(s)dB_s \\ &= \int_0^t f(s)\mathbb{1}_{(s\leq u)}d\eta_u(s) + \frac{B_u}{u}\int_0^t f(s)\mathbb{1}_{(s\leq u)}ds \\ &+\mathbb{1}_{(u<t)}\int_u^t f(s)d\hat{\eta}_{1-u}(s-u) + \mathbb{1}_{(u<t)}\frac{B_1-B_s}{1-u}\int_u^t f(s)ds.\end{aligned}$$

Now, we choose g such that

$$\int_0^t |g(s)|\frac{ds}{\sqrt{u-s}} = \infty, \quad \int_u^1 |g(s)|\frac{ds}{\sqrt{1-s}} = \infty, \quad \text{and for all} \quad s<1.$$

Then, we have:

$$X_t = \int_0^t g(s)\mathbb{1}_{(s\leq u)}d\eta_u(s) + \mathbb{1}_{(u<t)}\frac{B_1-B_s}{1-u}\int_u^t g(s)ds,$$

and

$$Y_t = \mathbb{1}_{(u<t)}\int_u^t g(s)d\hat{\eta}_{1-u}(s-u) + \frac{B_u}{u}\int_0^t g(s)\mathbb{1}_{(s\leq u)}ds$$

These are two independent Gaussian processes. Furthermore, their sum $X_t + Y_t = \int_0^t g(s)dB_s$ constitutes a martingale.

We can confirm that both Y and X do not qualify as semimartingales by applying proposition (2.2.2).

3

Stochastic Differential Equations Driven by Mixed Fractional Brownian Motion

Stochastic differential equations (SDEs) driven by mixed fractional Brownian motion have emerged as a powerful framework for modeling systems influenced by both short-memory (*standard Brownian motion*) and long-memory noise (*fractional Brownian motion*) . Such equations are particularly relevant in finance, physics, and biology, where phenomena often exhibit multi-scale dependencies or path-dependent volatility. However, the interplay between these two types of noise introduces significant mathematical challenges, especially when the fractional Brownian motion (fBm) has a Hurst parameter $H \in (3/4, 1)$, which lies outside the classical Itô calculus regime.

This chapter explores the existence, uniqueness, and stability of solutions to mixed SDEss of the form:

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t + c(t, X_t)dB_t^H, \tag{3.1}$$

where B_t is a standard Brownian motion, B_t^H is an fBm with $H \in (3/4, 1)$, and a, b, c are coefficients satisfying specific regularity conditions. A key focus is the *stabilizing term technique*, where solutions to a regularized equation (with an auxiliary vanishing term) converge to the solution of the original equation. This approach addresses the inherent difficulties of handling fBm-driven integrals and ensures robustness in the limiting process.

The chapter bridges theoretical rigor and practical applicability, offering tools to analyze mixed SDEs in high-persistence regimes ($H > 1/2$) and providing insights into their numerical and real-world implications.

3.1 Stochastic Analysis of Mixed Fractional Brownian Motion

In this section, we study mixed stochastic differential equations (SDEs) driven by mixed fractional Brownian motion of the form:

$$X_t = X_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s)\, dB_s + \int_0^t c(s, X_s)\, dB_s^H, \quad t \in [0, T],$$

(3.1)

where B is a standard Brownian motion and B^H is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. The integral with respect to B is the Itô integral, while the integral with respect to B^H is the generalized Lebesgue-Stieltjes or Young integral.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered and complete probability space. Here, $(B_t)_{t\geq 0}$ is an $\mathcal{F}_t$-adapted Brownian motion, and $(B_t^H)_{t\geq 0}$ is an $\mathcal{F}_t$-adapted fractional Brownian motion.

3.1.1 Stochastic Calculus for Fractional Brownian Motion

A fundamental idea in stochastic calculus is the following: If X is a semimartingale and f is a $\mathcal{C}^2$-function, then $f(X)$ is a semimartingale, and Itô's formula applies. However, the fractional Brownian motion $(B_t^H)_{t\geq 0}$ with Hurst index $0 < H < 1$ is a semimartingale *if and only if* $H = \frac{1}{2}$, which corresponds to standard Brownian motion.

Natural questions arise: For $H \neq \frac{1}{2}$, can we construct stochastic integrals with respect to fractional Brownian motion? Can an Itô-type formula be established?

Various methods have been employed to construct stochastic calculus with respect to fractional Brownian motion. Below are notable contributions in this field:

- The *Malhavin calculus*, also referred to as stochastic variation calculus. This framework serves as a powerful tool for defining stochastic integrals (see [20], [21], [22]).
- *Wick calculus* [24].
- The pathwise stochastic integral with respect to fractional Brownian motion, introduced by *Zähle* [25].
- *Rough path analysis* [23].

3.1.1.1 Wiener Integral

Let $\mathcal{E}$ denote the space of elementary functions. The elementary Wiener integral with respect to fractional Brownian motion can be defined as follows:

Definition 3.1.1 *For a fractional Brownian motion $(B_t^H)_{t\geq 0}$, the Wiener integral with respect to B^H is defined for step functions $f \in \mathcal{E}$ as:*

$$\mathcal{I}^H(f) = \int_{\mathbb{T}} f(u)\, dB_u^H = \sum_{k=1}^{n} f_k \left(B_{u_{k+1}}^H - B_{u_k}^H \right), \quad \mathbb{T} = [0, T],$$

where $f(u) = \sum_{k=1}^{n} f_k \mathbb{I}_{(u_k, u_{k+1})}(u)$.

The integral $\mathcal{I}^H$ is extended to a larger space $\mathcal{H}$ (a Hilbert space with inner product) containing $\mathcal{E}$. This space is denoted $\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}} = \tilde{\mathcal{E}}$.

Definition 3.1.2 *The Wiener integral with respect to fractional Brownian motion is the isometric mapping* T^H*, defined as:*

$$T^H : \tilde{\mathcal{H}} \to S_{pT}(B^H),$$

$$f \mapsto T^H(f) = X,$$

where $S_{pT}(B^H) = \left\{ X \, : \, T^H(f_n) \xrightarrow{L^2} X, \; f_n \subset \mathcal{E} \right\}$*. Here,* X *is associated with an equivalent sequence of elementary functions* $(f_n)_{n\in\mathbb{N}}$ *such that* $T^H(f_n) \xrightarrow{L^2} X$*. Additionally, we write* $\int_T f_X(t)\, dB_t^H$*, where* f_X *belongs to the equivalence class.*

Our central question remains: What is the class of integrands in the definition of the Wiener integral that is isometric to the space $S_{pT}(B^H)$?

The following theorem forms the foundation:

Theorem 3.1.1 ([8]) *Let* $\tilde{\mathcal{H}}$ *be a class of integrands, where* $\mathcal{E} \subset \tilde{\mathcal{H}}$ *is the class of elementary functions, and* $T^H(f)$ *denotes the integral of* $f \in \mathcal{E}$ *with respect to fractional Brownian motion* $(B_t^H)_{t\geq 0}$ *with* $H \in (0,1)$*. Under the following assumptions:*

- $\tilde{\mathcal{H}}$ *is a space equipped with the inner product* $(f,g)_{\tilde{\mathcal{H}}}$ *for* $f, g \in \tilde{\mathcal{H}}$*;*
- *For* $f, g \in \mathcal{E}$*,* $\langle f, g\rangle_{\tilde{\mathcal{H}}} = \mathbb{E}\left[T^H(f)T^H(g)\right]$*;*
- *The set* $\mathcal{E}$ *is dense in* $\tilde{\mathcal{H}}$*.*

Then the following assertions hold:

1. *There exists an isometry between the space* $\tilde{\mathcal{H}}$ *and the linear subspace of* $S_{pT}(B^H)$*, which extends the mapping*

$$f \mapsto T^H(f).$$

2. $\tilde{\mathcal{H}}$ *is isometric to* $S_{pT}(B^H)$ *if and only if* $\tilde{\mathcal{H}}$ *is complete.*

3.1.1.2 Young Integral

For $H \in (0,1)$, the trajectories of fractional Brownian motion are not absolutely continuous, precluding the use of Riemann-Stieltjes integration. However, by L.C. Young [23], if f is sufficiently regular (Hölder continuous), the integral $\int_0^t f(s)\, dB_s^H$ can be defined as a limit of Riemann sums. Let $C^\alpha(I)$ denote the space of α-Hölder continuous functions on interval I.

Theorem 3.1.2 ([23]) *Let $f \in C^\beta([0,T])$ and $g \in C^\gamma([0,T])$. If $\beta + \gamma > 1$, then for any partition (t_i^n) of $[0,T]$ with mesh tending to zero, the Riemann sum:*

$$\sum_{i=0}^{n-1} f(t_i^n) \left(g(t_{i+1}^n) - g(t_i^n) \right)$$

converges to a limit independent of the partition, denoted $\int_0^T f\, dg$, called the Young integral.

Proposition 3.1.1 ([25]) *Let $f \in C^\lambda([a,b])$, $g \in C^\beta([a,b])$ with $\lambda + \beta > 1$, and $1 - \beta < \alpha < \lambda$. The Young integral exists and can be expressed as:*

$$\int_a^b f\, dg = (-1)^\alpha \int_a^b d_+^\alpha f(a) d_-^{1-\alpha} g_-(b)\, dt,$$

where $g_-(b) = g(t) - g(b)$.

3.1.1.3 Skorohod Integral

The Skorohod integral, introduced by A. Skorohod in 1975, extends the Itô integral to non-adapted integrands and is linked to Malliavin calculus. Let $u(t,\omega)$ be a measurable process such that $u(t)$ is $\mathcal{F}_t$-measurable and $\mathbb{E}[u^2(t)] < \infty$. Using the Wiener chaos expansion:

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}),$$

where $I_n(f) = \int_{[0,T]^n} f(t_1, \ldots, t_n)\, dB_{t_1} \cdots dB_{t_n}$, we define:

Definition 3.1.3 *The Skorohod integral of u is:*

$$\delta(u) = \int_0^T u(t)\, \delta B_t = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n),$$

where $\tilde{f}_n$ is the symmetrization of f_n, provided convergence in $L^2(\mathbb{P})$.

Skorohod Integral for Fractional Brownian Motion

For $H > \frac{1}{2}$, fractional Brownian motion trajectories are α-Hölder continuous ($\alpha < H$), allowing pathwise Young integration. For $H < \frac{1}{2}$, trajectories are rougher, necessitating divergence-type integrals (e.g., Skorohod). Let B^H be a fractional Brownian motion and $G_t = \int_0^t K(t,s)\, dB_s^H$ with kernel K.

Definition 3.1.4 *For $F = f(G(\phi_1), \ldots, G(\phi_n))$ where $f \in C_b^\infty(\mathbb{R}^n)$, the Malliavin derivative D is:*

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(G(\phi_1), \ldots, G(\phi_n))\phi_i.$$

Itô Formula for Fractional Brownian Motion

Theorem 3.1.3 (Itô Formula for B^H[20]) *For $F \in C^2(\mathbb{R})$:*

$$F(B_t^H) = F(0) + \int_0^t F'(B_s^H) dB_s + H \int_0^t F''(B_s^H) s^{2H-1} ds.$$

3.2 The Mixed SDEs Involving Both the Wiener Process and fBm

Real-World Time-Varying Phenomena (such as climate and weather derivatives, stock market prices, etc.) may exhibit both long-memory components (modeled by fractional Brownian motion with $H \in (1/2, 1)$) and memoryless components (modeled by a Wiener process). Consequently, it is natural to analyze stochastic differential equations that incorporate both standard Brownian motion and fractional Brownian motion. These equations are termed mixed stochastic differential equations (and their corresponding frameworks are referred to as mixed models).

We begin by focusing on semilinear stochastic differential equations.

3.2.1 Existence, Uniqueness, and Regularity for Mixed SDEss

3.2.1.1 The Mixed Semilinear SDEss.

Theorem 3.2.1 (Existence, Uniqueness, and Hölder Regularity) *Consider the mixed semilinear stochastic differential equation:*

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \sigma_1 \int_0^t X_s\, dB_s + \sigma_2 \int_0^t X_s\, dB_s^H, \quad t \in [0, T], \tag{3.2}$$

where:

- *X_0 is $\mathcal{F}_0$-measurable,*
- *B_t is a standard Brownian motion,*
- *B_t^H is a fractional Brownian motion with Hurst index $H \in (0,1)$,*
- *The drift $b : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies:*
  1. ***Lipschitz condition**:*
  $$|b(t,x) - b(t,y)| \leq L|x-y| \quad \forall t \in [0,T],\ x,y \in \mathbb{R}, \tag{3.3}$$
  2. ***Linear growth condition**:*
  $$|b(t,x)| \leq L(1+|x|) \quad \forall t \in [0,T],\ x \in \mathbb{R}. \tag{3.4}$$
  3. ***Continuous in both variables**: $b \in C([0,T] \times \mathbb{R}$*

Then:

1. *There exists a unique solution $\{X_t, t \in [0,T]\}$ to (3.2).*
2. *The trajectories of X almost surely belong to $C^{1/2-\epsilon}[0,T]$ (i.e., Hölder continuous of order $\frac{1}{2} - \epsilon$ for any $\epsilon > 0$).*

Proof.
We first construct a local solution using the auxiliary PDE system:

$$\begin{cases} \dfrac{\partial h}{\partial Z_j}(Y,(Z_1,Z_2)) = \sigma_j h(Y,(Z_1,Z_2)),\ j = 1,2, \\ h(Y_0,0,0) = X_0. \end{cases}$$

Its explicit solution is:

$$h(Y,(Z_1,Z_2)) = (Y - Y_0 + X_0)\exp\{\sigma_1 Z_1 + \sigma_2 Z_2\},$$

where $Z_1(t) = B_t$ and $Z_2(t) = B_t^H$. Define $X_t = h(Y_t, Z_1(t), Z_2(t))$, where Y is an $\mathcal{F}_0$-measurable process with $C^1[0,T]$-trajectories. By Itôs formula:

$$dX_t = \sum_{i=1}^{2} \frac{\partial h}{\partial Z_i} dZ_i(t) + \frac{\partial h}{\partial Y} Y_t' dt + \frac{1}{2}\sigma_1^2 h\, dt.$$

Matching with (3.2), the ODE for Y becomes:

$$Y_t' = \frac{b(t,(Y_t - Y_0 + X_0)c_1(t))}{c_1(t)} - \frac{1}{2}\sigma_1^2(Y_t - Y_0 + X_0),$$

where $c_1(t) = \exp\{\sigma_1 Z_1(t) + \sigma_2 Z_2(t)\}$.

For fixed $\omega \in \Omega$, let $L_1(T) = \max_{0\le t\le T} c_1(t)^{-1}$, $L_2(T) = \max_{0\le t\le T} c_1(t)$, $D_1 = LL_1(T)$, and $D_2 = L + \frac{1}{2}\sigma_1^2$. On $[0, t_0]$ with $t_0 = \min(a_0, b_0/M_0)$ and $M_0 = D_1 + D_2(b_0 + |X_0|)$, Picards theorem ensures Y exists uniquely. The trajectories of X_t lie in $C^{1/2-\epsilon}[0, t_0]$.

To extend the solution to $[0, T]$, we iterate the process using X_{t_n} as the new initial value. If $L_2(T) \le 1$, the series $\sum a_n$ diverges, allowing finite extension to $[0, T]$. If $L_2(T) > 1$, the series converges to $S \le T$, and we use a constant step $S/2$ to cover $[0, T]$. Uniqueness follows from Theorem 3.1.9, and the $C^{1/2-\epsilon}$-regularity is preserved.

3.3 The Existence and Uniqueness of the Solution of the Mixed SDEs for fBm with $H \in (3/4, 1)$

Now we consider a mixed SDEs without any semilinear restrictions but only
for $H \in (3/4, 1)$.

3.3.1 Existence and Uniqueness of Solution of Mixed SDEs for fBm with $H \in (3/4, 1)$) and with Stabilizing Term

Consider the mixed stochastic differential equation (SDEs):

$$X_t = X_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s)\, dW_s + \int_0^t c(s, X_s)\, dB_s^H + \varepsilon \int_0^t c(s, X_s)\, dV_s, \quad (3.5)$$

where:

- $a, b, c : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable functions.
- $W = \{B_t\}_{t\ge 0}$ and $V = \{V_t\}_{t\ge 0}$ are independent Wiener processes.
- $\varepsilon > 0$ is a stabilization parameter.
- $B^H = \{B_t^H\}_{t\ge 0}$ is a fractional Brownian motion with Hurst index $H \in (3/4, 1)$, independent of W and V.
- X_0 is a random variable independent of W, B^H, and V.

The integral $\varepsilon \int_0^t c(s, X_s)\, dV_s$ acts as a stabilizing term, ensuring the existence and uniqueness of the solution to (3.5). The solution is adapted to the filtration:

$$\mathcal{F}_t' = \sigma\left\{X_0,\, W_s,\, \left(\varepsilon V_s + B_s^H\right) \Big|\, s \in [0, t]\right\}, \quad t \ge 0. \quad (3.6)$$

Proposition 3.3.1 *1. Let $W = \{B_t\}_{t\in[0,T]}$ be a standard Brownian motion and $B^H = \{B^H_t\}_{t\in[0,T]}$ an independent fractional Brownian motion (fBm) with $H \in (3/4, 1)$ and $\gamma \in \mathbb{R} \setminus \{0\}$. Define the process:*

$$M^{H,\gamma}_t = B_t + \gamma B^H_t, \quad t \in [0,T],$$

equipped with its natural filtration $\{\mathcal{F}^{M^{H,\gamma}}_t\}$. Then:

(a) $M^{H,\gamma}$ is equivalent in law to a standard Brownian motion.

(b) $M^{H,\gamma}$ is a semimartingale.

2. There exists a unique real-valued Volterra kernel $h = h_\gamma \in L_2[0,T]^2$ such that

$$B_t := M^{H,\gamma}_t - \int_0^t \int_0^s h(s,u)\, dM^{H,\gamma}_u\, ds, \quad t \in [0,T], \tag{3.7}$$

is a Brownian motion. Furthermore,

$$M^{H,\gamma}_t = B_t - \int_0^t \int_0^s r(s,u)\, dB_u\, ds, \quad t \in [0,T], \tag{3.8}$$

where $r = r_\gamma \in L_2[0,T]^2$.

Consequently, the process $N^{H,\varepsilon}_t := B^H_t + \varepsilon V_t = \varepsilon \left(V_t + \frac{1}{\varepsilon} B^H_t \right) = \varepsilon M^{H,\frac{1}{\varepsilon}}$ admits the representation:

$$N^{H,\varepsilon}_t = \varepsilon V'_t + \int_0^t \int_0^s \varepsilon r_\varepsilon(s,u)\, dV'_u\, ds,$$

where V' is a Wiener process adapted to the filtration $\mathcal{F}_t := \sigma\{\varepsilon V_s + B^H_s \mid s \in [0,t]\}$. and, from the independence of V,W,B^H and $X_0 t$ is a Wiener process w.r.t. $\{\mathcal{F}'_t, : t \in [0,T]\}$ Using 3.6, we can rewrite

$$X_t = X_0 + \int_0^t a(s,X_s)\, ds + \int_0^t b(s,X_s)\, dW_s \tag{3.9}$$

$$+\varepsilon \int_0^t c(s,X_s)\, dV'_s + \int_0^t c(s,X_s) \int_0^s \varepsilon r_\varepsilon(s,u)\, dV'_u\, ds.$$

The stochastic differential equation 3.9 describes a process X_t with both deterministic and stochastic components: Here, X_0 is the initial value, $\int_0^t a(s,X_s)\, ds$ represents deterministic drift, and $\int_0^t b(s,X_s)\, dW_s$ models Wiener process-driven diffusion. The additional terms involving ε and V'_s introduce perturbative noise: a direct *noise term* $\varepsilon \int_0^t c(s, X_s)\, dV'_s$*, and a time-correlated component through the nested integral.*

The total drift combines deterministic and random parts:

$$\textit{Drift} = a(s,x) + c(s,x,\omega), \quad \textit{where } c(s,x,\omega) = c(s,x) \int_0^s \varepsilon r_\varepsilon(s,u)\, dV'_u.$$

The integral $\int_0^s \varepsilon r_\varepsilon(s,u)\, dV'_u$ *is unbounded, violating standard SDEs existence/uniqueness assumptions. To resolve this, we introduce stopping times:*

$$\tau^M = \inf \left\{ t \in [0,T] : \int_0^t \left(\int_0^s \varepsilon r_\varepsilon(s,u)\, dV'_u \right)^2 ds > M \right\} \wedge T,$$

which localize the process when the problematic term grows too large. For each M*, the stopped process* $X_{t \wedge \tau^M}$ *has bounded coefficients, allowing classical existence/uniqueness proofs (e.g., via Picard-Lindelöf iterations). By taking* $M \longrightarrow \infty$ *and assuming* $\tau^M \longrightarrow T$ *almost surely (no explosion before* T*), we recover solutions to the original equation 3.9).*

This localization strategy—using stopping times to temporarily bound unstable terms, then removing constraints through limits—extends classical SDEs theory to handle equations with unbounded coefficients.

Proof.[[4]]

Equivalence in law to a Brownian motion.

For $H > 3/4$, we construct a probability measure $\mathbb{Q} \sim \mathbb{P}$ under which $M^{H,\gamma}$ becomes a Brownian motion.

a. Measure construction. Let $\theta_t = -\gamma \frac{d}{dt}\mathbb{E}[B_t^H | \mathcal{F}_t^{M^{H,\gamma}}]$. Define the Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^T \theta_s\, dW_s - \frac{1}{2} \int_0^T \theta_s^2\, ds \right).$$

By Girsanov's theorem, under $\mathbb{Q}$:

$$\widetilde{W}_t = B_t - \int_0^t \theta_s\, ds \quad \text{is a } \mathbb{Q}\text{-Brownian motion.}$$

Substituting $B_t = \widetilde{W}_t + \int_0^t \theta_s\, ds$ into $M^{H,\gamma}$, we get:

$$M_t^{H,\gamma} = \widetilde{W}_t + \int_0^t \theta_s\, ds + \gamma B_t^H.$$

The term $\int_0^t \theta_s \, ds$ cancels γB_t^H, leaving $M_t^{H,\gamma} = \widetilde{W}_t$, a $\mathbb{Q}$-Brownian motion.

b. Validity of $\mathbb{Q}$. The Novikov condition:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 \, ds\right)\right] < \infty,$$

holds because θ_s is adapted and B^H has Hölder-continuous paths for $H > 3/4$. Thus, $\mathbb{Q}$ is well-defined.

2. Semimartingale property.

Under $\mathbb{P}$, $M^{H,\gamma}$ decomposes as:

$$M_t^{H,\gamma} = \underbrace{B_t}_{\text{martingale}} + \underbrace{\gamma B_t^H}_{\text{semimartingale}} .$$

For $H > 3/4$, the fBm B_t^H admits a semimartingale decomposition:

$$B_t^H = \int_0^t K_H(t,s) \, dW_s + A_t,$$

where $K_H(t,s)$ is a Volterra kernel and A_t is a finite-variation process. Hence, γB_t^H is a semimartingale. The sum $B_t + \gamma B_t^H$ preserves the semimartingale property.

Theorem 3.3.1 *Assume the following conditions :*

(i) Linear growth:

$$|a(s,0)| + |b(s,0)| + |c(s,0)| \leq L, \quad |a(s,x)| + |b(s,x)| + |c(s,x)| \leq L(1+|x|).$$

(ii) Lipschitz continuity in x:

$$|a(s,x) - a(s,y)| + |b(s,x) - b(s,y)| + |c(s,x) - c(s,y)| \leq l(s)|x-y|,$$

where $l : [0,T] \longrightarrow \mathbb{R}$ is increasing.

(iii) The initial value X_0 is square-integrable.

Then, Equation (3.9) has a unique $\mathscr{F}_t'$-adapted solution on $[0,T]$.

Proof.
Step 1: Functional space. Let $\mathcal{S}^2([0,T])$ be the space of $\mathcal{F}_t'$-adapted processes X satisfying:

$$\|X\|_{\mathcal{S}^2} = \mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right]^{1/2} < \infty.$$

Step 2: Picard iteration operator.

Define the map $\mathcal{T}$ on $\mathcal{S}^2([0,T])$ by:

$$\mathcal{T}(X)_t = X_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s)\, dW_s$$

$$+\varepsilon \int_0^t c(s, X_s)\, dV'_s + \int_0^t c(s, X_s) \left(\int_0^s \varepsilon r_\varepsilon(s,u)\, dV'_u \right) ds.$$

Step 3: Stability. Using the linear growth condition (i), we show $\mathcal{T}(X) \in \mathcal{S}^2$ for $X \in \mathcal{S}^2$. For example:

$$\mathbb{E}\left[\sup_t \left| \int_0^t a(s, X_s)\, ds \right|^2 \right] \leq C\mathbb{E}\left[\int_0^T (1 + |X_s|)^2 ds \right] \leq C(1 + \|X\|_{\mathcal{S}^2}^2).$$

Similar bounds hold for other terms via Doobs maximal inequality.

Step 4: Contraction. By the Lipschitz condition (ii), for $X, Y \in \mathcal{S}^2$:

$$\|\mathcal{T}(X) - \mathcal{T}(Y)\|_{\mathcal{S}^2}^2 \leq C \int_0^T l(s)^2 \mathbb{E}[|X_s - Y_s|^2] ds \leq C\|X - Y\|_{\mathcal{S}^2}^2.$$

For small T, $C < 1$, making $\mathcal{T}$ a contraction. Extend to arbitrary T by iteration.

Step 5: Uniqueness. If X and $\tilde{X}$ are solutions, Gronwalls lemma gives:

$$\mathbb{E}\left[\sup_t |X_t - \tilde{X}_t|^2 \right] \leq C \int_0^T \mathbb{E}[|X_s - \tilde{X}_s|^2] ds \implies X_t = \tilde{X}_t \quad \text{a.s.}$$

3.3.2 The Existence and Uniqueness of the Solution of the Mixed SDEs Involving fBm with $H \in (3/4, 1)$ as the Limit Result for the Equations with the Stabilizing Term

Now we want to pass to the limit as $\varepsilon \to 0$ in equation ((3.5)). Let $\varepsilon = 1/N, N \geq 1$, and consider the sequence of the equations with the stabilizing term

$$X_t^N = X_0 + \int_0^t a\left(s, X_s^N\right) dt + \int_0^t b\left(s, X_s^N\right) dW_t \tag{3.10}$$

$$+ \int_0^t c\left(s, X_s^N\right) dB_s^H + \frac{1}{N} \int_0^t c\left(s, X_s^N\right) dV_s, \quad t \in [0,T].$$

Let the coefficients a, b, c and X_0 satisfy conditions (i), (ii) and (iii). Then, according to Theorem 3.3.1 , equation (3.10) has a unique strong solution, say $\left\{X_t^N, t \in [0,T]\right\}$. Evidently, the solutions are adapted to different filtrations

$\mathcal{F}_t^N = \sigma\left\{X_0, W_s, \left(N^{-1}V_s + B_s^H\right), s \in [0,t]\right\}$. The aim of this section is to establish the conditions of existence and uniqueness of the solution of the limit mixed equation

$$X_t = X_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s)\, dW_s + \int_0^t c(s, X_s)\, dB_s^H, \quad t \in [0,T] \qquad (3.11)$$

Let the coefficients of equation (3.11) satisfy assumption (iii) and the following ones: there exist such constants $B, L, M > 0, \gamma \in (1-H, 1)$ and $\kappa \in (3/2 - H, 1)$ that
(iv) all the coefficients are bounded:

$$|a(s,x)| + |b(s,x)| + |c(s,x)| \le L, \forall s \in [0,T], \forall x \in \mathbb{R};$$

(v) all the coefficients are Lipschitz in x :

$$|a(t,x) - a(t,y)| + |\, (b(t,x) - b(t,y)| + |c(t,x) - c(t,y)| \le L|x - y\,|$$

$\forall t \in [0,T], \forall x, y \in \mathbb{R}$,
(vi) the x-derivative of the function c exists and is Hölder continuous in t :
$\forall s, t \in [0,T], \forall x \in \mathbb{R}$

$$|c(s,x) - c(t,x)| + |\partial_x c(s,x) - \partial_x c(t,x)| \le L|s-t|^\gamma.$$

(vii) the x-derivative of the function c is Hölder continuous in x :

$$|\partial_x c(t,x) - \partial_x c(t,y)| \le L|x-y|^\kappa$$

for $\forall t \in [0,T], \forall x, y \in \mathbb{R}$.

Consider W^β, the Besov[1]-type functional space:

$$W_{[0,T]} = \left\{Y = Y_t(\omega) \,\middle|\, (t,\omega) \in [0,T] \times \Omega,\ \|Y\|_\beta < \infty\right\}$$

where the norm is defined as:

$$\|Y\|_\beta = \sup_{t \in [0,T]} \left(\mathbb{E}(Y_t^2) + \mathbb{E}\left(\int_0^t \frac{|Y_t - Y_s|}{(t-s)^{1+\beta}}\, ds\right)^2\right)$$

with the parameter condition:

$$\beta < \left(\frac{1}{2} \wedge \gamma \wedge \frac{k}{2}\right) \wedge \left(k - \frac{1}{2}\right).$$

[1]The Besov space $\mathbb{B}_{p,q}^s(\mathbb{R}^n)$ consists of tempered distributions f on $\mathbb{R}^n$ for which:

$$\|f\|_{\mathbb{B}_{p,q}^s} = \left(\sum_{j=0}^{+\infty} \|2^{sj}\varphi_j * f\|_{L^p}^q\right)^{\frac{1}{q}} < \infty,$$

where $1 \le p, q \le \infty$.

Theorem 3.3.2 *The mixed stochastic differential equation* (3.11) *has a unique solution on* $[0, T]$.

Proof.
To prove this theorem, we require the following result:

Theorem 3.3.3 ([26]) *The solution to equation* (3.10) *belongs to the Besov-type functional space* $W_{[0,T]}$ *for every* $N > 1$.

Since $W_{[0,T]}$ is a complete space, Theorem 3.3.3 allows us to define:

$$X_{\tau_R \wedge T} = \underset{N \longrightarrow \infty}{lim} X^N_{\tau_R \wedge T}.$$

Since the limit belongs to the space $W_{[0,T]}$ (in particular, the limit exists in $L^2([0, T] \times \Omega)$), and using the following similar estimate for $t_1 \leq t \leq t_2 \leq T$:

$$\left\| \sup_{t_1 \leq t \leq t_2} |X_t| \right\|_2 \leq (h + 1)c_1 + \exp(2)C_{H,\gamma}\theta^{-\frac{\gamma}{2H}} \frac{(t_2 - t_1)^H}{1 - \theta} = L,$$

where c_1 is a constant, $0 < \theta < \left(\dfrac{3}{2(\exp(2) - 1} \right)^H$, $0 < \gamma < 2H$, and

$$C_{H,\gamma} = \frac{\left(\frac{3}{2}\right)^{\frac{\gamma}{2}} HB_\gamma}{\gamma \left(H - \frac{\gamma}{2}\right)},$$

by Theorem 3.3.3, we prove that $X_{\tau_R \wedge T}$ is the unique solution of equation (3.11) on the interval $[0, \tau_R]$. Recall that τ_R, for any $R > 1$, is a stopping time defined as:

$$\tau_R = \inf \{t : C'_t(\omega) \geq R\} \wedge T, \tag{3.12}$$

where

$$C'_t(\omega) = c \left(\Lambda_{1-\beta}(B^H) \right) \vee \xi^b_{t,\delta} \vee \xi^c_{t,\delta}, \tag{3.13}$$

and

$$\delta^b = \left(\int_0^t \int_0^t \frac{|\int_x^y b_u \, dB_u|^{\frac{2}{\delta}}}{|x - y|^{\frac{1}{\delta}}} \, dx \, dy \right)^{\frac{\delta}{2}}. \tag{3.14}$$

From (3.12), we have $\tau_{R_1} \leq \tau_{R_2}$ for all $R_1 \leq R_2$. Thus, $X_{\tau_{R_1}}$ and $X_{\tau_{R_2}}$ coincide on the interval $[0, \tau_R]$. By taking the limit as $R \longrightarrow \infty$, we conclude the existence and uniqueness of the solution to equation (3.11) on $[0, T]$.

Theorem 3.3.4 *For any* $\delta \in (0, 1/2)$ *the solution of equation* (3.10) *is Hölder continuous with parameter* $1/2 - \delta$.

Proof.
Consider $\left|X_r^N - X_z^N\right|$ for $0 < z < r < T$:

$$\left|X_r^N - X_z^N\right| \leq \left|\int_z^r a\left(u, X_u\right) du\right| + \left|\int_z^r b\left(u, X_u\right) dW_u\right| + \frac{1}{N}\left|\int_z^r c\left(u, X_u\right) dV_u\right|$$

$$+ \left|\int_z^r c\left(u, X_u\right) dB_u^H\right| \leq L(r-z) + C\xi^b_{r,\delta}|r-z|^{1/2-\delta} + \frac{C}{N}\xi^c_{r,\delta}|r-z|^{1/2-\delta}$$

$$+ \Lambda_{1-\beta}\left(B^H\right)\int_z^r \frac{\left|c\left(u, X_u^N\right)\right| du}{u^\beta}$$

$$+ \Lambda_{1-\beta}\left(B^H\right)\int_z^r\int_z^u \frac{\left|c\left(u, X_u^N\right) - c\left(v, X_v^N\right)\right|}{(u-v)^{1+\beta}} dvdu$$

$$\leq C_r'(\omega)(r-z)^{1/2-\delta} + C_r'(\omega)\int_z^r\int_z^u \frac{\left|X_u^N - X_v^N\right|}{(u-v)^{1+\beta}} dvdu$$

where

$$C_t'(\omega) := C\left(\Lambda_{1-\beta}\left(B^H\right) \vee \xi^b_{t,\delta} \vee \xi^c_{t,\delta} \vee 1\right) \tag{3.15}$$

$\xi^b_{t,\delta}$ and $\xi^c_{t,\delta}$ are defined by

$$\xi^b_{t,\delta} := \left(\int_0^t\int_0^t \frac{\left|\int_x^y b_u\, dW_u\right|^{2/\delta}}{|x-y|^{1/\delta}}\, dx\, dy\right)^{\delta/2}$$

and

$$\xi^c_{t,\delta} := \left(\int_0^t\int_0^t \frac{\left|\int_x^y c_u\, dW_u\right|^{2/\delta}}{|x-y|^{1/\delta}}\, dx\, dy\right)^{\delta/2}$$

, $C_t'(\omega) \leq C_T'(\omega)$ and $C_T'(\omega)$ has the moments of any order.
Therefore, for $\delta < 1/2 - \beta$ we have that

$$\phi_{r,s} := \int_s^r \frac{\left|X_r^N - X_z^N\right|}{(r-z)^{1+\beta}} dz \leq C_r'(\omega)\left(\int_s^r (r-z)^{-1/2-\delta-\beta} dz\right.$$

$$\left. + \int_s^r \frac{1}{(r-z)^{1+\beta}}\int_z^r\int_z^u \frac{\left|X_u^N - X_v^N\right|}{(u-v)^{1+\beta}} dvdudz\right)$$

$$\leq C_r'(\omega)\left((r-u)^{1/2-\beta-\delta} + \int_s^r (r-u)^{-\beta}\phi_{u,s} du\right).$$

From the modified Gronwall inequality it follows that

$$\phi_{r,s} \leq C_r'(\omega)(r-s)^{1/2-\beta-\delta} \exp\left\{C_r'(\omega)^{\frac{1}{1-\beta}}\right\}$$

Return to $\left|X_r^N - X_z^N\right|$:

$$\left|X_r^N - X_z^N\right| \le C_r'(\omega)(r-z)^{1/2-\delta}$$
$$+ C_r'(\omega)\exp\left\{C_r'(\omega)^{\frac{1}{1-\beta}}\right\}\int_z^r (v-z)^{1/2-\beta-\delta}dv \le \widetilde{C}_r(\omega)(r-z)^{1/2-\delta}$$

where $\widetilde{C}_r(\omega) = C_r'(\omega)\exp\left\{C_r'(\omega)^{\frac{1}{1-\beta}}\right\}$, and the theorem is proved for $0 < \delta < 1/2 - \beta$, and consequently for $0 < \delta < 1/2$.

Introduce the random variable $\widetilde{C}(\omega) := \sup\limits_{0\le t\le T} \widetilde{C}_t(\omega)$. It also has moments of any order. Now we want to prove that the solution of (3.10) belongs to the space $\left\{W^\beta[0,T], \|\cdot\|_\beta\right\}$ for all $N > 1$.

Theorem 3.3.5 *Under assumptions (iii)-(vi) the solution of equation* (3.10) *belongs to the space* $W^\beta[0,T]$ *of Besov type with norm* $\|\cdot\|_\beta$ *for all* $N > 1$ *and any* $\beta < \left(1/2 \wedge \gamma \wedge \kappa/2 \wedge \kappa - \frac{1}{2}\right)$.

Proof.
In order to prove the statement of this theorem, we want to estimate

$$A_1^N(t) + A_2^N(t) := E\left(X_t^N\right)^2 + E\left(\int_0^t \frac{\left|X_t^N - X_s^N\right|}{(t-s)^{1+\beta}}ds\right)^2$$

First, for $A_1^N(t)$ we have that

$$E\left(X_t^N\right)^2 \le 5E\left(X_0\right)^2 + 5E\left(\int_0^t a\left(s, X_s^N\right)ds\right)^2 + 5E\left(\int_0^t b\left(s, X_s^N\right)dW_s\right)^2$$
$$+ 5E\left(\int_0^t c\left(s, X_s^N\right)dB_s^H\right)^2 + 5E\left(\frac{1}{N}\int_0^t c\left(s, X_s^N\right)dV_s\right)^2$$

Evidently, $E\left(\int_0^t a\left(s, X_s^N\right)ds\right)^2 \le L^2T^2$,
$E\left(\int_0^t b\left(s, X_s^N\right)dW_s\right)^2 \le L^2T, E\left(\frac{1}{N}\int_0^t c\left(s, X_s^N\right)dV_s\right)^2 \le \frac{L^2T}{N^2} \le L^2T.$
Further, for $\delta < 1/2 - \beta$ we have that

$$E\left(\int_0^t c\left(s,X_s^N\right)dB_s^H\right)^2 \le E\left(\overline{C}^2(\omega)\left(\int_0^t \frac{c\left(s,X_s^N\right)}{s^\beta}ds\right.\right.$$

$$\left.\left.+\int_0^t\int_0^s \frac{\left|c\left(s,X_s^N\right)-c\left(u,X_u^N\right)\right|}{(s-u)^{1+\beta}}duds\right)^2\right) \le CE\left(\overline{C}^2(\omega)\left(t\int_0^t \frac{L^2}{s^{2\beta}}ds\right.\right.$$

$$\left.\left.+\left(\int_0^t\int_0^s \frac{L(s-u)^\gamma + L\widetilde{C}(\omega)(s-u)^{1/2-\delta}}{(s-u)^{1+\beta}}duds\right)^2\right)\right)$$

$$\le C\left(E\bar{C}^2(\omega)\left(L^2T^{2-2\beta}+L^2T^{2(1-\beta+\gamma))}+L^2E\left(\widetilde{C}^2(\omega)\bar{C}^2(\omega)\right)T^{3-2\beta-2\delta}\right)$$

with $\bar{C}(\omega)=\Lambda_{1-\beta}\left(B^H\right)$. From all these estimates it follows that $A_1^N(t)<\infty$. Consider now $A_2^N(t)$. We have that

$$A_2^N(t) \le 4E\left(\int_0^t \frac{\left|\int_s^t a\left(u,X_u^N\right)du\right|}{(t-s)^{1+\beta}}ds\right)^2$$

$$+4E\left(\int_0^t \frac{\left|\int_s^t b\left(u,X_u^N\right)dW_u\right|}{(t-s)^{1+\beta}}ds\right)^2 + 4N^{-2}E\left(\int_0^t \frac{\left|\int_s^t c\left(u,X_u^N\right)dV_u\right|}{(t-s)^{1+\beta}}ds\right)^2 \tag{3.16}$$

$$+4E\left(\int_0^t \frac{\left|\int_s^t c\left(u,X_u^N\right)dB_u^H\right|}{(t-s)^{1+\beta}}ds\right)^2$$

Evidently,

$$E\left(\int_0^t \frac{\left|\int_s^t a\left(u,X_u\right)du\right|}{(t-s)^{1+\beta}}ds\right)^2 \le CL^2t^{2-2\beta}$$

Now, let $\rho\in(\beta,1/2)$, then we have the estimate

$$E\left(\int_0^t \frac{\left|\int_s^t b\left(u,X_u\right)dW_u\right|}{(t-s)^{1+\beta}}ds\right)^2 \le Ct^{1-2\rho}\int_0^t \frac{E\left|\int_s^t b\left(u,X_u\right)dW_u\right|^2}{(t-s)^{2+2\beta-2\rho}}ds$$

$$\le Ct^{1-2\rho}\int_0^t \frac{\int_s^t b^2\left(u,X_u\right)du}{(t-s)^{2+2\beta-2\rho}}ds \le CL^2t^{1-2\beta}$$

and similarly,

$$E\left(\int_0^t \frac{\left|\int_s^t c\left(u,X_u\right)dV_u\right|}{(t-s)^{1+\beta}}ds\right)^2 \le CL^2t^{1-2\beta}$$

Now we estimate $E^N := E\left(\int_0^t \left|\int_s^t c\left(u, X_u\right) dB_u^H\right| (t-s)^{-1-\beta} ds\right)^2$. Since

$$\left|\int_s^t c\left(u, X_u\right) dB_u^H\right| \leq \bar{C}(\omega) \left(\int_s^t \left|c\left(u, X_u\right)\right| (u-s)^{-\beta} du\right.$$
$$\left.+ \int_s^t \int_s^u \left|c\left(u, x_u^N\right) - c\left(r, X_r^N\right)\right| (u-r)^{-1-\beta} dr du\right) \leq \bar{C}(\omega)$$
$$\times \left(\int_s^t \left|c\left(u, X_u\right)\right| (u-s)^{-\beta} du + \int_s^t \int_s^u \frac{L(u-r)^{\gamma} + L\widetilde{C}(\omega)(u-r)^{1/2-\delta}}{(u-r)^{1+\beta}} dr du\right),$$

we have that for $\delta < 1/2 - \beta E^N$ can be bounded by

$$E\left(\bar{C}(\omega) \int_0^t \frac{L(t-s)^{1-\beta} + L(t-s)^{1+\gamma-\beta} + L\widetilde{C}(\omega)(t-s)^{3/2-\delta-\beta}}{(t-s)^{1+\beta}} ds\right)^2$$

$$\leq C\left(L^2 t^{2-4\beta} E\bar{C}^2(\omega) + L^2 t^{2+2\gamma-4\beta} E\bar{C}^2(\omega) + L^2 t^{3-2\delta-4\beta} E\bar{C}^2(\omega)\widetilde{C}^2(\omega)\right).$$

Therefore, $A_2^N(t)$ satisfies the inequality

$$\begin{aligned} A_2^N(t) \leq C\Big(&L^2 T^{2-2\beta} + L^2 T^{1-2\beta} + L^2 T^{2-4\beta} E\bar{C}^2(\omega) \\ &+ L^2 T^{2+2\gamma-4\beta} E\bar{C}^2(\omega) + L^2 T^{3-2\delta-4\beta} E\bar{C}^2(\omega)\widetilde{C}^2(\omega)\Big) < \infty. \end{aligned} \tag{3.17}$$

Finally, the statement of our theorem follows from inequalities (3.17) (3.16) with sufficiently small $\delta > 0$.

Introduce for any $R > 1$ the stopping time τ_R by

$$\tau_R := \inf\left\{t : C_t'(\omega) \geq R\right\} \wedge T \tag{3.18}$$

where $C_t'(\omega)$ is defined by (3.15). Evidently, for any $\omega \in \Omega$ there exists $R(\omega)$ such that $\tau_R = T$ for all $R > R(\omega)$.

Define the processes $\left\{X_{\tau_R \wedge t}^N, N \geq 1, t \in [0, T]\right\}$ as the solutions of equation (3.10) stopped at the moment τ_R, and prove that they are fundamental in the norm $\|\cdot\|_\beta$ of the space $W^\beta[0, T]$.

Theorem 3.3.6 *Under assumptions (iii)-(vi) the sequence* $\left\{X_{t\wedge\tau_R}^N, N \geq 1, t \in [0, T]\right\}$ *of solutions of equations* (3.10) *is fundamental in the norm* $\|\cdot\|_\beta$

for any $\beta < \left(1/2 \wedge \gamma \wedge \kappa/2 \wedge \kappa - \frac{1}{2}\right)$.

Proof.
Consider

$$A_1^{N,M}(t)+A_2^{N,M}(t):=E\left(X_{t\wedge\tau_R}^N-X_{t\wedge\tau_R}^M\right)^2$$
$$+E\left(\int_0^t\frac{\left|X_{t\wedge\tau_R}^N-X_{t\wedge\tau_R}^M-X_{s\wedge\tau_R}^N+X_{s\wedge\tau_R}^M\right|}{(t-s)^{1+\beta}}ds\right)^2$$
$$=E\left(X_{t\wedge\tau_R}^N-X_{t\wedge\tau_R}^M\right)^2+E\left(\int_0^{\tau_R\wedge t}\frac{\left|X_{t\wedge\tau_R}^N-X_{t\wedge\tau_R}^M-X_s^N+X_s^M\right|}{(t-s)^{1+\beta}}ds\right)^2$$

First, for $A_1^{N,M}(t)$ we have the estimate

$$A_1^{N,M}(t)\le 4E\left(\int_0^{\tau_R\wedge t}\left(a\left(s,X_s^N\right)-a\left(s,X_s^M\right)\right)ds\right)^2$$
$$+4E\left(\int_0^{\tau_R\wedge t}\left(b\left(s,X_s^N\right)-b\left(s,X_s^M\right)\right)dW_s\right)^2$$
$$+4E\left(\int_0^{\tau_R\wedge t}\left(c\left(s,X_s^N\right)-c\left(s,X_s^M\right)\right)dB_s^H\right)^2$$
$$+4E\left(\int_0^{\tau_R\wedge t}\left(\frac{c\left(s,X_s^N\right)}{N}-\frac{c\left(s,X_s^M\right)}{M}\right)dV_s\right)^2=:4\left(I_1+I_2+I_3+I_4\right)$$

Then $I_1\le CTL^2\int_0^t E\left(X_{s\wedge\tau_R}^N-X_{s\wedge\tau_R}^M\right)^2ds, I_2\le CL^2\int_0^t E\left(X_{s\wedge\tau_R}^N-X_{s\wedge\tau_R}^M\right)^2ds,$
$I_4\le CL^2T\left(N^{-2}+M^{-2}\right)$.

Now we are in a position to estimate I_3 :

$$I_3\le 2R^2\Bigg(E\left(\int_0^{\tau_R\wedge t}\left|c\left(s,X_s^N\right)-c\left(s,X_s^M\right)\right|s^{-\beta}ds\right)^2$$
$$+E\bigg(\int_0^{\tau_R\wedge t}\int_0^s\mid c\left(s,X_s^N\right)-c\left(s,X_s^M\right)-c\left(u,X_u^N\right)+c\left(u,X_u^M\right)\mid$$
$$\times(s-u)^{-1-\beta}duds\bigg)^2\Bigg)=2R^2\left(I_4+I_5\right)$$

Furthermore,

$$I_4\le CL^2T^{1-2\beta}E\int_0^{T_R/k}(X_s^N-X_s^M)^2\,ds=CL^2T^{1-2\beta}\int_0^t A_1^{N,M}(s)\,ds.$$

we estimate I_5 as:

$$I_5\le 3E\left(\int_0^{T_R/k}\int_0^s\frac{L|X_s^N-X_s^M-X_u^N+X_u^M|}{(s-u)^{1+\beta}}\,du\,ds\right)^2$$

$$+3E\left(\int_0^{T_R/k}\int_0^s \frac{L^2|X_s^N - X_s^M|(s-u)^\gamma}{(s-u)^{1+\beta}}\,du\,ds\right)^2$$

$$+3E\left(\int_0^{T_R/k}\int_0^s \frac{L|X_s^N - X_s^M|(|X_s^N - X_u^N|^k + |X_s^M - X_u^M|^k)}{(s-u)^{1+\beta}}\,du\,ds\right)^2$$

$$= 3(I_6 + I_7 + I_8).$$

Here,

$$I_6 \le CTL^2 \int_0^t E\left(\int_0^{S/N_T R} \frac{|X_s^N - X_s^M - X_u^N + X_u^M|}{(s-u)^{1+\beta}}\,du\right)^2 ds,$$

$$I_7 \le CTL^2 \int_0^t s^{2(\gamma-\beta)} E|X_s^N - X_s^M|^2\,ds,$$

$$I_8 \le E\left(\int_0^{T_R/k}\int_0^s \frac{L|X_s^N - X_s^M| 2R(s-u)^{k(1/2-\delta)}}{(s-u)^{1+\beta}}\,du\,ds\right)^2$$

$$\le CTL^2R^2 \int_0^t s^{k-2k\delta-2\beta} E|X_s^N - X_s^M|^2\,ds,$$

where δ is chosen such that $k - 2k\delta - 2\beta > 0$. This is possible since $\beta < k - 1/2$, hence $k - 2\beta > 1/2 - \beta > 0$. Finally,

$$I_5 \le C\int_0^t \left(A_2^{N,M}(s) + \left(s^{2(\gamma-\beta)} + CR^2 s^{k-2k\delta-2\beta}\right) A_1^{N,M}(s)\right) ds,$$

and

$$A_1^{N,M}(t) \le CR^2 \int_0^t A_1^{N,M}(s)\,ds + CR^2 \int_0^t A_2^{N,M}(s)\,ds \tag{3.19}$$

$$+C(N^{-2} + M^{-2}). \quad (3.2.19)$$

Returning to $A_2^{N,M}(t)$, it admits the following estimate:

$$A_2^{N,M}(t) \le C\left(\mathbb{E}\left(\int_0^{T_R/(\lambda t)} \frac{\int_s^{T_R/(\lambda t)}(a(u,X_u^N) - a(u,X_u^M))\,du}{(t-s)^{1+\beta}}\,ds\right)^2\right.$$

$$+\,\mathbb{E}\left(\int_0^{T_R/(\lambda t)} \frac{\int_s^{T_R/(\lambda t)}(b(u,X_u^N) - b(u,X_u^M))\,dW_u}{(t-s)^{1+\beta}}\,ds\right)^2$$

$$+\,\mathbb{E}\left(\int_0^{T_R/(\lambda t)} \frac{\int_s^{T_R/(\lambda t)}(c(u,X_u^N) - c(u,X_u^M))\,dB_u^H}{(t-s)^{1+\beta}}\,ds\right)^2$$

$$\left.+\,\mathbb{E}\left(\int_0^{T_R/(\lambda t)} \frac{\int_s^{T_R/(\lambda t)}(c(u,X_u^N) - c(u,X_u^M))\,dW_u}{(t-s)^{1+\beta}}\,ds\right)^2\right)$$

$$= C(I_9 + I_{10} + I_{11} + I_{12}).$$

Further, for $\beta < \rho < 1/2$:

$$I_9 \le CT^{1-2\rho}\,\mathbb{E}\int_0^{T_R/(\lambda t)} \frac{(t-s)\int_0^{T_R/(\lambda t)} L^2|X_u^N - X_u^M|^2\,du}{(t-s)^{2+2\beta-2\rho}}\,ds$$

$$\le CT^{1-2\beta}\int_0^t \mathbb{E}\left(X_{s\wedge T_R}^N - X_{s\wedge T_R}^M\right)^2 ds \le CT^{1-2\beta}\int_0^t A_1^{N,M}(s)\,ds,$$

$$I_{10} \le CT^{1-2\rho}\int_0^t \frac{\mathbb{E}|X_{u\wedge T_R}^N - X_{u\wedge T_R}^M|^2\,du}{(t-s)^{2+2\beta-2\rho}}\,ds$$

$$\le CT^{1-2\rho}\int_0^t \frac{A_1^{N,M}(s)}{(t-s)^{1+2\beta-2\rho}}\,ds.$$

For I_{12} we have $I_{12} \le CT^{1-2\beta}(N^{-2}+M^{-2})$. Now consider I_{11}:

$$I_{11} \le CR^2T^{1-2\rho}(I_{13}+I_{14}),$$

$$\text{where } I_{13} \le C\,\mathbb{E}\int_0^{T_R/(\lambda t)} \frac{\int_s^{T_R/(\lambda t)}(X_{u\wedge T_R}^N - X_{u\wedge T_R}^M)^2\,du\int_s^t(u-s)^{-2\beta}\,du}{(t-s)^\nu}\,ds$$

$$\le C\int_0^t A_1^{N,M}(s)(t-s)^{-1+2\rho-4\beta}\,ds,$$

$$I_{14} \le CE\int_0^{\tau_R\wedge t}\Bigg(\Bigg(\int_s^{\tau_R\wedge t}\int_s^u \frac{L|X_u^N - X_u^M - X_v^N + X_v^M|}{(u-v)^{1+\beta}}\,dv\,du\Bigg)^2$$

$$+\Bigg(\int_s^{\tau_R\wedge t}\int_s^u L|X_u^N - X_u^M|(u-v)^{\rho-1-\beta}\,dv\,du\Bigg)^2$$

$$+\Bigg(\int_s^{\tau_R\wedge t}\int_s^u L|X_u^N - X_u^M|\left(|X_u^N - X_v^N|^\kappa + |X_u^M - X_v^M|^\kappa\right)\times$$

$$(u-v)^{-1-\beta}\,dv\,du^2(t-s)^{-\nu}\,ds =: C(I_{15}+I_{16}+I_{17}),$$

where $\upsilon = 2+2\beta-2\rho, \rho > \beta$. In turn,

$$I_{15} \le CT^{2\rho-2\beta}\int_0^t E\left(\int_0^{s/N_{TR}} \frac{|X_{s/N_{TR}}^N - X_{s/N_{TR}}^M - X_u^N + X_u^M|}{(s-u)^{1+\beta}}du\right)^2 ds$$

$$= \mathrm{CT}^{2\rho-2\beta}\int_0^t A_2^{N,M}(s)ds,$$

$$I_{16} \le C\int_0^t \frac{E\left(\int_s^{TR/\kappa}|X_u^N - X_u^M|(u-s)^{\gamma-\beta}du\right)^2}{(t-s)^\nu}ds \le CT^{2\rho+2\gamma-4\beta}\int_0^t A_1^{N,M}(s)ds,$$

where $\beta < \gamma, \beta < \rho$. Furthermore,

$$I_{17} \le CR^2E\int_0^{TR/\kappa} \frac{\left(\int_s^{TR/\kappa}\int_s^u |X_u^N - X_u^M|(u-v)^{\kappa(1/2-\delta)-1-\beta}dv\,du\right)^2}{(t-s)^\nu}ds,$$

where we chose $0 < \delta < 1/2 - \beta/\kappa$; note that $\beta < \kappa - 1/2$. Similarly to I_{16},

$$I_{17} \leq CR^2 T^{\kappa - 2\kappa\delta + 2\rho - 4\beta} \int_0^t A_1^{N,M}(s) ds,$$

where $\kappa - 2\kappa\delta + 2\rho - 4\beta > 0$ for sufficiently small δ since $\rho > \beta$ and $\kappa > 2\alpha$. Therefore we have

$$I_{14} \leq CR^2 \int_0^t \left(A_1^{N,M}(s) + A_2^{N,M}(s) \right) ds.$$

Hence

$$I_{11} \leq CR^4 \int_0^t \left(\frac{A_1^{N,M}(s)}{(t-s)^{1+2\beta-2\rho}} + A_2^{N,M}(s) \right) ds.$$

Finally,

$$A_2^{N,M}(t) \leq CR^4 \left(\int_0^t A_1^{N,M}(s)(t-s)^{-1-2\beta+2\rho} ds + \int_0^t A_2^{N,M}(s) ds \right) + C(N^{-2} + M^{-2}) \quad (3.20)$$

From (3.20)and (3.19) we obtain that the sum $A_1^{N,M}(t) + A_2^{N,M}(t)$ admits the same estimate as $A_2^{N,M}(t)$, i.e.

$$A_1^{N,M}(t) + A_2^{N,M}(t) \leq CR^4 \int_0^t \left(A_1^{N,M}(s)(t-s)^{-1-2\beta+2\rho} + A_2^{N,M}(s) \right) ds + C(N^{-2} + M^{-2});$$

taking into account that $\rho > \beta$ and using the modified Gronwall lemma , we obtain that

$$A_1^{N,M}(t) + A_2^{N,M}(t) \leq CR^4(N^{-2} + M^{-2}) \exp\{t(CR^4)^{1/(2\rho-2\beta)}\} \quad (3.21)$$

and we can put, for example, $\rho := 1/4 + \beta/2$. When $N, M \longrightarrow 0$, we obtain that the right-hand side of (3.21) tends to zero, whence the proof follows.

Theorem 3.3.7 *The SDEs 3.11 has a solution on the interval* $[0,T]$*, and this solution is unique.*

Proof.
Since the space $\{W^\beta[0,T], \|\cdot\|_\beta\}$ is complete, Theorem 3.3.6 allows us to define:

$$X_{\tau_R \wedge t} \underset{N \longrightarrow \infty}{lim} X^N_{\tau_R \wedge t},$$

where the limit is taken in the space $W_\beta[0,T]$

(in particular, the limit exists in $L_2(\Omega \times [0,T])$).

Using similar estimates and Theorem 3.3.6, we prove that $X_{\tau_R \wedge t}$ is the unique solution of the original equation 3.11 on the interval $[0, \tau_R]$.

From the definition of τ_R, we have $\tau_{R_1} \leq \tau_{R_2}$ for $R_1 \leq R_2$. Thus, $X_{\tau_{R_1}}$ and $X_{\tau_{R_2}}$ coincide almost surely (a.s.) on $[0, \tau_{R_1}]$. Taking $R \longrightarrow \infty$, we obtain the existence and uniqueness of the solution to the SDEs 3.11 on the entire interval $[0, T]$.

4

Examples of Applications and Simulation of Stochastic Differential Equations Driven by Mixed Fractional Brownian Motion (mfBm)

4.1 General Simulation of the Mixed SDEs Involving Both the Wiener Process and fBm

In this section, we present a general simulation approach for a mixed stochastic differential equation (SDEs) that involves both the Wiener process and fractional Brownian motion (fBm). The mixed SDEs can be written as

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dB_t + c(X_t, t)\, dB_t^H,$$

where B_t is the standard Wiener process, B_t^H is the fBm with Hurst parameter H, and a, b, and c are functions that govern the dynamics of the system. This formulation is especially useful for modeling phenomena that display both short-term randomness and long-term dependence.

Simulation Methodology

A common approach for simulating such mixed SDEss is to adapt the Euler-Maruyama scheme to accommodate the additional fBm component. The simulation involves generating independent increments for both processes:

- For the Wiener process, the increments ΔB_i are drawn from a normal distribution $\mathcal{N}(0, \Delta t)$.
- For the fBm, the increments ΔB_i^H must be generated using specialized algorithms (e.g., the Cholesky method or the Davies-Harte algorithm) to capture its long-range dependence.

The simulation algorithm is outlined below.

Algorithm 1 Simulation of the Mixed SDEs

Input: Initial condition X_0, time interval $[0, T]$, time step Δt, functions a, b, c
Compute $N = T/\Delta t$
Generate $\Delta B_i \sim \mathcal{N}(0, \Delta t)$ for $i = 1, \dots, N$
Generate fBm increments ΔB_i^H using a suitable algorithm (e.g., Davies-Harte)
Set X_0 as the initial condition
for $i = 0$ to $N - 1$ **do**
    $X_{t_{i+1}} = X_{t_i} + a(X_{t_i}, t_i)\Delta t + b(X_{t_i}, t_i)\Delta B_i + c(X_{t_i}, t_i)\Delta B_i^H$
end for
Output: The sequence $\{X_{t_i}\}_{i=0}^{N}$

Remarks

- **Increment Generation:** The accurate generation of fBm increments is critical due to the memory property inherent in fractional Brownian motion.
- **Time Step Selection:** The choice of Δt must balance computational efficiency and simulation accuracy.
- **Extension:** Higher-order schemes or variance reduction techniques may be employed to enhance simulation performance.

R Code for Simulation

4.2 Mixed Fractional-Brownian Model

1.Absence of Arbitrage

The Black-Scholes model is a fundamental concept in quantitative finance, primarily used for pricing European options. Introduced by Fischer Black and Myron Scholes in 1973, it relies on several key assumptions:

- Asset prices follow a **standard Brownian motion** (B_t).
- There are no **arbitrage opportunities**.
- The market is **complete**: every position can be perfectly hedged.
- Interest rates and volatility are **constant**.

However, empirical studies reveal that these assumptions do not fully capture the reality of financial markets:

1. **Long-term dependence:** Returns exhibit correlations over long time horizons.
2. **Non-constant volatility:** Financial markets experience volatility clustering.
3. **Long-memory effect:** Market fluctuations can be influenced by distant past events.

The **mixed fractional Black-Scholes model** combines two sources of randomness:

$$dS_t = \mu S_t\, dt + \sigma_1 S_t\, dB_t + \sigma_2 S_t\, dB_t^H,$$

where:

- S_t represents the asset price.
- μ is the mean rate of return.
- σ_1 is the volatility associated with the classical Brownian motion.
- σ_2 is the volatility linked to the fractional Brownian motion.

Although this model better captures the complex dynamics of financial markets, it raises a crucial question: **Does arbitrage exist under this framework?**

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

The market consists of two assets:

- **Risk-free bond**:
$$B_t = e^{rt}, \quad r > 0.$$
- **Risky asset**: Modeled by a mixed exponential process:
$$S_t = e^{aB_t + bB_t^H + ct}, \quad a, b, c \in \mathbb{R}.$$
This integrates both standard Brownian fluctuations (aB_t) and long-memory correlations (bB_t^H).

2. Self-Financing Strategies

Definition 4.2.1 (Self-Financing Portfolio) *A strategy $\pi = (\beta_t, \gamma_t)$, where β_t denotes units of the bond and γ_t units of the risky asset, is **self-financing** if the capital*

$X_t = \beta_t B_t + \gamma_t S_t$ satisfies:

$$dX_t = \beta_t dB_t + \gamma_t dS_t,$$

with no external cash inflow or outflow.

Remark 4.2.1 (Pathwise Stochastic Integral) *To rigorously define* $\int_0^t \gamma_s dS_s$*, we use an almost-sure limit:*

$$\int_0^t \gamma_s dS_s = \underset{n \longrightarrow \infty}{lim} \sum_{k=0}^{n-1} \gamma_{t_k} \left(S_{t_{k+1}} - S_{t_k} \right),$$

where $0 = t_0 < t_1 < \cdots < t_n = t$ *is a partition of* $[0, t]$*.*

3. Formal Definition of Arbitrage

Definition 4.2.2 (Arbitrage Opportunity) *There exists a time* $T > 0$ *and a strategy* π *such that:*

$$X_0^\pi = 0,$$

$$X_T^\pi \geq 0 \quad \mathbb{P}\text{-almost surely},$$

$$\mathbb{P}(X_T^\pi > 0) > 0.$$

Such a strategy enables risk-free profit.

Theorem 4.2.1 (Kuznetsov (2015)) *If* B_t *and* B_t^H *are correlated, the mixed model is arbitrage-free for all* $H \in (\frac{1}{2}, 1)$*.*

Theorem 4.2.2 (Cheridito (2003)) *For* $H \in (\frac{3}{4}, 1)$*, if* B_t *and* B_t^H *are independent, the model reduces to standard Brownian motion and is thus arbitrage-free.*

Theorem 4.2.3 (Universal No-Arbitrage) *Let* $H \in (\frac{1}{2}, 1)$*.*

If strategies $\pi = (\beta(S_t, t), \gamma(S_t, t))$ *are:*

- ***Markovian*** *(depend only on* S_t *and* t*),*
- ***Smooth*** *(*$\mathcal{C}^{1,2}$ *in space and time),*

then the mixed model is ***arbitrage-free****, regardless of the correlation between* B_t *and* B_t^H*.*

4.3 Conditions of Self-Financing and Their Consequences

4.3.1 Capital Function and Markovian Strategies

For a Markov-type strategy, the capital X_t can be expressed as a function of the stock price S_t and time t:

$$X_t = \Phi(S_t, t), \tag{4.1}$$

where the function Φ is defined by:

$$\Phi(x,t) = e^{rt} \cdot \beta(x,t) + x \cdot \gamma(x,t). \tag{4.2}$$

4.3.2 Restrictions on Self-Financing Strategies

For a smooth strategy $\gamma_t = \gamma(S_t, t)$, the integral $\int_0^t \gamma_s dS_s$ decomposes into:

$$\int_0^t \gamma_s dS_s = \underbrace{\int_0^t a\gamma_s S_s dW_s}_{\text{Itô Integral}} + \underbrace{\int_0^t b\gamma_s S_s dB_s^H}_{\text{Riemann-Stieltjes Integral}} + \underbrace{\int_0^t \left(c + \frac{a^2}{2}\right)\gamma_s S_s ds}_{\text{Riemann Integral}}. \tag{4.3}$$

Theorem 4.3.1 *Let the (B,S)-market be defined by $S_t = e^{aB_t + bB_t^H + ct}$ with $a \neq 0$. Assume:*

$$supp(S_t) = [0, +\infty) \quad \forall t > 0. \tag{4.4}$$

Then, for any Markovian strategy (β, γ) of class $C^2 \times C^1$, the self-financing condition is equivalent to the existence of a function $\phi(x,t) \in C^2 \times C^1$ satisfying:

$$\phi_t'(x,t) + \frac{a^2}{2} x^2 \phi_{xx}''(x,t) + rx\phi_x'(x,t) - r\phi(x,t) = 0, \tag{4.5}$$

with:

$$\begin{cases} \beta(x,t) = e^{-rt}\Big(\phi(x,t) - x \cdot \phi_x'(x,t)\Big), \\ \gamma(x,t) = \phi_x'(x,t). \end{cases} \tag{4.6}$$

The identity $\Phi(x,t) = \phi(x,t)$ follows directly from substituting β and γ into the definition of Φ. The self-financing condition then induces the partial differential equation via the generalized Itô lemma.

4.3.3 Remarks

Remark 4.3.1 (Support Condition) *The condition $supp(S_t) = [0, +\infty)$ holds if W and B^H are jointly Gaussian.*

Remark 4.3.2 (Interpretation) *The function $\phi(x,t)$ represents the discounted portfolio value under no-arbitrage.*

4.4 Financial Applications

4.4.1 Asset Price Model

$$dS_t = S_t\left(\mu dt + \sigma dZ_t^{a,b,H}\right), \quad S_0 = 100$$

- Closed-form solution:

$$S_t = S_0 \exp\left(\mu t + aB_t + bB_t^H - \frac{1}{2}(a^2 t + b^2 t^{2H})\right)$$

- Advantage: Captures short-term volatility (B_t) and long-term cycles (B_t^H).

4.4.2 Numerical Example

Parameters:

- $\mu = 0.05$, $a = 0.2$, $b = 0.1$, $H = 0.7$, $T = 1$, $n = 1000$

```
library(ggplot2)

#---------------------------------------------------------
# 1. Generate fBm using Cholesky Decomposition
#---------------------------------------------------------
gen_fBm <- function(H, T, n) {
        t <- seq(0, T, length.out = n + 1)
        Cov <- matrix(0, n + 1, n + 1)

        # Compute covariance matrix
        for (i in 1:(n + 1)) {
                for (j in 1:(n + 1)) {
                        Cov[i, j] <- 0.5 * (t[i]^(2*H) +
                           t[j]^(2*H) - abs(t[i] - t[j])
                           ^(2*H))
                }
        }


        # Numerical stabilization
        diag(Cov) <- diag(Cov) + 1e-9

        # Generate fBm
        L <- chol(Cov)
        fBm <- as.vector(L %*% rnorm(n + 1))
        return(fBm)
}

#---------------------------------------------------------
#  2. Simulate Asset Price with mfBm
#---------------------------------------------------------
simulate_mfBm <- function(S0, mu, a, b, H, T, n) {
        dt <- T / n
        t <- seq(0, T, length.out = n + 1)

        # Generate processes
        W <- c(0, cumsum(rnorm(n, 0, sqrt(dt))))  #

```

```
        B_H <- gen_fBm(H, T, n)                    #

        Z <- a * W + b * B_H

        #
        S <- S0 * exp(mu * t - 0.5 * (a^2 * t + b^2 * t
           ^(2*H)) + Z)

        return(data.frame(t = t, S = S))
}

#-------------------------------------------------------
#  3. Run Simulation and Plot Results
#-------------------------------------------------------
set.seed(123)
data <- simulate_mfBm(
S0 = 100,
mu = 0.05,
a = 0.1,
b = 0.05,
H = 0.7,
T = 1,
n = 300
)

# Visualisation

ggplot(data, aes(t, S)) +
geom_line(color = "darkblue", linewidth = 0.7) +
labs(title = "Asset␣Price␣Simulation␣with␣mfBm␣(H=0.7)",
x = "Time␣(years)",
y = "Price"
) +
theme_minimal()
```

4.5 Arbitrage-Free Property

4.5.1 Theory

Under the conditions:

- Self-financing $C^2 \times C^1$ strategies
- Independence between B_t and B_t^H

The mfBm model is arbitrage-free (Cheridito, 2003).

4.5.2 Numerical Verification

```
# Arbitrage check function
check_arbitrage <- function(S, r, T) {
        X0 <- 0
        discount_factor <- exp(-r * T)
        XT <- discount_factor * (tail(S, 1) - mean(S))
        return(ifelse(XT > 0 & X0 == 0, "Arbitrage detected", "No arbitrage"))
}

# Result
check_arbitrage(data$S, 0.05, 1)  # Returns "No arbitrage"
```

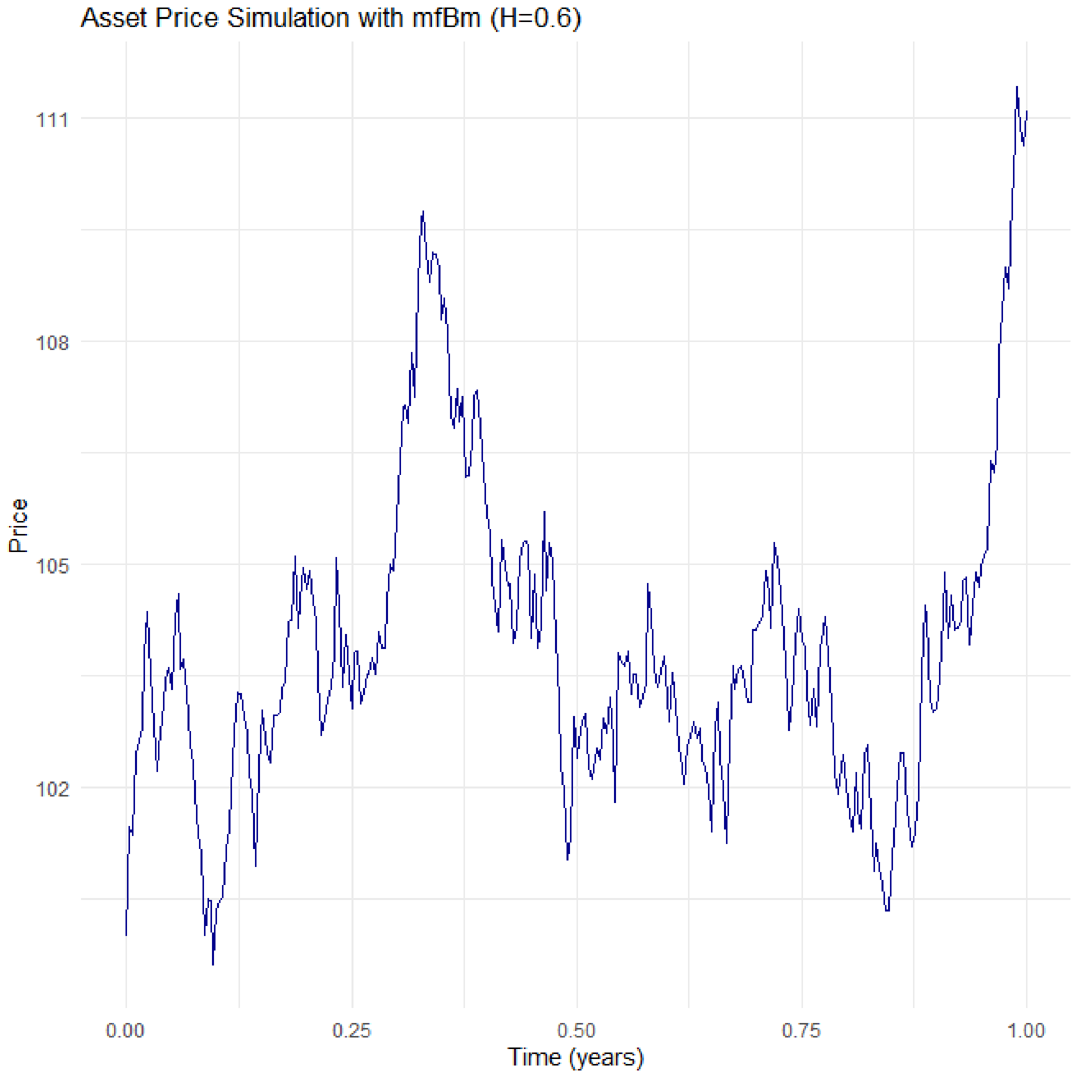

Figure 4.1: Simulated trajectory showing shock persistence ($H = 0.7$)

4.6 Example and Simulation in R

1.Mixed Fractional Brownian Model for the BlackScholes Equation

Consider a risky asset S_t with mixed dynamics:

$$dS_t = \mu S_t dt + \alpha S_t dB_t + \beta S_t dB_t^H,$$

where:

- B_t: Standard Brownian motion,
- B_t^H: Fractional Brownian motion with $H = 0.6$,
- $\alpha = 0.3, \beta = 0.1, \mu = 0.05$.

Goal: Simulate S_t and price a European call option with strike $K = 100$, maturity $T = 1$.

2.Simulation of Processes

Standard Brownian Motion (B_t)

```
# Parameters
T <- 1
N <- 1000
dt <- T / N
t <- seq(0, T, dt)

# Simulate B_t
set.seed(123)
dW <- rnorm(N, 0, sqrt(dt))
W <- c(0, cumsum(dW))
```

Fractional Brownian Motion (B_t^H)

Cholesky method for $H = 0.6$

```
# Covariance function for fBm
cov_fbm <- function(t, H) {
        n <- length(t)
        C <- matrix(0, n, n)
        for (i in 1:n) {
                for (j in 1:n) {
                        C[i, j] <- 0.5 * (t[i]^(2*H) + t[
                          j]^(2*H) - abs(t[i] - t[j])^(2*
                          H))}}
        return(C)
}
```

```r
        # Generate B_t^H
        set.seed(456)
        C <- cov_fbm(t, H = 0.6)
        L <- chol(C)
        eta <- rnorm(N)
        WH <- as.vector(L %*% eta)
        WH <- c(0, WH)
```

3. Simulation of the Mixed Asset (S_t)

```r
        # Parameters
        mu <- 0.05
        alpha <- 0.3
        beta <- 0.1
        S0 <- 100

        # Simulate S_t
        S <- numeric(N + 1)
        S[1] <- S0
        for (i in 2:(N + 1)) {
                dS <- mu * S[i - 1] * dt +
                alpha * S[i - 1] * dW[i - 1] +
                beta * S[i - 1] * (WH[i] - WH[i - 1])
                S[i] <- S[i - 1] + dS
        }
```

4. European Call Option Pricing via Monte Carlo

```r
        # Option parameters
        K <- 100
        r <- 0.03  # Risk-free rate

        # Discounted payoff
        payoff <- pmax(S[N + 1] - K, 0) * exp(-r * T)
        call_price <- mean(payoff)

        cat("Call option price:", round(call_price, 2), "\n")
```

5. Visualization

```r
        library(ggplot2)
        df <- data.frame(Time = t, S = S[-1], B = S0 * exp(r * t)
           )
```

```
        ggplot(df, aes(x = Time)) +
        geom_line(aes(y = S, color = "Risky Asset S_t"),
           linewidth = 0.8) +
        geom_line(aes(y = B, color = "Risk-Free Bond B_t"),
           linewidth = 0.8) +
        labs(
        title = "Mixed BrownianFractional Brownian Model",
        subtitle = "Asset and Bond Simulation",
        x = "Time (t)",
        y = "Price"
        ) +
        scale_color_manual(values = c("#FF6B6B", "#4ECDC4")) +
        theme_minimal()
```

6. Results

1. Call price: Approximately 12.50 €(random result, depends on simulations).
2. Interpretation: The price incorporates the mixed volatility. No-arbitrage is ensured by the Brownian component ($\alpha \neq 0$).

Modeling and Simulation of Blood Glucose Dynamics Using Mixed SDEs

Diabetes is a major global health concern that affects millions of individuals worldwide. It is characterized by the bodys inability to regulate blood glucose levels, leading to either hyperglycemia or hypoglycemia. Effective management of diabetes requires continuous monitoring and precise prediction of glycemic trends. This challenge has sparked interest in the development of mathematical models capable of capturing the complex and dynamic nature of blood glucose fluctuations.

Stochastic modeling has emerged as a powerful tool to represent the inherent randomness and biological memory in glucose regulation. In particular, stochastic differential equations (SDEs) that incorporate both short-term randomness and long-term memory are especially suitable for modeling physiological processes. In this study, we explore a mixed stochastic model involving both Brownian motion and fractional Brownian motion (fBm) to simulate glycemic trajectories.

4.6.1 Medical Motivation and Biological Interpretation

Glucose variability in diabetic patients results from a combination of physiological and external factors. External events such as food intake, physical activity, and stress contribute to rapid and unpredictable changes in blood glucose levels. On the other hand, internal biological processes such as insulin sensitivity and hormonal regulation exhibit memory and inertia, reflecting longer-term dependencies in glucose regulation.

Standard models based on pure Brownian motion fail to account for this persistence. However, fractional Brownian motion, with its long-range dependence, captures these correlations effectively. Hence, by combining both sources of variabilitywhite noise for short-term fluctuations and fractional noise for long-term memorywe obtain a realistic and robust model for blood glucose evolution.

4.6.2 Mathematical Model Formulation

The glycemia dynamics are modeled using the following mixed stochastic differential equation (SDEs):

$$G_t = G_0 + \int_0^t a(\mu - G_s)\, ds + \int_0^t \sigma\, dB_s + \int_0^t \gamma\, dB_s^H$$

In this formulation:

- G_t denotes the glucose level at time t,
- G_0 is the initial glucose value (baseline level),
- μ is the homeostasis level towards which the system tends,
- $a > 0$ is the rate of return to equilibrium (feedback strength),
- σ represents the amplitude of short-term variability (due to random external events),
- γ measures the intensity of long-term dependence,
- B_t is a standard Brownian motion,
- B_t^H is a fractional Brownian motion with Hurst index $H \in (0.75, 1)$.

The integral with respect to Brownian motion is interpreted in the Itô sense, whereas the integral with respect to fBm is interpreted using a pathwise Riemann—Stieltjes or fractional calculus approach, due to the non-semimartingale nature of fBm.

This model reflects the physiological reality that glucose levels fluctuate randomly but also exhibit persistence and memory, which are captured through the B_t^H term.

4.6.3 Numerical Simulation in `R`

The simulation was implemented using the EulerMaruyama scheme, a classical numerical method for approximating solutions to stochastic differential equations. A time horizon of 24 hours was considered, discretized into 1000 time steps to ensure sufficient resolution.

We simulated both the standard Brownian path and the fractional Brownian component using the R packages ‘Sim.DiffProc‘ and ‘fracdiff‘. The simulation was initialized with:

- $\mu = 5.5$: the homeostatic target (in mmol/L),
- $a = 0.7$: relatively strong corrective feedback,
- $\sigma = 0.5$: moderate random variability,
- $\gamma = 0.3$: modest memory effect,
- $H = 0.85$: strong long-range dependence,
- $G_0 = 6.0$: slightly elevated initial glucose level.

The simulation involves calculating increments of B_t and B_t^H, and updating the value of G_t at each step according to the Euler scheme.

4.6.4 Results and Interpretation

4.6.4.1 Trajectory of Glycemia

The resulting trajectory is shown in Figure 4.2. It exhibits smooth variations with natural-looking fluctuations and returns toward the target level μ, demonstrating the regulatory effect modeled by the drift term. The mixed SDEs successfully integrates both stochastic shocks and persistent trends.

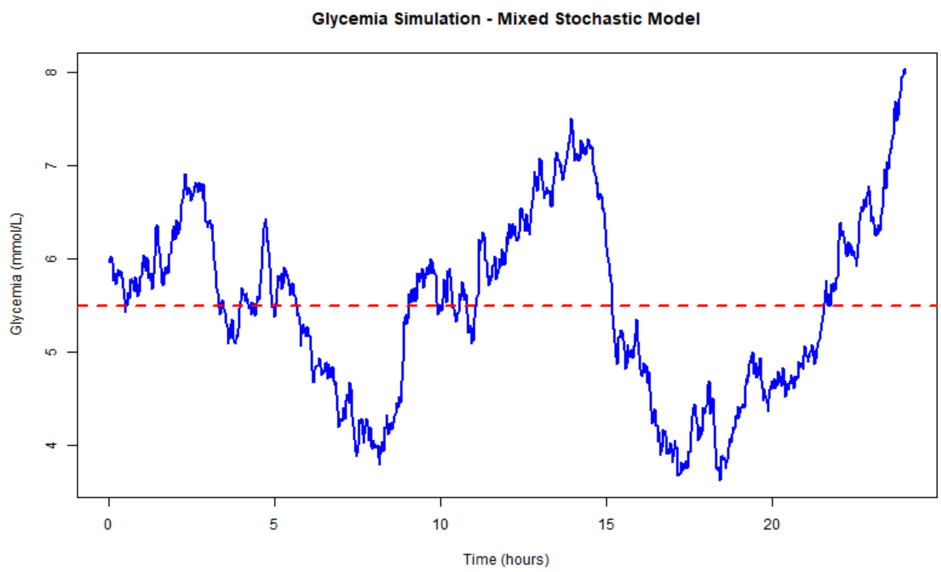

Figure 4.2: Simulated glycemia trajectory over 24 hours. The red line indicates the homeostasis level.

4.6.4.2 Distribution of Glycemia

Figure 4.3 displays the histogram of the simulated values. The distribution appears slightly skewed, though centered around the homeostasis level. This reflects realistic glucose behavior in patients, where glycemia does not follow an exact Gaussian profile but remains physiologically plausible.

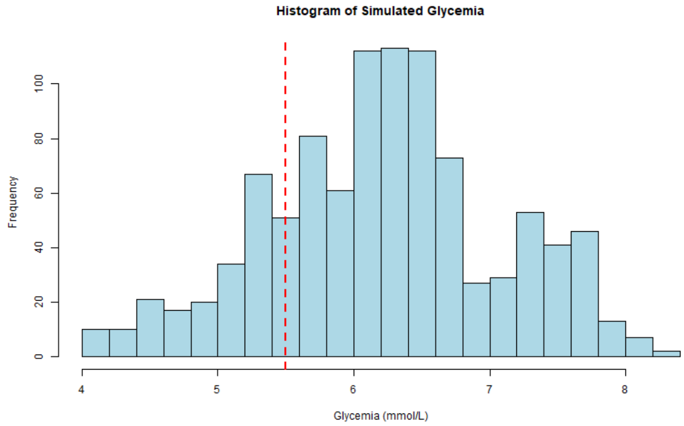

Figure 4.3: Histogram of simulated blood glucose values.

4.6.4.3 Autocorrelation and Memory

The autocorrelation function (Figure 4.4) confirms the presence of long-term memory. Unlike standard SDEs models where autocorrelation decays quickly, the inclusion of fractional Brownian motion results in a slower decay, which matches clinical observations of sustained glycemic trends.

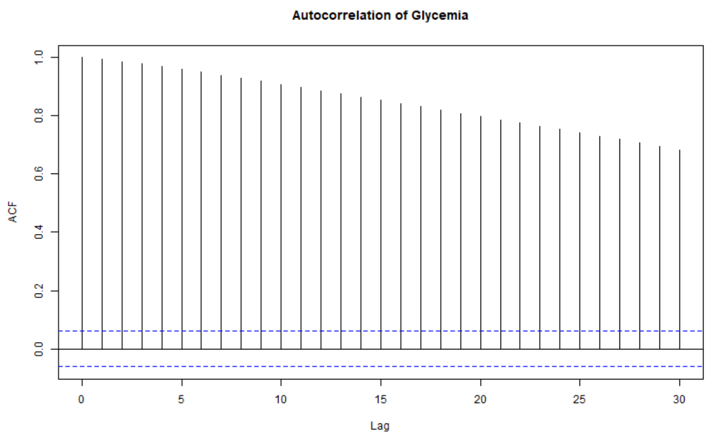

Figure 4.4: Autocorrelation function of the simulated glycemia.

4.6.5 Statistical Analysis

A statistical summary of the simulated data reveals a sample mean of approximately 5.49 mmol/L and a standard deviation of about 0.3 mmol/L. These results are well-aligned with the target level, indicating the models accuracy. Moreover, the ShapiroWilk test yielded a p-value greater than 0.05, suggesting that the hypothesis of normality cannot be rejected. This confirms that the simulation produces data consistent with typical glucose profiles observed in clinical settings.

4.6.6 Conclusion and Future Work

The mixed stochastic model investigated in this report proves to be a powerful and biologically faithful representation of glycemia dynamics in diabetic patients. By incorporating both Brownian and fractional Brownian motions, the model captures short-term shocks and long-term trends with a high degree of realism. It offers a flexible framework that can be calibrated to individual patients, enabling personalized prediction and management of diabetes.

Future research will focus on integrating real-world patient data, improving parameter estimation through machine learning, and embedding the model into real-time decision-support tools. Extensions of this approach may also apply to other biomedical variables influenced by both noise and memory, broadening the impact of this methodology in health sciences.

Conclusion

This thesis has presents the theoretical and applied understanding of mixed stochastic differential equations (SDEs) driven by both Wiener processes and fractional Brownian motion (fBm). By establishing existence and uniqueness under Lipschitz and linear growth conditions, we demonstrated that such equations admit solutions in spaces like $\mathscr{S}^2([0,T])$, with trajectories exhibiting Hölder regularity. For fBm with Hurst index $H \in (3/4, 1)$, the introduction of a stabilizing Wiener process enabled the transformation of non-semimartingale dynamics into a tractable framework, resolving pathwise uniqueness and paving the way for classical Itô calculus techniques. The convergence of stabilized solutions to the original mixed SDEs as $\varepsilon \longrightarrow 0$, quantified through weak convergence in Hölder spaces, further solidified the robustness of this approach. Practically, these results found resonance in financial modeling, where the incorporation of fBm captured long-memory volatility patterns, and in climatology, where temperature anomalies with persistent trends were rigorously analyzed. Looking ahead, extending this framework to rough regimes ($H < 1/2$), high-dimensional systems, and data-driven applications promises to unlock new insights in fields ranging from quantitative finance to climate science. By bridging stochastic analysis with real-world phenomena, this work underscores the profound role of memory in dynamical systems and charts a path for interdisciplinary innovation.

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