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Theme:

Predator-prey model with uncertainty

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Abstract

A stochastic fish prey predator model with functional response based on the changes of water level in an artificial lake is proposed and investigated. We show there is a unique positive solution to the model with positive initial value and we show that the positive solution to the stochastic system is stochastically bounded. Besides, a condition for the system to be extinct is given and persistent conditions are established. We further investigate the stability of our system. To investigate the effect of harvesting on the prey population a stochastic optimal control is introduced. Finally theoretical results are illustrated using numerical simulations.
Keywords: Stochastic prey-predator model, existence of positive global solution, Persistent in mean, Extinction, Stability, Stochastic optimal control.

Resumé

Un modèle stochastique de prédateurs de poissons avec une réponse fonctionnelle basée sur les variations du niveau d'eau d'un lac artificiel est proposé et étudié. Nous démontrons l'existence d'une solution positive unique au modèle avec une valeur initiale positive et démontrons que la solution positive du système stochastique est stochastiquement bornée. De plus, une condition d'extinction du système est donnée et des conditions persistantes sont établies. Nous étudions également la stabilité de notre système. Pour étudier l'effet de la récolte sur la population de proies, un contrôle optimal stochastique est introduit. Enfin, les résultats théoriques sont illustrés par des simulations numériques.
Mots clés: Modèle stochastique proie-prédateur, Modèle stochastique, existence d'une solution globale positive, Persistant en moyenne, Extinction, Stabilité, Contrôle optimal stochastique.

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Introduction

Studies of food chains are on the edge of two domains of ecology: population and community ecology. The properties of food chains are determined by the nature of their basic link, the interaction of two species, a consumer and its resource, a predator and its prey. The study of this basic link of the chain is part of population ecology while the more complex food webs belong to community ecology. This is one of the main reasons why understanding the dynamics of predation is important for many ecologists working at different scales [3]. Another stronger reason is that there is no organism can live, grow and reproduce without consuming resources (Murdoch et al. 2003, 1–2). To better understand and explain real-world phenomena. Mathematical analysis is becoming essential for understanding our environment and its evolution, and for exploring phenomena on time and space scales which was difficult to access experimentally. The mathematical model allows to numerically determine certain phenomena and predict certain behaviors. Because the study of these problems requires the collaboration of scientists from different disciplines (biology, mathematics, computer science). It can be seen that in recent decades, mathematical modeling in biology has been widely applied. Thus, we can consider mathematical modeling to be an essential step in biological research.

The first mathematical models in ecology date back to the 1920s, with the scientists A.J. Lotka and V. Volterra. These models are based on a representation of interactions between species using systems of ordinary differential equations, which is well explained in subsection (1.4.2), the Lotka-Volterra model can be considered as the founding model of modern ecology. Numerous studies have contributed to describe the growth rates of both populations and their interactions, such as logistic growth and Holling type I, II, and III functional responses, in various ways. As early as 1933, Nicholson and Bailey criticized the Lotka-Volterra model because the linear relationship between attack rates and predator abundance seemed unrealistic. Canadian ecologist C.S. Holling concluded that understanding predation requires a more realistic and empirical approach. The Leslie-Gower prey-predator model was introduced in 1948 by Leslie and was discussed by Leslie and Gower in 1960 and Pielou in 1969. Leslie (1948) used the logistic equation to obtain a more realistic model based on the following assumptions:

– The prey population follows logistic growth.

– The functional response corresponds to a linear relationship with the density of the prey population.

– The predator population follows logistic growth and the growth of predators is limited by the prey density.

In the modeling of predator–prey interactions, the attacking rate of the predator cannot always be constant. Thus, seasonal variations and human activities enhance the amplitude, frequency, and unpredictability of the natural fluctuations. Recently, researchers extended their investigation beyond previous work and consider water level fluctuations as a phenomenon that can influence the structure and dynamics of fish communities. The impact of water level fluctuations in aquatic ecosystems has been studied by many researchers. Chiboub et al. [27] have worked on a non-autonomous system using two species: the pike (predator) and the roach (prey) in Pareloup Lake, and showed that the seasonal variations in water level exert a strong influence on prey–predator interactions. Moussaoui and collaborators [28] investigated a complex interaction between three species of fish in a lake and showed that only under some conditions is the system permanent. Gownaris et al. [19] studied the consequences of water fluctuations on ecosystem attributes of 13 African lakes. They found that in most cases water fluctuation enhanced primary and overall lake production, including fisheries production, but with important consequences on species diversity [31].

Population systems are often affected by environmental noise, and hence stochastic differential equation models play a significant role in various branches of applied sciences, including biology and population dynamics, as they provide some additional degree of realism compared to their deterministic counterparts. In reality, due to continuous fluctuations in the environment (e.g., variation in intensity of sunlight, temperature, water level, etc.), parameters involved in models are not absolute constants, but they always fluctuate around some average value [21]. There are several papers on stochastic predator–prey models that can be seen as an extension of deterministic models. For example, in [21], the authors investigate a stochastic ratio-dependent predator–prey model. The paper [25] developed a stochastic predator–prey system with a Holling II functional response, and authors in [17] proposed a predator–prey model with modified Leslie–Gower and Holling type II schemes with stochastic perturbation.

In the present work, we developed a stochastic fish predator–prey model built upon previous studies in the deterministic case to enhance the model's realism and analyze the effect of stochastic perturbation on population dynamics. The model we have considered is influenced by water level fluctuations, and we added an optimal control on the harvesting policy to incorporate human activities into the model. Accordingly, this thesis is divided into three main parts.

- The first chapter discuss the fundamental tools and concepts of population dynamics and this including the mathematical modeling, and explain some important ecological concepts.
- The second chapter is the core of this thesis, presenting the main findings of the research. In this chapter, we introduce the mathematical model that forms the foundation of the study, which incorporates the effects of water level fluctuations and harvesting, in both deterministic and stochastic frameworks. A qualitative analysis of the model, its long term behavior and stability are also provided.
- In the last chapter, we aim to investigate the effect of harvesting on the prey population. We consider an optimal harvesting policy based on previous studies in the deterministic case and carry out numerical simulations to support the theoretical analysis.

Chapter 1

Basics of predator-prey system modeling

First, this chapter will review some basic notions of population dynamics. Next, it will introduce some classical models proposed in the framework of mathematical biology in order to model the time evolution of animal species populations.

1.1 Fundamental concepts of population dynamics

This section introduces some definitions related to population dynamics.

Definition 1.1. *(Individuals) Individuals are organisms and possess life. They have growth and development, presenting different stages of life. They reproduce and die, mostly having a shorter span than the systems they are part of. To survive, they need resources, and to get these resources they alter their environment. Individuals vary from each other even when belonging to the same species and age, thus interacting differently with their environment. Most importantly, individuals are adaptive: all their actions growth, development, resource acquisition, reproduction, and interactions are shaped by both their internal and external environments. They possess traits that enable them to adapt to changes within themselves and their surroundings in ways that enhance their fitness.*

Definition 1.2. ***(Population)***
Population terminology tends to keep with the norm of interbreeding. In ecology populations are typically defined as all the organisms of the same group or species, which live in a particular geographical area, and have the capability of interbreeding. The importance of the population concept in the understanding of ecological processes consists in understanding the dynamics of the species in a determined space and time.

The notion of a population is fundamentally spatial: the locality where the individuals reside and move. This area may be difficult to characterize as it is a function of the movement and dispersion patterns of the species of interest. Population size is affected by birth rates, death rates, immigration and emigration.

Definition 1.3. ***(Population dynamics)***
Population dynamics and population change tend to be used to refer specifically to the change in the number of individuals, and emphasis should be made on the distinction between these and size, structure and composition. These changes can be influenced by the following five significant ecological processes:

- ***Reproduction:*** *The reproduction of individuals determines the growth of the population.*
- ***Survival:*** *A population decreases or remains stable due to the death rate.*
- ***Migration:*** *The introduction or removal of individuals to/from a population, thus disrupting its genetic composition, it leads to an altered population structure.*
- ***Dispersal:*** *Related to movement of individuals within their home ranges or to new locations which is very important in mobile breeds and contribute on the population spatial dynamics.*

Additionally, populations evolve depending on the different interactions they have with other species influencing their dynamics. The major forms of interactions are:

1. ***Competition:*** *It is interactions between species for the same resources such as food, leading to a reduction in their growth or reproductive rates.*
2. ***Mutualism or symbiosis:*** *Represents interactions between species that benefit from each other, and that result in increased growth or reproductive rates. For example bees and flowers have a mutualistic relationship.*
3. ***Predation:*** *Interactions between predator and prey where the population growth of one affects the population growth of another, the predator depend on prey for survive*

Definition 1.4. *(Persistence) In the real sense of prey-predator models, persistence relates to coexistence of both population, the prey and their predator remain in time with no occurrence of extinction among the individuals. without reaching extinction.*
The permanence means that, in addition to long-term survival, it also takes into account the

limits of species growth. Mathematically there are several definition of persistence, definitions for weak, strong, and uniform persistence have been given by [8, 9] for dynamical systems in a locally compact metric space.

1.2 The basics of modeling in ecology

1.2.1 Preliminary concepts

The continuous time evolution of a population or a system of populations can be simply modeled by an equation governing its variation in $\mathbb{R}^n$*:*

$$\dot{x} = \frac{dx}{dt} = Births + Immigrations - Deaths - Emigrations, \tag{1.1}$$

which can be expressed as an ordinary differential equation:

$$\dot{x} = f(x), \tag{1.2}$$

where x *is a variable of interest such as the density of a population, and* $f : D \longrightarrow \mathbb{R}^n$ *is a locally Lipschitz function where* D *is the domain of definition of* f *(in the context of population dynamics,* $D = \mathbb{R}^n_+$ *typically) [2].*

1.2.2 Equilibrium points and stability

It is often difficult to calculate an explicit solution to (1.2), and sometimes the model can only be analyzed through a qualitative study. In this case, it is possible to focus on particular points called equilibrium [26], whose existence and properties are directly deduced from (1.2).

Definition 1.5. *[26] We define an equilibrium point (or critical or stationary point) of the system* (1.2) *as any point* x^* *such that:* $f(x^*) = 0$.

Among other things, the long-term dynamics of the system (1.2) can be provided by studying the stability of its equilibrium [2].

Definition 1.6. *[26] The equilibrium point* x^* *of* (1.2) *is:*

1. ***stable** if, for every* $\epsilon > 0$*, there exists* $\delta = \delta(\epsilon) > 0$ *such that :*

$$\| x_0 - x^* \| < \delta \Longrightarrow \| x(t) - x^* \| < \epsilon, \forall t \geq 0.$$

2. ***asymptotically stable*** *if it is stable and* δ *can be chosen such that:*

$$\parallel x_0 - x^* \parallel < \delta \Longrightarrow \lim_{t \longrightarrow \infty} x(t) = x^*.$$

3. ***globally asymptotically*** *stable if it is asymptotically stable and :*

$$\forall x_0 \in D, \lim_{t \longrightarrow \infty} x(t) = x^*.$$

Furthermore, the theorem of the indirect method of Lyapunov allows us to define the stability of an equilibrium.

Theorem 1.1. *Let* x^* *be an equilibrium of the nonlinear system* (1.2)*, where* $f : D \longrightarrow \mathbb{R}^n$ *is continuous and differentiable, and* D *is a neighborhood of* x^**. Let the Jacobian matrix be* $J = \nabla_x f(x^*)$*.*

1. x^* *is asymptotically stable if* $\Re(\lambda_i) < 0$ *for all eigenvalues* λ_i *of* J*.*
2. x^* *is unstable if* $\Re(\lambda_i) > 0$ *for at least one eigenvalue* λ_i *of* J*.*

By examining the eigenvalues of the Jacobian matrix at the equilibrium point, one can determine the behavior of the system in the vicinity of that point.

$$\begin{cases} \dot{x}(t) = Jx(t), \\ x(0) = 0. \end{cases} \tag{1.3}$$

The study will be restricted to the analysis of the nature of critical points in the case where the state space is two-dimensional. Consider the system (1.3), where J is a (2×2) invertible matrix ($det(J) \neq 0$). The characteristic equation associated with the system (1.3) is given by :

$$\lambda^2 - tr(J)\lambda + det(J) = 0, \tag{1.4}$$

then the eigenvalues of this equation are λ_1 and λ_2 solutions of (1.4) Depending on the nature of the eigenvalues, we have the following results:

1. If both eigenvalues λ_1 and λ_1 are distinct **real numbers**. The system (2.2) transforms into: $\dot{Y} = AY$ with $Y = (Y_1, Y_2)$ and $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

- If $\lambda_1 > 0$ and $\lambda_2 < 0$, then $\lim\limits_{t\longrightarrow\infty} Y_1 = \infty$, and $\lim\limits_{t\longrightarrow\infty} Y_2 = 0$, thus the critical point x^* is a **saddle point** which is always **unstable**.
- If λ_1 and λ_2 are real numbers with the same sign, there are two cases:
  (a) If $\lambda_1 < \lambda_2 < 0$, then $\lim\limits_{t\longrightarrow\infty} Y_1 = 0$ and $\lim\limits_{t\longrightarrow\infty} Y_2 = 0$, the critical point x^* is a **stable node**.
  (b) If $0 < \lambda_1 < \lambda_2$, then $\lim\limits_{t\longrightarrow\infty} Y_1 = \infty$ and $\lim\limits_{t\longrightarrow\infty} Y_2 = \infty$, the critical point x^* is an **unstable node**.

2. If the two eigenvalues are **complex conjugates** $\lambda_{1,2} = \alpha \pm \mathrm{i}\beta$, the matrix J is written as:

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

   (a) If $\beta \neq 0$ then the critical point x^* is a **focus**. It is **stable** if $\Re(\lambda_{1,2}) < 0$ and **unstable** if $\alpha > 0$.
   (b) If λ_1 and λ_2 are purely imaginary, then the critical point x^* is a **center**. It is **stable** but **not asymptotically stable**.

1.3 Model of single population dynamics

1.3.1 The exponential model

The simplest model of population growth is the exponential model, otherwise known as the Malthusian model. It was first proposed by Euler in 1760 as a geometric series, and later by Malthus in 1798 to illustrate the growth of a population. In this model, the population has continuous births and deaths at any time. The dynamics of the population are represented by the birth rate β and death rate δ. In addition, the change in a population over time equals a difference between births and deaths. This can be defined with the following differential equation [1]

$$\dot{x} = \beta x - \delta x = ax, \tag{1.5}$$

where a is the population growth rate. The equation (1.5) has an exact solution obtained by integrating:

$$\dot{x} = ax \Longrightarrow x(t) = x_0 e^{at},$$

this model leads to three possible types of behavior depending on the values of the birth and death rates. Indeed, the sign of a.

1. If $a > 0 \Longrightarrow \lim\limits_{t \longrightarrow +\infty} x(t) = +\infty$, the population will grow exponentially.
2. If $a = 0 \Longrightarrow \lim\limits_{t \longrightarrow +\infty} x(t) = x_0$, the population remains constant.
3. If $a < 0 \Longrightarrow \lim\limits_{t \longrightarrow +\infty} x(t) = 0$, the population will decrease (extinction of the population).

The Malthusian model is unrealistic over a long period, even if it correctly describes certain episodes of population growth.

1.3.2 The logistic model

In the exponential model, the birth and death rates are constant. The idea of the logistic model, introduced by Verhulst (1838), is that it is more realistic to consider rates that change with the state of the population and that involve a limitation on population growth [4]. Thus, these rates can depend partly on population density:

$$\begin{aligned} \beta =& \beta_0 - \beta_x x, \\ \delta =& \delta_0 + \delta_x x, \end{aligned}$$

where β_0 and δ_0 are the birth and death rates when x (population's size) is very small. β_x and δ_x correspond to the strength of density dependence when births and deaths rates are influenced by population density. This means that population growth is not constant but varies depending on the number of individuals present of the birth and death rates. These rates can be introduced into (1.5)

$$\dot{x} = \left[(\beta_0 - \delta_0) - (\beta_x + \delta_x)x\right] x, \tag{1.6}$$

Assuming $a = \beta_0 - \delta_0 > 0$, this allows the model to be biologically relevant, with the growth rate of the population when x is very small and $K = \dfrac{a}{\beta_x + \delta_x}$.
K is called the carrying capacity of the environment and it represents the maximum population that an environment can support. This parameter is also commonly referred to as the population limit capacity. The number of parameters in (1.6) is reduced to :

$$\dot{x} = ax\left(1 - \frac{x}{K}\right) = f(x), \tag{1.7}$$

the model (1.7) is called the logistic model.

- **The equilibrium points**: The differential equation (1.7) has two equilibrium points the origin and K.

- **The stability**: Let's first calculate the derivative of the function $f(x)$.

$$\frac{df}{dx} = a - 2a\frac{x}{K}$$

$\frac{df}{dx}(0) = a > 0$, then the origin is unstable.

$\frac{df}{dx}(K) = -a < 0$, then K is stable equilibrium.

- **The solution of the differential equation**: The explicit solution x can be easily calculated using the method of separation of variables

$$x(t) = \frac{K}{1 + (\frac{K}{x_0} - 1)e^{-at}},$$

for any positive initial condition, we have :

$$\lim_{t \longrightarrow \infty} x(t) = K.$$

1.4 Model of dynamics of two interacting populations

1.4.1 Predator prey model

Definition 1.7. *[37] Consider a class of predator-prey models. Let the state variables represent the total densities of both species, assumed to be spatially homogeneous and without consideration of age or size. Models consider dynamics in continuous time: Let $y(t)$ be the predator density (either in terms of number or biomass) it gives the number of predator individuals per unit space at time t, while $x(t)$ is the prey density, it gives the number of prey individuals per unit area or volume at time t. Then, the general form of such models can be written (for instance, Yodzis 1989) :*

$$\begin{cases} \frac{dx}{dt} = f(x) - F(x,y)y, \\ \\ \frac{dy}{dt} = G(x,y)y. \end{cases} \tag{1.8}$$

To define a model, we need to specify three functions of density, namely:

- $f(x)$: *This is the function that gives the growth rate of the prey population when it is free of predation [37]. It is describing how the prey population would change over time in an environment with no predators that may reduce their numbers. This function depends on the density of the prey and is usually modeled as a logistic growth rate, as it takes into account resource limitation and environmental carrying capacity.*

- $F(x,y)$: *the predator's functional response, which is the number of prey individuals consumed per unit area per unit time by an individual predator [37]. Holling (1959) defined the term 'functional response' (after Solomon, 1949) as an increase in the number of prey consumed in response to increasing prey population density and recognized three major types of predator responses [29].*

- $G(x,y)$: *This is the function of the numerical response by the predator, which gives the rate at which predators reproduce in response to variation in the prey density. Both Solomon (1949) and Holling (1959) demonstrated that predators have numerical responses defined as changes in predator numbers based on reproductive recruitment in response to increasing prey density [29].*

An important aspect of biology is how individual predators interact with each other. One possible scenario is where predators do not interfere with one another in their activities (at least not in a way that affects the population dynamics), a situation referred to as "laissez-faire" by Caughley and Lawton (1976). In this case, neither the functional response nor the numerical response depends on predator density y*, meaning the functions only depend on prey density* x*:*

$$\begin{cases} F(x,y) = F(x), \\ \\ G(x,y) = G(x). \end{cases} \tag{1.9}$$

In reality, however, predators may interact in various ways that could affect population growth. For instance [37]:

1. ***Impeding feeding activities****: The predators can interfere with the feeding activity of others by competing for the same prey.*
2. ***Competition for mates, safe refuges****: They may also compete for essential resources such as mates for reproduction and safe refuges.*

3. ***Disease transmission**: High predator density may aid in disease transmission and weaken the population.*
4. ***Cannibalism**: It is when some predators eat members of their own species, including adults feeding on juveniles, a common sight among fishes.*
5. ***Territorial behavior**: Some may be territorial, leading to conflicts with other conspecifics, limiting area use, or resource availability.*
6. ***Migration**: Some predators may also emigrate out of the study area, due to competition and stress, for example.*

All of these are forms of predator interference, in which individuals reduce each other's survival or reproduction. On the other hand, some of these effects could run in the other direction. For example, predators may sometimes facilitate each other while feeding, such as when hunting in coordination or in defense with one another in order to help capture prey. This sort of facilitation is considered rarely in predator-prey theory, but it should be kept in mind, particularly for social animals such as marine mammals [37].

Remark 1. *[16] The predator-prey system can also take the following form :*

$$\begin{cases} \dfrac{dx}{dt} = f(x) - F(x,y)y, \\ \\ \dfrac{dy}{dt} = eF(x,y)y - qy. \end{cases} \tag{1.10}$$

The production of prey in absence of predators is described by the function $f(x)$, whereas $F(x,y)$ is the function response. The quantity e is the trophic efficiency (is a factor that describes how efficiently the predator converts consumed prey into new predator biomass) and predators are assumed to die with a constant death rate q.

Functional response

In this section, we define the various expressions related to the functional response $F(x,y)$ of system (1.8) that are used in the literature. Based on the expression the functional response, there are three types of predator-prey models [6]:

- A model is called **prey-dependent** when the functional response depends only on the prey density, i.e., $F(x,y) = F(x)$. Additionally, this type of model corresponds to the "laissez-faire" situation introduced by Caughley and Lawton [37], where there is no interference between predators in their feeding activities. Holling's and Lotka-Volterra's functional responses belong to this category.

- A model is called **ratio-dependent** [33] when the functional response takes the form $F(x,y) = F\left(\frac{x}{y}\right)$.
- A model is called **density-dependent (or predator-dependent)** [33] when the functional response depends on both x and y.

In the late 1950s, C. S. Holling conducted seminal experiments to investigate the functional response which is, how a predator's rate of prey capture is related to prey density. The results of these experiments were presented in a series of classic papers that divided functional responses into three types: Type I, Type II, and Type III. These types give different ways of describing how a predator's feeding rate varies with increasing prey density [12].

1. **The type I functional response** [12] is characterized by a linear relationship between the predator's capture rate and prey density x

$$F(x) = \alpha x,$$

   where α is a proportionality constant set by the rate at which predators encounter prey.

2. **The type II functional response** [12] Holling derived a mathematical expression for F in which handling time t_h is taken into account (t_h is assumed to be greater than zero), this functional response describes a predator's capture rate that increases with prey density but gradually slows down, approaching a maximum. This occurs because the predator spends time handling and consuming prey, limiting its feeding rate at high prey densities. Expressed in its currently accepted form, Holling's Type II relationship is:

$$F(t_h, x) = \frac{\alpha x}{1 + \alpha t_h x},$$

   where α play the same role as α the Type I functional response.

3. **The type III functional response** [12] is similar to Type II, but at low prey densities, the capture rate is initially slow and then accelerates as prey density increases. This response suggests that predators may learn or improve their hunting efficiency as they

encounter more prey.

$$F(t_h, x) = \frac{\alpha x^2}{1 + \alpha t_h x^2},$$

where where α play the same role as α the Type I functional response and t_h is is the handling time similar to Type II.

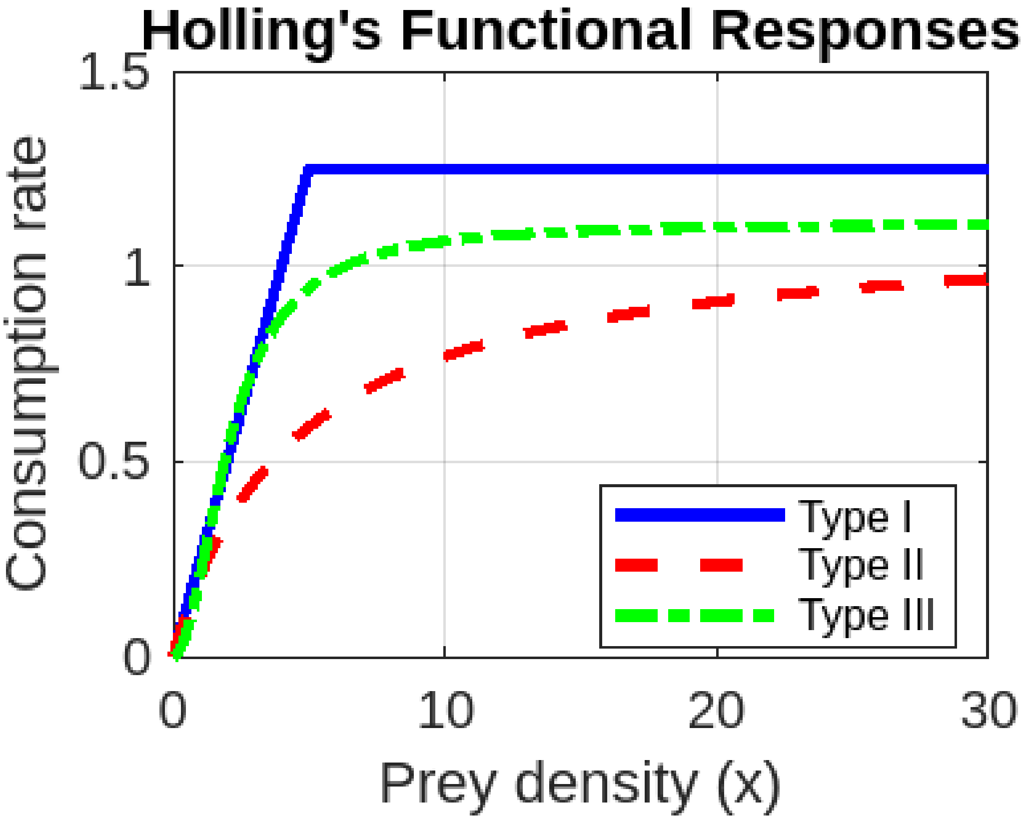

Figure 1.1: The three types of Holling functional response with $\alpha = 0.25$ and $t_h = 0.9$.

1.4.2 The Lotka-Volterra prey predator model

The earliest mathematical models in ecology trace back to the 1920s, developed by scientists A.J. Lotka and V. Volterra. These models represent species interactions through systems of ordinary differential equations. They extend the work of Malthus and Verhulst, particularly the idea of exponential growth, but with a variable growth rate for populations. The Lotka-Volterra model is treated as the conceptual basis of modern ecology. It is an attempt to explain the oscillatory dynamics with a phase shift in the populations of predators and prey by using the following model of Lotka-Volterra (see [35]) :

$$\begin{cases} \dot{x} = \alpha x - \gamma x y, \\ \dot{y} = e \gamma x y - \beta y. \end{cases} \tag{1.11}$$

where:

1. x and y represent the quantities of prey and predators at a given time, respectively, and $\dot{x}$, $\dot{y}$ represent the growth of both populations at time t.
2. α is the intrinsic growth rate of the prey population when there are no predators.
3. γ is a measure of how effective the predators are at catching and consuming prey.
4. β represents the natural death rate of the predator population.
5. e trophic efficiency.

The Lotka-Volterra assumptions

The model is simplified with the following assumptions [32]

1. There are only two species: one predator and one prey.
2. The preys are considered to have an unlimited food supply and are growing exponentially in the absence of predation. Such exponential growth of prey population is represented by the term αx in the system (1.2).
3. If either of the terms x or y is zero, then there can be no predation.
4. Prey are born and can die either from predation or natural causes.
5. Predators are born, and their birth rate is positively affected by the rate of predation, while they also die naturally.

By setting $e\gamma = \zeta$, the system (1.11) can be rewritten as follows :

$$\begin{cases} \dot{x} = \alpha x - \gamma x y, \\ \dot{y} = \zeta x y - \beta y. \end{cases} \tag{1.12}$$

- **The equilibrium points**

  We define $f(x, y) = x(\alpha - \gamma y)$ and $g(x, y) = y(-\beta + \zeta x)$. To find the equilibrium points, we need to find the solutions to the following system according to (1.11):

$$\begin{cases} f(x, y) = 0, \\ g(x, y) = 0. \end{cases} \tag{1.13}$$

Therefore:

$$\begin{cases} x = 0 \vee x = \dfrac{\beta}{\zeta}, \\[2ex] y = 0 \vee y = \dfrac{\alpha}{\gamma}. \end{cases} \tag{1.14}$$

- **The stability** Let's calculate the Jacobian matrix to determine the local stability properties of the equilibrium.

$$J(x,y) = \begin{pmatrix} \alpha - \gamma y & -\gamma x \\ \zeta y & \zeta x - \beta \end{pmatrix}.$$

1. **Stability for the point (0,0)** :

$$J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix},$$

J is a diagonal matrix and thus has two distinct real eigenvalues: $\lambda_1 = \alpha > 0$ and $\lambda_2 = -\beta < 0$, with opposite signs. This is hence an **unstable saddle point**.

2. **Stability for the point** $\left(\dfrac{\beta}{\zeta}, \dfrac{\alpha}{\gamma}\right)$:

$$J\left(\frac{\beta}{\zeta}, \frac{\alpha}{\gamma}\right) = \begin{pmatrix} 0 & -\gamma\left(\dfrac{\beta}{\zeta}\right) \\ \zeta\left(\dfrac{\alpha}{\gamma}\right) & 0 \end{pmatrix},$$

$\lambda_1 = \mathrm{i}\sqrt{\beta\alpha}$ and $\lambda_2 = -\mathrm{i}\sqrt{\beta\alpha}$ are purely imaginary, then the critical point x^* is a **stable center**.

1.4.3 The Lotka-Volterra model and logistic growth

The classic Lotka-Volterra model (1.12) is built under the assumption that prey can grow exponentially. However, in nature, populations are often limited by their environment, and this property tends to stabilize them around their carrying capacity. As a result, by considering logistic growth for the prey, the modified Lotka-Volterra model gains stability, as demonstrated by its analysis [36]. Which leads to the following model:

$$\begin{cases} \dot{x} = ax\left(1 - \dfrac{x}{K}\right) - \gamma xy, \\ \\ \dot{y} = \zeta xy - \beta y. \end{cases} \tag{1.15}$$

with K being the carrying capacity of the environment.

- **The equilibrium points**

  The behavior of the system is qualitatively studied through its equilibria and their stability. The model at equilibrium is written as follows:

  $$\begin{cases} \dot{x} = ax\left(1 - \dfrac{x}{K}\right) - \gamma xy = 0, \\ \\ \dot{y} = \zeta xy - \beta y = 0. \end{cases} \tag{1.16}$$

  The second equation of the algebraic system (2.1) is written as:

  $$\dot{y} = y(\zeta x - \beta) = 0,$$

  so, we have two cases :

  1. $y = 0$ then the first equation gives:

     $$ax\left(1 - \frac{x}{K}\right) = 0,$$

     Case (1) : $x = 0$, $(0, 0)$ is the first equilibrium point.

     Case (2) : $x = K$, $(K, 0)$ is the second equilibrium point.

  2. $x = \dfrac{\beta}{\zeta}$ then the first equation gives:

     $$y = \frac{a}{\gamma}\left(1 - \frac{\beta}{\zeta K}\right),$$

     $\left(\dfrac{\beta}{\zeta}, \dfrac{a}{\gamma}\left(1 - \dfrac{\beta}{\zeta K}\right)\right)$ is the third equilibrium point, and exist if $\zeta K > 0$.

- **The stability:** Let's calculate the Jacobian matrix to determine the local stability properties of the equilibrium.

$$J(x,y) = \begin{pmatrix} a\left(1-\dfrac{2x}{K}\right)-\gamma y & -\gamma x \\ \\ \zeta y & \zeta x-\beta \end{pmatrix},$$

1. **Stability for The Point (0,0):**

$$J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix},$$

which has eigenvalues: $\lambda_1 = \alpha$ and $\lambda_2 = \beta$, these eigenvalues have opposite signs, which means that (0 , 0) is a **saddle point**, implying that it is **unstable**.

2. **Stability for The Point** $(K,0)$:

$$J(K,0) = \begin{pmatrix} -a & -\gamma K \\ 0 & \zeta K-\beta \end{pmatrix},$$

and it has eigenvalues: $\lambda_1 = -a$ and $\lambda_2 = \zeta K - \beta$. So It is clear that $\lambda_1 = -a < 0$.

- If $K < \dfrac{\beta}{\zeta} \Longrightarrow \lambda_2 < 0$, both eigenvalues are negative, and the equilibrium $(K,0)$ is a **stable node**.
- If $K > \dfrac{\beta}{\zeta} \Longrightarrow \lambda_2 > 0$, the two eigenvalues have opposite signs, and the equilibrium $(K,0)$ is an **unstable saddle point**.

3. **Stability for the point** $\left(\dfrac{\beta}{\zeta}, \dfrac{a}{\gamma}\left(1-\dfrac{\beta}{\zeta K}\right)\right)$:

$$J\left(\frac{\beta}{\zeta}, \frac{a}{\gamma}\left(1-\frac{\beta}{\zeta K}\right)\right) = \begin{pmatrix} -\dfrac{a\beta}{\zeta K} & -\dfrac{\gamma\beta}{\zeta} \\ \dfrac{\zeta a}{\gamma}\left(1-\dfrac{\beta}{\zeta K}\right) & 0 \end{pmatrix},$$

and it has eigenvalues:

$$\lambda_1 = \frac{\frac{-a\beta}{\zeta K} - \sqrt{\left(\frac{a\beta}{\zeta K}\right)^2 + 4a\beta\left(1 - \frac{\beta}{\zeta K}\right)}}{2},$$

$$\lambda_2 = \frac{\frac{-a\beta}{\zeta K} + \sqrt{\left(\frac{a\beta}{\zeta K}\right)^2 + 4a\beta\left(1 - \frac{\beta}{\zeta K}\right)}}{2},$$

it is clear that $\lambda_1 < 0$, so we study the second eigenvalue.

(a) If $\beta - \zeta K < 0$, then $\lambda_2 < 0$, which implies that $\left(\frac{\beta}{\zeta}, \frac{a}{\gamma}\left(1 - \frac{\beta}{\zeta K}\right)\right)$ is a **stable node**.

(b) If $\beta - \zeta K > 0$, then $\lambda_2 > 0$, which implies that $\left(\frac{\beta}{\zeta}, \frac{a}{\gamma}\left(1 - \frac{\beta}{\zeta K}\right)\right)$ is an **unstable saddle point**.

(c) If $\beta - \zeta K < 0$, and if more $\left(\frac{a\beta}{\zeta K}\right)^2 + 4a\beta\left(1 - \frac{\beta}{\zeta K}\right)$. So, $\Re(\lambda_2) < 0$ which implies that $\left(\frac{\beta}{\zeta}, \frac{a}{\gamma}\left(1 - \frac{\beta}{\zeta K}\right)\right)$ is an **stable focus**. Finally, the equilibrium point $\left(\frac{\beta}{\zeta}, \frac{a}{\gamma}\left(1 - \frac{\beta}{\zeta K}\right)\right)$ is **stable** if :

$$\beta - \zeta K < 0.$$

1.5 The long-term behavior of the system

In this subsection, we analyze the persistence (weak and strong) and permanence behavior of a system in both deterministic and stochastic cases.

1.5.1 Deterministic case

Definition 1.8. *(Persistent system) [5] A system is said to be weakly persistent if every solution $(x(t), y(t))$ satisfies two conditions:*

- $x(t) \geqslant 0$, $y(t) \geqslant 0$, $\forall t \geqslant 0$.
- $\limsup_{t \longrightarrow +\infty} x(t) > 0$, $\limsup_{t \longrightarrow +\infty} y(t) > 0$.

A system is said to be strongly persistent if every solution $(x(t), y(t))$ satisfies the following condition along with the first condition of the weak persistence:

- $\liminf_{t\longrightarrow+\infty} x(t) > 0$, $\liminf_{t\longrightarrow+\infty} y(t) > 0$.

Definition 1.9. *(Permanence and non-permanence) [5] A system is said to be permanent if there exist positive constants $0 < m \leqslant M$ such that,*

$$\min\left\{\liminf_{t\longrightarrow+\infty} x(t)\ ,\ \liminf_{t\longrightarrow+\infty} y(t)\right\} \geqslant m\ ,\ \max\left\{\limsup_{t\longrightarrow+\infty} x(t)\ ,\ \limsup_{t\longrightarrow+\infty} y(t)\right\} \leqslant M,$$

for all solutions $(x(t), y(t))$ of the system with positive initial values.

A system is said to be non-permanent if there is a positive solution $(x(t), y(t))$ such that,

$$\min\left\{\liminf_{t\longrightarrow+\infty} x(t)\ ,\ \liminf_{t\longrightarrow+\infty} y(t)\right\} = 0.$$

1.5.2 Stochastic case

Definition 1.10. *(Persistence in mean)[22] The system is said to be persistent in mean, if:*

$$\liminf_{t\longrightarrow+\infty} \frac{1}{t}\int_0^t x(s)ds > 0\ \ a.s,\ \ \liminf_{t\longrightarrow+\infty} \frac{1}{t}\int_0^t y(s)ds > 0\ \ a.s.$$

Definition 1.11. *(Extinction)[22] The prey population $x(t)$ goes extinct if:*

$$\lim_{t\to\infty} x(t) = 0. \quad a.s.$$

The predator population $y(t)$ goes extinct if:

$$\lim_{t\to\infty} y(t) = 0. \quad a.s.$$

1.6 Existence and uniqueness theorem for stochastic differential equations

Considering the following stochastic differential equation (SDE)

$$dx(t) = f(t, x(t))\, dt + g(t, x(t))\, dB(t), \tag{1.17}$$

- B denoting the Brownian motion process.
- $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is the **drift** coefficient.
- $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is the **diffusion** coefficient.

Theorem 1.2. *If the coefficient functions* f, g *of the SDE* (1.17) *satisfy the conditions*

1. ***Lipschitz condition***

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \leq k|x-y|, \tag{1.18}$$

2. ***Linear growth condition***

$$|f(t,x)|^2 + |g(t,x)|^2 \leq k^2(1+|x|^2), \tag{1.19}$$

for some constant k *and all* $t \in [t_0, T]$, $T > 0$, *then there exists a continuously adapted solution* $(x(t))_{t\in[t_0,T]}$ *of the SDE* (1.17) *such that*
$x(t_0) = x_0$.

1.7 The comparison theorem of stochastic equations

Theorem 1.3. *[17] Let the differential equations be:*

$$dx_j(t) = a_j(t, x(t))dt + \sum_{k=1}^{r} \sigma_{jk}(t, x_j(t))dB_k(t),$$
$$dy_j(t) = b_j(t, y(t))dt + \sum_{k=1}^{r} \sigma_{jk}(t, y_j(t))dB_k(t),$$

where $j = 1, \dots, d$. *Suppose that:*

- *A.1* $x_0 \leq y_0$.
- *A.2 The functions* a_j, b_j *and* σ_{jk} *aare continuous functions in both variables, and are Lipschitz continuous in the second variable uniformly with respect to the first variable. For all* $j \in \{1, \dots, d\}$ *and* $k \in \{1, \dots, r\}$.
- *A.3 For all* $t \geq 0$, $a_j(t,x) \leq b_j(t,y)$ *if* $x_i \leq y_i$ *and* $x_j = y_j$.

Then, we have

$$\mathbb{P}\{x(t) \leq y(t) : t \geq 0\} = 1.$$

1.8 The diffusion processes

Theorem 1.4. *A diffusion is a stochastic process* $(x(t))_{t\geq 0}$ *that satisfies a stochastic differential equation of the following form:*

$$dx(t) = b(t, x)dt + \sigma(t, x)dB(t).$$

Chapter 2

The effect of water level and harvesting in a prey predator interactions

In this chapter, we will examine the impact of water level on local community dynamics, focusing on how seasonal fluctuations in water level affect prey-predator interactions in Parloup Lake in the south of France.
The Pareloup lake, also known as Lac de Salles-Curan, is the largest lake in the Midi-Pyrenees region and the 5th largest lake in France (1,200 hectares) [27]. The Pareloup lake is a reservoir where water is taken partly from Vioulou, a river located at 30 km in the south- east of Rodez [16]. The water of the Pareloup lake is used by EDF, the French national electricity company, to generate electricity [27]. The management of this lake is of considerable ecological importance. Significant variations of the water level of the lake can have a strong impact on the persistence of some species [16]. In order to explain the influence of changing water level fluctuations in this artificial lake on fish predator prey dynamics. Two interdependent species are considered: the pike (brochet in French) which is the most important predator and the roach (gardon in French) which is the prey [5].

Introduction and mathematical model

2.1 Deterministic model

We investigate a mathematical model that describes the interaction between roach and pike, focusing on a predator-prey system (1.10). The function f is of logistic type given by (1.7). Many questions in prey-predator theory revolve around the expression that is used for the functional response $F(x,y)$. The well known models of Lotka–Volterra and Holling do not include predator density [16].

The assumption of environmental periodicity plays an important role. Many researchers have explored cases where the parameters exhibit periodic behavior (see [7, 11]).

In [11], the function $F(x,y)$ is expressed as follows:

$$F(x,y) = F(x) = \frac{\alpha x}{x+D},$$

where α and D are periodic functions. At high prey densities, food is more abundant, and predators spend more time foraging to maximize their intake. Additionally, the interference between predators increases at higher predator densities. In [33] under appropriate conditions, the functional response $F(x,y)$ accounts for predator interference and is given by:

$$F(x,y) = \frac{\alpha x}{y+D}.$$

Predator prey dynamics are usually represented by a functional response, which is the amount of prey eaten per predator and per unit of time [5]. This response functional is based on the idea that the predator's access to food depends on the water level. In autumn, when the water level is low, the predator is more in contact with the prey, leading to increased predation. Conversely, in spring, when the water level is high, it becomes harder for the predator to find prey, resulting in reduced predation. It is assumed that the accessibility function b for the prey is continuous and 1-periodic, the minimum value b_1 is reached in spring and the maximum value b_2 is attained during autumn. The predator needs a quantity γ as food, but it has access to a quantity

$$F(t,x,y) = \frac{b(t)x}{y+D},$$

which depends on the water level, where D measures other causes of mortality outside of predation. Thus, if

$$F(t,x,y) \geqslant \gamma,$$

then the predator will be satisfied with the quantity γ for his food. Otherwise, if

$$F(t,x,y) < \gamma,$$

the predator will content itself with $F(t,x,y)$. To summarize, the quantity of food received per predator is

$$\min\left(\frac{b(t)x}{y+D},\gamma\right).$$

The authors in [16] studied the following non-autonomous prey–predator model

$$\begin{cases} \dfrac{dx(t)}{dt} = a\,x(t)\left(1-\dfrac{x(t)}{K}\right) - \min\left(\dfrac{b(t)x(t)}{y(t)+D},\gamma\right)y(t), \\ \dfrac{dy(t)}{dt} = -q\,y(t) + e\min\left(\dfrac{b(t)x(t)}{y(t)+D},\gamma\right)y(t). \end{cases} \tag{2.1}$$

All previously mentioned constants are positive. The prey grow logistically with carrying capacity K and intrinsic growth rate a, The quantity e is the trophic efficiency and predators are assumed to die with a constant death rate q. we focus on the autonomous case and use as predation rate, the mean function $\bar{b} = \int_0^1 b(t)dt$. Moreover, to explore the impact of harvesting on the prey–predator ecosystem, we build upon and extend the work presented in [16]. The change in stock size can be generalized to

$$\begin{cases} \dfrac{dx(t)}{dt} = a\,x(t)\left(1-\dfrac{x(t)}{K}\right) - \min\left(\dfrac{\bar{b}x(t)}{y(t)+D},\gamma\right)y(t) - h, \\ \dfrac{dy(t)}{dt} = -q\,y(t) + e\ \min\left(\dfrac{\bar{b}x(t)}{y(t)+D},\gamma\right)y(t). \end{cases} \tag{2.2}$$

where h is the harvest. Harvest is often assumed to be linear in effort, u, and stock x. $h = m\,ux$, where m is the catchability coefficient of the prey species and u denotes the effort devoted to the harvesting. This results in a catch per unit effort, which is proportional to the level of biomass for all levels of u and x. This hypothesis relies on several conditions including non saturation of fishing gear, non congestion of fishing vessels, i.e. basically no decreasing returns to scale, and uniform distribution of fish [13]. Throughout this chapter, we consider u to be constant. Therefore, we study the following prey–predator model:

$$\begin{cases} \dfrac{dx(t)}{dt} = a\,x(t)\left(1-\dfrac{x(t)}{K}\right) - \min\left(\dfrac{\bar{b}x(t)}{y(t)+D},\gamma\right)y(t) - m\,u\,x(t), \\ \dfrac{dy(t)}{dt} = -q\,y(t) + e\ \min\left(\dfrac{\bar{b}x(t)}{y(t)+D},\gamma\right)y(t). \end{cases} \tag{2.3}$$

Qualitative analysis of the system and the main results

From a biological perspective, we are only concerned with the dynamics of system (2.3) in the closed first octant $\mathbb{R}^2_+$, where both state variables are nonnegative. Thus, we consider the biologically meaningful initial condition $x_0 \geq 0$ and $y_0 \geq 0$.
If $u > \dfrac{a}{m}$ then $dx < 0$, hence, throughout this analysis, we make the following assumption on the fishing effort:

$$u < \frac{a}{m} \tag{H_0}$$

Biologically, assumption (H_0) means that if the fishing effort increases beyond a threshold value (that is if $u > \dfrac{a}{m}$), then the two species vanish eventually [5].
We also assume throughout this work that the predation rate $\bar{b}$ satisfies:

$$\bar{b} < \min\left\{\frac{\gamma(y_0+D)}{x_0}, \frac{4\,a\,\gamma\,D\,q}{K(a+q-m\,u)^2}\right\} := \tilde{b} \tag{H_1}$$

Assumption (H_1) means that if the predation rate is less than a threshold value $\tilde{b}$, all the species are present and none of them will go to extinction [5].

2.1.1 Existence, uniqueness and boundedness of the positive solution

We begin by showing that the solutions of (2.3), initially within $\mathbb{R}^2_+$, exist, remain in this domain, and are uniformly bounded. To do so, we first present the following lemme:

Lemma 2.1. *[5] Let $h : (x, y) \to \min(g(x, y), \gamma)$. If g is locally Lipschitz, then it is also goes for h.*

Consequently, local existence and uniqueness of solutions of system (2.3) are obtained for the corresponding Cauchy problem.[5]
We now present the following lemma regarding the positivity and boundedness of the solution to system (2.3).

Lemma 2.2. *[5]*

1. *The positive cone $\mathbb{R}^2_+$ is positively invariant for* (2.3)*.*
2. *All the solutions of system* (2.3) *which initiate in $\mathbb{R}^2_+$ are bounded, with ultimate bound.*

2.1.2 Persistence, permanence and extinction

Geometrically, persistence means that trajectories that initiate in a positive cone are eventually bounded away from coordinate planes. On the other hand, permanently coexistence (uniform persistence) implies the existence of a region in the phase space at a non-zero distance from boundary in which population vectors must lie ultimately. The last ensures the survival of species in biological sense [5].

First we denote:

$$c_1 \overset{\text{def}}{=} \frac{K}{a}(a - m\,u - \bar{b}),$$

$$c_2 \overset{\text{def}}{=} \frac{e\,\bar{b}\,c_1}{q} - D,$$

$$C_1 \overset{\text{def}}{=} K,$$

$$C_2 \overset{\text{def}}{=} \frac{e\,\bar{b}\,C_1}{q} - D,$$

then $C_i > c_i$, $i = 1, 2$. We will show that $\max\{c_i, 0\}$ $(i = 1, 2)$ are the lower bounds for the limiting bounds of species, as time t goes to infinity. This is obvious when $c_i \leq 0$. Therefore, it is assumed that:

$$c_i > 0, \quad i = 1, 2. \tag{H_2}$$

Proposition 2.1. *[5] In addition to* (H_0), (H_1), *assume further that* (H_2) *holds. Then system* (2.3) *is permanent, i.e., any positive solution* $(x(t), y(t))_{t\geq 0}$ *of system* (2.3) *satisfies*

$$0 < c_1 \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq C_1,$$

$$0 < c_2 \leq \liminf_{t\to\infty} y(t) \leq \limsup_{t\to\infty} y(t) \leq C_2.$$

For the proof of this proposition, the authors in [5] were unable to demonstrate strong persistence. Therefore, they considered an equivalent system, given by the following proposition.

Proposition 2.2. *[5] Under hypothesis* (H_1), *we have for all* $t \geq 0$,

$$\bar{b}x(t) < \gamma(y(t) + D).$$

Consequently, under hypothesis (H_1), system (2.3) is reduced to the simple form

$$\begin{cases} \dfrac{dx(t)}{dt} = a\,x(t)\left(1 - \dfrac{x(t)}{K}\right) - \dfrac{\bar{b}\,x(t)\,y(t)}{y(t) + D} - m\,u\,x(t), \\ \dfrac{dy(t)}{dt} = -q\,y(t) + \dfrac{e\,\bar{b}\,x(t)\,y(t)}{y(t) + D}. \end{cases} \tag{2.4}$$

In the next proposition gives sufficient conditions under which the given system is not persistent.

Proposition 2.3. *[5] If* $C_2 < 0$, *then* $\lim_{t\to\infty} y(t) = 0$, *that is, the predator goes to extinction.*

Remark 2. *From a biological perspective, it means that, when the predation rate is enough small, the predator disappears.*

2.1.3 Steady states and their existence

System (2.4) possesses the following three equilibrium points[5]:

(i) The trivial equilibrium $P^0 = (0, 0)$.

(ii) The predator free equilibrium $P^1 = (\bar{x}, 0)$, where $\bar{x} = \dfrac{K}{a}(a - mu)$.

(iii) The steady state of coexistence (interior equilibrium point) $P^* = (x^*, y^*)$.

The last is the point of intersection of the prey zero growth rate isocline (i.e., $\dfrac{dx}{dt} = 0$) and the predator zero growth rate isocline (i.e., $\dfrac{dy}{dt} = 0$) given by

$$\begin{cases} a\left(1 - \dfrac{x^*}{K}\right) - \dfrac{\bar{b}\, y^*}{y^* + D} - m\, u = 0, \\ -q + \dfrac{e\, \bar{b}\, x^*}{y^* + D} = 0, \end{cases} \tag{2.5}$$

where $y^* = \dfrac{1}{2}\left(-B + \sqrt{B^2 - 4C}\right)$, $x^* = \dfrac{q}{e\,\bar{b}}(y^* + D)$,
and

$$B = 2D - \frac{e\,\bar{b}\,K(a - m\,u - \bar{b})}{a\,q} = D - c_2, \quad C = D^2 - \frac{e\,\bar{b}\,K\,D(a - m\,u)}{a\,q}.$$

Observe that $\sqrt{B^2 - 4C}$ is always positive, then, the interior equilibrium is positive if one of the two cases holds:

1. $B < 0$, ($c_2 > D$) which is equivalent to $0 < u < \dfrac{a}{m}\left(1 - \dfrac{2\,q\,D}{e\,\bar{b}\,K}\right) - \dfrac{b}{m}$,
2. $B \geq 0$ and $C < 0$, that is $0 < \dfrac{a}{m}\left(1 - \dfrac{2\,q\,D}{e\,\bar{b}\,K}\right) - \dfrac{\bar{b}}{m} \leq u < \dfrac{a}{m}\left(1 - \dfrac{q\,D}{e\,\bar{b}\,K}\right)$.

Hence, the interior equilibrium is positive if and only if condition

$$0 < u < \frac{a}{m}\left(1 - \frac{q\,D}{e\,\bar{b}\,K}\right), \tag{H_3}$$

holds.

2.1.4 Dynamical behavior: stability analysis

To assess the stability of an equilibrium state, one must analyze the eigenvalues of the Jacobian matrix linearized around that point.

Proposition 2.4. *[5]*

1. *The equilibrium point P^0 is always a saddle point.*
2. *The point P^1 is stable iff*
$$u > \frac{a}{m}\left(1 - \frac{qD}{e\,\bar{b}K}\right). \tag{H_3^c}$$
3. *The Steady state P^* is locally asymptotically stable when it exists.*

Proof. To obtain the local stability results, we use the Jacobian matrix associated to system (2.4)

$$J(x,y) = \begin{pmatrix} a - mu - \dfrac{2\,a\,x}{K} - \dfrac{\bar{b}\,y}{y+D} & -\dfrac{\bar{b}\,D\,x}{(y+D)^2} \\ \dfrac{e\,\bar{b}\,y}{y+D} & -q + \dfrac{e\,\bar{b}\,D\,x}{(y+D)^2} \end{pmatrix}.$$

- The Jacobian matrix of the equilibrium $P_0 = (0,0)$ is

$$J(0,0) = \begin{pmatrix} a - m\,u & 0 \\ 0 & -q \end{pmatrix}.$$

Hence, P_0 is a saddle point.

- The Jacobian matrix of the equilibrium $P_1 = (\bar{x}, 0)$ is

$$J(\bar{x},0) = \begin{pmatrix} -(a - m\,u) & -\dfrac{\bar{b}K(a - m\,u)}{Da} \\ 0 & -q + \dfrac{e\,\bar{b}K(a - m\,u)}{D\,a} \end{pmatrix}.$$

If (H_3^c) holds, then P_1 is stable, and there is no interior equilibrium, otherwise, if

$$u < \frac{a}{m}\left(1 - \frac{q\,D}{e\,\bar{b}\,K}\right),$$

then P_1 is unstable.

- The Jacobian matrix for P^* is

$$J(x^*,y^*) = \begin{pmatrix} -\dfrac{a\,x^*}{K} & -\dfrac{\bar{b}\,D\,x^*}{(y^*+D)^2} \\ \dfrac{e\,\bar{b}\,y^*}{y^*+D} & -\dfrac{e\,\bar{b}\,x^* y^*}{(y^*+D)^2} \end{pmatrix}.$$

The trace of $J(x^*,y^*)$ can be directly observed as

$$trJ(x^*,y^*) = -\frac{a\,x^*}{K} - \frac{e\,\bar{b}\,x^*y^*}{(y^*+D)^2} < 0,$$

and its determinant is

$$detJ(x^*,y^*) = \frac{a\,e\,\bar{b}\,x^{*2}y^*}{K(y^*+D)^2} + \frac{e\,\bar{b}^2D\,x^*y^*}{(y^*+D)^3} > 0.$$

Consequently, the equilibrium point (x^*, y^*) is locally asymptotically stable under the condition of its existence ☐

Theorem 2.1. *[5] If conditions* $(H_0) - (H_3)$ *hold, then, the co-existing equilibrium point* P^* *is globally asymptotically stable.*

Remark 3. *Global stability implies that exploitation will not irreversibly change the system. As long as the prey are not made extinct by excessive exploitation of their food resources, the system is able to recover.*

2.1.5 Bionomic equilibrium

Bionomic equilibrium is a concept that integrates the ideas of biological and economic equilibrium. When the total income received from selling gathered biomass (the total revenue obtained by selling the harvested biomass) equals the total cost of harvesting effort, the economic equilibrium is considered to have been established [30].

Let c be a constant fishing cost per unit effort and p the constant price per unit biomass of prey fish. As a result, at any time t, the net income or financial rent (net revenue) is,

$$\pi = (pmx - c)u$$

The bionomic equilibrium $P_\infty(x_\infty, y_\infty, u_\infty)$ is calculated using the following equations:

$$\begin{aligned} a\left(1 - \frac{x}{K}\right) - \frac{\bar{b}y}{y+D} - m\,u &= 0, \\ -q + \frac{e\,\bar{b}\,x}{y+D} &= 0, \\ \pi = (p\,m\,x - c)u &= 0. \end{aligned}$$

Solving the above equations, we get,

$$\begin{aligned} x_\infty &= \frac{c}{p\,m}, \\ y_\infty &= \frac{e\,\bar{b}\,c}{p\,m\,q} - D, \\ u_\infty &= \frac{1}{m}\left[a\left(1-\frac{c}{p\,m\,K}\right) - \frac{\bar{b}y_\infty}{y_{\infty+D}}\right]. \end{aligned}$$

• If $u_\infty < u$ the total cost of harvesting the prey population would exceed the total revenue generated from the fishery. As a result, some fishermen would incur losses and exit the fishery.
• If $u_\infty > u$ the fishery becomes more profitable, attracting additional fishermen in an open-access context. This leads to a continuous increase in harvesting effort.
In both cases, the effort level cannot be sustained in the long term.

2.1.6 Optimal harvesting policy

The core challenge in establishing an optimal harvest policy for a commercial fishery context is the management of renewable resources, which can be viewed as a dynamic allocation problem. How much of a resource should be harvested today and how much should be left for tomorrow? A well defined resource management problem needs a clear objective. There are many potential objectives. A reasonably general approach is to define $\pi_t = \pi(x(t), h(t))$ to be the net benefits at instant or period t from having a resource stock of size $x(t)$ and harvest at rate $h(t)$ [10].
In order to investigate an optimal harvesting policy, the authors in [5] consider the present value J given by

$$J(u) = \int_0^\infty e^{-\delta t}\pi(x(t), y(t), u)dt$$

where π is given by $\pi = (pmx - c)u$ and δ is the instantaneous annual rate of discount. Time discounting is the primary technique for resolving concerns of intertemporal economic gains. Although much debate exists regarding the concept's societal justification, time discounting is a common technique in company management to calculate the ideal harvesting strategy [30].
Let u_δ be an optimal control associated with the corresponding state x_δ and y_δ. They define $A_\delta = (x_\delta, y_\delta)$ as the optimal equilibrium point. The aim is to determine the optimal control $J(u_\delta)$ such that:

$$J(u_\delta) = \sup\left\{J(u), u \in V\right\}$$

where $V = [0, u_{max}]$ is the control set, and u_{max} represents the feasible upper bound for the harvesting effort.Lastly, the optimal harvesting policy is obtained by using the Pontryagin's maximum principle.

2.2 Stochastic model

In reality, population dynamics is inevitably affected by environmental white noise which is an important component in an ecosystem. But the deterministic models assume that parameters in the model system are all deterministic irrespective environmental fluctuations. Hence they have some limitations in mathematical modelling of ecological systems, besides they are quite difficult to fitting data perfectly and to predict the future dynamics of the system [22], and hence stochastic differential equation models play a significant role in various branches of applied sciences including biology and population dynamics, as they provide some additional degree of realism compared to their deterministic counterpart. In reality, due to continuous fluctuations in the environment (e.g. variation in intensity of sunlight, temperature, water level, etc.), parameters involved in models are not absolute constants, but they always fluctuate around some average value. As a result, the population density never attains a fixed value with the advancement of time but rather exhibits continuous oscillation around some average values. Based upon these factors, stochastic population models have received more and more attention [21].

Throughout this chapter, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let B^1 and B^2 denote two correlated standard Brownian motions defined on this probability space. In [21], environmental fluctuations are assumed to primarily affect the intrinsic growth rate of the prey population and the death rate of the predator population. In the present work, we incorporate stochastic perturbations into the first equation of the deterministic system (2.4), such that the intensity of the noise increases or decreases with the size of the prey population. Additionally, the predator's death rate is perturbed according to $q \longrightarrow q + dB^2(t)$, where $\alpha x(t)$ and $-\beta y(t)$ represent the diffusion coefficients. For future work, we plan to introduce a stochastic perturbation in the intrinsic growth rate of the prey population. Consequently, the autonomous stochastic system corresponding to (2.4) takes the following form:

$$\begin{cases} dx(t) = \left[a\,x(t)\left(1 - \dfrac{x(t)}{K}\right) - \min\left(\dfrac{\bar{b}\,x(t)}{y(t)+D}, \gamma\right) y(t)\right] dt + \alpha\,x(t)\,dB^1(t), \\ dy(t) = \left[-q\,y(t) + e\,\min\left(\dfrac{\bar{b}\,x(t)}{y(t)+D}, \gamma\right) y(t)\right] dt - \beta\,y(t)\,dB^2(t). \end{cases} \tag{2.6}$$

We extend the deterministic controlled system (2.3) by introducing stochastic dynamics to account for environmental variability and uncertainty. The stochastic controlled system is constructed as a natural counterpart to the deterministic model, preserving its core structure while incorporating random fluctuations. This formulation allows for a more realistic and robust analysis of the fishery dynamics under uncertainty.

$$\begin{cases} dx(t) = \left[a\, x(t)\left(1 - \dfrac{x(t)}{K}\right) - \min\left(\dfrac{\bar{b}\, x(t)}{y(t)+D}, \gamma\right) y(t) - m\, u\, x(t)\right] dt + \alpha\, x(t)\, dB^1(t), \\ dy(t) = \left[-q\, y(t) + e\, \min\left(\dfrac{\bar{b}\, x(t)}{y(t)+D}, \gamma\right) y(t)\right] dt - \beta\, y(t)\, dB^2(t). \end{cases} \tag{2.7}$$

In this section, we show some properties of the stochastic differential equation including: the global existence, uniqueness and boundedness of positive solution, which is one of the most important topics in population system because it guarantees the system's validity.

Since stochastic system (2.7) describes population dynamics, it is necessary for the solution to be positive and not to explode to infinity in a finite time. Moreover, in order for a stochastic differential equation to have a unique global (i.e. no explosion in a finite time) solution for any given initial value $x(0) = x_0 > 0$, $y(0) = y_0 > 0$.

2.2.1 Existence and uniqueness of the positive global solution

In order for a stochastic differential equation to have a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of equation are generally required to satisfy the linear growth condition and local Lipschitz condition. In this section, by the Lyapunov analysis method, we will show there is a unique positive global solution with positive initial value of system (2.7), then a stochastic ultimate boundedness is studied.

Theorem 2.2. *For any initial value $(x_0, y_0) \in \mathbb{R}_+^2$ there exists a unique solution $(x(t), y(t))$ of system on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^2$ a.s.*

Proof. Since the coefficients of (2.7) are locally Lipschitz and satisfy the linear growth condition uniqueness of the solution until explosion time is guaranteed for any initial condition i.e. (there is a unique solution $(x(t), y(t))$ on $t \in [0, \tau_e)$).

Let us now prove global existence of the solution by showing that $\tau_e = \infty$. Let $n_0 > 0$ be sufficiently large for $(x_0, y_0) \in D_{n_0} = \left[\dfrac{1}{n_0}, n_0\right] \times \left[\dfrac{1}{n_0}, n_0\right]$ for each integer $n > n_0$, we define the following stopping time:

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{x(t), y(t)\} \leq \frac{1}{n} \text{ or } \max\{x(t), y(t)\} \geq n\right\},$$

We let $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set). From the definition of stopping time, it is easy to know that τ_n is increasing as $n \to \infty$. Set $\tau_\infty = \lim\limits_{n\to\infty} \tau_n$, then $\tau_\infty \leq \tau_e$ almost surely.
Now we want to show that $\tau_\infty = \infty$. We assume that this is false. If the statement is false, then there exists a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that:

$$\mathbb{P}(\tau_\infty \leq T) > \epsilon. \tag{2.8}$$

Consequently, by denoting $\Omega_n = \{\tau_n \leq T\}$, then is an integer such that, for all $n_1 \geq n_0$,

$$\mathbb{P}(\Omega_n) \geq \epsilon, \ \forall n \geq n_1. \tag{2.9}$$

Define a C^2 - function $V : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ by: $V(x, y) = (x - 1 - lnx) + (y - 1 - lny)$ which is non-negative.
Applying Itô's formula to our model (2.7), we obtain:

$$dV(x(t), y(t)) = \mathcal{L}V(x(t), y(t))dt + \alpha(x(t) - 1)dB^1(t) - \beta(y(t) - 1)dB^2(t),$$

where

$$\begin{aligned}
\mathcal{L}V(x(t, y(t))) \quad = \quad & \left(1 - \frac{1}{x(t)}\right)\left(a\,x(t)\left(1 - \frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}\,x(t)}{y(t) + D}, \gamma\right)y(t) - m\,u\,x(t)\right) + \frac{\alpha^2}{2} \\
& + \left(1 - \frac{1}{y(t)}\right)\left(- q\,y(t) + e\,y(t)\min\left(\frac{\bar{b}\,x(t)}{y(t) + D}, \gamma\right)\right) + \frac{\beta^2}{2}, \\
\\
= \quad & (x(t) - 1)\left(a\left(1 - \frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}\,x(t)}{y(t) + D}, \gamma\right)\frac{y(t)}{x(t)} - m\,u\right) \\
& + (y(t) - 1)\left(- q + e\,\min\left(\frac{\bar{b}\,x(t)}{y(t) + D}, \gamma\right)\right) + \frac{\alpha^2 + \beta^2}{2}.
\end{aligned} \tag{2.10}$$

It is easy to show that $\mathcal{L}V(x, y)$ is bounded above, say by N (that is to say $|\mathcal{L}V(x, y)| \leq N$). So we have

$$dV(x(t), y(t)) \leq Ndt + \alpha(x(t) - 1)dB^1(t) - \beta(y(t) - 1)dB^2(t). \tag{2.11}$$

Integrating both sides of (2.11) from 0 to $\tau_n \wedge T = \min\{\tau_n, T\}$ and then taking the expectation, one gets:

$$\mathbb{E}V(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq V(x_0, y_0) + N\mathbb{E}((\tau_n \wedge T)),$$

therefore

$$\mathbb{E}V(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq V(x_0, y_0) + NT. \tag{2.12}$$

From (2.9), we obtain $\mathbb{P}(\Omega_n) > \epsilon$. For each $w \in \Omega_n$, $x(\tau_n)$ or $y(\tau_n)$ equals either n or $\frac{1}{n}$. So $V(x(\tau_n), y(\tau_n))$ is not less than either

$$n - 1 - \ln n \text{ or } \frac{1}{n} - 1 - \ln\frac{1}{n} = \frac{1}{n} - 1 + \ln n,$$

thus,

$$V(x(\tau_n), y(\tau_n)) \geq (n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right),$$

holds. According (2.12), we have

$$V(x_0, y_0) + NT \geq \mathbb{E}\Big[I_{\Omega_n} V(x(\tau_n), y(\tau_n))\Big] \geq \epsilon\,(n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right),$$

in which I_{Ω_n} denotes the indicator function of Ω_n. Letting $n \to \infty$, then we obtain

$$\infty > V(x_0, y_0) + NT = \infty,$$

which leads to a contradiction, so we must have $\tau_\infty = \infty$ so we have that $\tau_e = \infty$ almost surely. □

2.2.2 Stochastically ultimate boundednes

Theorem (2.2) shows that the solution of System (2.7) remains in the positive cone $\mathbb{R}^2_+$. However, this nonexplosion property in a population dynamical system is often not good enough. Therefore, the property of ultimate boundedness is more desired. First we recall the definition of stochastically ultimate boundedness.

Definition 2.1. *The solution $(w(t))_{t\geq 0} = (x(t), y(t))_{t\geq 0}$ of system (2.7) is said to be stochastically ultimately bounded, if for any $\epsilon \in (0, 1)$, there is a constant $H := H(\epsilon)$ such that for any initial data $(x_0, y_0) \in \mathbb{R}^2_+$,*

$$\limsup_{t\longrightarrow\infty} \mathbb{P}\{||w(t)||_2 > H\} < \epsilon.$$

Theorem 2.3. *The solutions of system* (2.7) *are stochastically ultimately bounded for any initial data* $(x_0, y_0) \in \mathbb{R}^2_+$.

Proof. By Theorem (2.2), the solution $(x(t), y(t))$ remains in $\mathbb{R}^2_+$ for all $t \geq 0$. Define the function $V_1(t, x) = e^t x^\theta$ for $\theta > 0$.
By Itô's formula, we have

$$\begin{aligned}
L_1(t, x(t), y(t)) &= e^t x^\theta(t)\left[1 + \left(a\left(1 - \frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}\,x(t)}{y(t)+D}, \gamma\right)\frac{y(t)}{x(t)} - m\,u\right)\theta + \frac{(\theta-1)\theta}{2}\alpha^2\right],\\
&\leq e^t\left[\left(1 + a\theta + \frac{(\theta-1)\theta}{2}\alpha^2\right)x^\theta(t) - \frac{a\,\theta}{K}x^{\theta+1}(t)\right],\\
&\leq M_1(\theta)e^t.
\end{aligned} \tag{2.13}$$

Integrating both sides of (2.13) from 0 to t and then taking expectations, we get

$$e^t\left(\mathbb{E}(x^\theta(t)) - \mathbb{E}(x_0^\theta)\right) \leq M_1(\theta)e^t.$$

Hence, we have:

$$\limsup_{t \longrightarrow \infty} \mathbb{E}\left(x^\theta(t)\right) \leq M_1(\theta) < +\infty.$$

Similarly, defining the function $V_2(t, y) = e^t y^\theta$ for $\theta > 0$ and applying the Itô's formula, we get

$$\begin{aligned}
L_2(t, x(t), y(t)) &= e^t y^\theta(t)\left[1 + \left(-q + e\,\min\left(\frac{\bar{b}\,x(t)}{y(t)+D}, \gamma\right)\right)\theta + \frac{(\theta-1)\theta}{2}\beta^2\right],\\
&\leq e^t y^\theta(t)\left[\left(1 + \gamma\,\theta - q\theta + \frac{(\theta-1)\theta}{2}\alpha^2\right)x^\theta(t),\right.\\
&\leq M_2(\theta)e^t.
\end{aligned} \tag{2.14}$$

Then, $e^t\mathbb{E}(y^\theta(t)) - \mathbb{E}(y_0^\theta) \leq M_2(\theta)e^t$. So we have $\limsup\limits_{t \longrightarrow \infty} \mathbb{E}\left(y^\theta(t)\right) \leq M_2(\theta) < +\infty$.
For $(w(t)) \in \mathbb{R}^2_+$, we obtain:

$$\begin{aligned}
||w(t)||^\theta &\leq \left(2\max\left\{x^2(t), y^2(t)\right\}\right)^{\frac{\theta}{2}},\\
&\leq 2^{\frac{\theta}{2}}\left(x^\theta(t), y^\theta(t)\right).
\end{aligned}$$

Consequently,

$$\limsup_{t \longrightarrow \infty} \mathbb{E}||w(t)||^\rho \leq M_3(\theta) < +\infty,$$

where $M_3(\theta) = 2^{\frac{\theta}{2}}\left(M_1(\theta) + M_2(\theta)\right)$. By Chebyshev's inequality, we get that all solutions are stochastically bounded. ☐

The original stochastic predator prey system given by (2.7) is quite complex and difficult to study directly. To make the analysis and simulations easier, we follow a similar strategy to the one used by the authors in the deterministic case in [5]. For this reason, we propose an equivalent system that keeps the main features of our stochastic model but is simpler to handle. This allows us to study the long-term behavior of the system and perform simulations more effectively.

Proposition 2.5. *If* $\bar{b} < \dfrac{\gamma(y_0 + D)}{x_0}$, *we have for all* $t \geq 0$, $\bar{b}x(t) < \gamma(y(t) + D)$ *a.s.*

Proof. Let $f(x(t), y(t)) = bx(t) - \gamma(y(t) + D)$ and $\tau^+ = \inf\{t > 0, f(t) > 0\}$.
Let $n_0 > 0$ be sufficiently large such that $f(0) \in \left[-n_0, -\dfrac{1}{n_0}\right]$. For each integer $n \geq n_0$, define the stopping time as

$$\tau_n = \inf\left\{t \in [0, \tau^+), f(t) \notin \left[-n, -\frac{1}{n}\right]\right\}.$$

Obviously, τ_n is increasing as $n \to \infty$. Set $\tau_\infty = \lim\limits_{n\to\infty} \tau_n$, hence $\tau_\infty \leq \tau^+$ almost surely.
Now, we want to show that $\tau_\infty = \infty$. We assume that this is false. If the statement is false, then there are a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}(\tau_\infty \leq T) > \varepsilon. \tag{2.15}$$

Thus, there is an integer $n_1 \geq n_0$ such that

$$\mathbb{P}\{\tau_n \leq T\} \geq \varepsilon, \ \forall n \geq n_1. \tag{2.16}$$

Define a function $\widetilde{V} : \mathbb{R}^2_- \longrightarrow \mathbb{R}$ by:
$\widetilde{V}(x(t), y(t)) = (\bar{b}x(t) - \gamma(y(t) + D)) - 1 - \ln(-\bar{b}x(t) + \gamma(y(t) + D))$, for
$f(x(t), y(t)) = \bar{b}x(t) - \gamma(y(t) + D)$, the form of $\widetilde{V}$ will be $\widetilde{V}(z) = (z - 1 - ln(-z))$ which is non-negative.

$$
\begin{aligned}
d\widetilde{V}(x(t),y(t)) &= \sigma_t\widetilde{V} + \sigma_x\widetilde{V}dx(t) + \sigma_y\widetilde{V}dy(t) + \frac{1}{2}\sigma_{xx}\widetilde{V} < x(t),x(t) > \\
&+ \sigma_{xy}\widetilde{V} < x(t),y(t) > + \frac{1}{2}\sigma_{yy} < y(t),y(t) > \\
&= \left(\bar{b} + \frac{\bar{b}}{\gamma(y(t)+D)-\bar{b}x(t)}\right)dx + \left(-\gamma - \frac{\gamma}{\gamma(y(t)+D)-\bar{b}x(t)}\right)dy \\
&+ \left(\frac{\bar{b}}{2} + \frac{\bar{b}}{(\gamma(y(t)+D)-\bar{b}x(t))^2}\right)\alpha^2x^2(t)dt + \left(-\frac{\gamma}{2}\left(\frac{-\gamma}{\gamma(y(t)+D)-\bar{b}x(t))^2}\right)\right)\beta^2y^2(t)dt \\
&+ \bar{b}\left(\frac{-\gamma}{\gamma(y(t)+D)-\bar{b}x(t))^2}\right)\alpha\beta x(t)y(t)dt \\
&= \left(\bar{b} - \frac{\bar{b}}{f(t)}\right)\left[a\left(1-\frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}x(t)}{y(t)+\gamma},\gamma\right)\frac{y(t)}{x(t)} - mu\right]x(t) \\
&- \left(\gamma - \frac{\gamma}{f(t)}\right)\left[-d + e\min\left(\frac{\bar{b}x(t)}{y(t)+D},\gamma\right)\right]y(t) + \frac{\bar{b}^2}{2f(t)^2}\alpha^2x^2(t) \\
&+ \frac{\gamma^2}{2f(t)^2}\beta^2y^2(t)dt + \left(b - \frac{b}{f(t)}\right)\alpha x(t)dB^1(t) + \left(\gamma - \frac{\gamma}{f(t)}\right)\beta y(t)dB^2(t).
\end{aligned}
\tag{2.17}
$$

$$
d\widetilde{V}(x(t),y(t)) = \mathcal{L}\tilde{V}(x(t),y(t))dt + \alpha\left(\bar{b} - \frac{\bar{b}}{f(t)}\right)x(t)dB^1(t) + \beta\left(\frac{\gamma}{f(t)} - \gamma\right)y(t)dB^2(t),
$$

where

$$
\begin{aligned}
\mathcal{L}\widetilde{V}(x(t),y(t)) &= \left(\bar{b} - \frac{\bar{b}}{f(t)}\right)\left[x(t)\left(1-\frac{x(t)}{K}\right) - x(t)\min\left(\frac{\bar{b}x(t)}{y(t)+D},\gamma\right)\frac{y(t)}{x(t)} - mu\right] \\
&\quad + \left(1 - \frac{1}{y(t)}\right)\left[-qy(t) + ey(t)\min\left(\frac{\bar{b}x(t)}{y(t)+D},\gamma\right)\right] + \frac{\beta^2}{2} \\
&= (x(t)-1)\left(a\left(1-\frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}x(t)}{y(t)+D},\gamma\right)\frac{y(t)}{x(t)} - m\,u\right) \\
&\quad +(y(t)-1)\left(-q + e\min\left(\frac{\bar{b}x(t)}{y(t)+D},\gamma\right)\right) + \frac{\alpha^2+\beta^2}{2} \\
&\leq \tilde{P},
\end{aligned}
\tag{2.18}
$$

with $\tilde{P}$ is a positive constant. So we have

$$
d\widetilde{V}(x(t),y(t)) \leq \widetilde{P}dt + \alpha(x(t)-1)dB^1(t) + \beta(y(t)-1)dB^2(t). \tag{2.19}
$$

Integrating both sides of (2.19) from 0 to $\tau_n \wedge T$ and then taking the expectation, one gets

$$
\mathbb{E}\widetilde{V}(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq \widetilde{V}(x_0,y_0) + \widetilde{P}\mathbb{E}(\tau_n \wedge T)),
$$

therefore

$$\mathbb{E}\widetilde{V}(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq \widetilde{V}(x_0, y_0) + \widetilde{P}T. \tag{2.20}$$

Let $\Omega_n = \{\tau_n \leq T\}$ for $n > n_1$. From (2.16), we obtain $\mathbb{P}(\Omega_n) > \varepsilon$. For each $w \in \Omega_n$, $x(\tau_n)$ or $y(\tau_n)$ equals either n or $\dfrac{1}{n}$. So $\widetilde{V}(x(\tau_n), y(\tau_n))$ is not less than either

$$n - 1 - \ln n \text{ or } \frac{1}{n} - 1 - \ln\frac{1}{n} = \frac{1}{n} - 1 + \ln n.$$

Thus

$$\widetilde{V}(x(\tau_n), y(\tau_n)) \geq (n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right)$$

holds. According (2.20), we have

$$\widetilde{V}(x_0, y_0) + NT \geq \mathbb{E}\Big[I_{\Omega_n}\widetilde{V}(x(\tau_n), y(\tau_n))\Big] \geq \varepsilon(n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right),$$

in which I_{Ω_n} denotes the indicator function of Ω_n. Letting $n \to \infty$, then we obtain

$$\infty > \widetilde{V}(x_0, y_0) + \widetilde{P}T = \infty,$$

which leads to a contradiction, so we must have $\tau_\infty = \infty$ almost surely. □

Hence, system (2.7) is reduced to the simple form

$$\begin{cases} dx(t) = x(t)\Big[a\Big(1 - \dfrac{x(t)}{K}\Big) - \dfrac{\bar{b}\, y(t)}{y(t) + D} - m\, u\Big]dt + \alpha\, x(t) dB^1(t) \\ \\ dy(t) = y(t)\Big[-q + e\dfrac{\bar{b}\, x(t)}{y(t) + D}\Big]dt - \beta\, y(t) dB^2(t). \end{cases} \tag{2.21}$$

2.2.3 The long time behavior

Persistence of the stochastic predator-prey model is crucial in studying its long time behavior. In a view of ecology, the bad thing happens when a species disappears and a good situation occurs when all species co-exist. In this section, we will consider another persistence that is stochastic persistence in mean and condition for system (2.21) to be persistent [22]. By analyzing the stochastic differential equations for the system. We give criteria which ensure that prey and predator populations do not asymptotically approach extinction. Specifically, we show that under certain parameter settings, the species survive in the long run even in the presence of environmental noise. This is the proof of system robustness, and it implies that the predator and prey can coexist forever with a positive probability, maintaining long-time ecological stability. Next, we give out the condition for the system (2.21) to be extinct [22].

2.2.4 Persistence

In this part, we always assume:

Assumption 1. $a-\bar{b}-m\,u-\frac{\alpha^2}{2}>0.$

Assumption 2. $e\,\bar{b}-D\left(q+\frac{\beta^2}{2}\right)>0.$

Theorem 2.4. *Suppose that assumption 1 holds, for any $x_0>0$, the first component x of the solution to* (2.21) *satisfies*

$$\liminf_{t\longrightarrow+\infty}\frac{1}{t}\int_0^t x(s)ds\geq\frac{\left(a-m\,u-\bar{b}-\frac{\alpha^2}{2}\right)K}{a}. \tag{2.22}$$

Befor moving to proof Theorem (2.4) we first present the following lemma proposed in [22, 21].

Lemma 2.3. *Consider one dimensional stochastic differential equation*

$$dX(t)=X(t)\left[(a-b\,X(t))dt+\sigma dB(t)\right], \tag{2.23}$$

where parameters a, b and σ are positive, $a>\frac{\sigma^2}{2}$ and B is a standard Brownian motion. Suppose $a>\frac{\sigma^2}{2}$ and X is the solution of system (2.23) *with any initial value $X_0>0$. Then*

$$\lim_{t\to\infty}\frac{\log X(t)}{t}=0\quad a.s. \tag{2.24}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)ds=\frac{a-\frac{\sigma^2}{2}}{b}\quad a.s. \tag{2.25}$$

Proof. By Itô's formula, we can see

$$X(t)=\frac{e^{(a-\frac{\sigma^2}{2})t+\sigma B(t)}}{\frac{1}{X_0}+b\int_0^t e^{(a-\frac{\sigma^2}{2})s+\sigma B(s)}ds}$$

is the solution of system (2.23) with initial value $X_0 > 0$. Then it follows that

$$\begin{aligned}
\frac{1}{X(t)} &= \frac{1}{X_0} e^{-(a-\frac{\sigma^2}{2})t-\sigma B(t)} + b\int_0^t e^{-(a-\frac{\sigma^2}{2})(t-s)-\sigma(B(t)-B(s))} ds \\
&= e^{-\sigma B(t)} \left[\frac{1}{X_0} e^{-(a-\frac{\sigma^2}{2})t} + b\int_0^t e^{-(a-\frac{\sigma^2}{2})(t-s)} e^{-\sigma B(s)} ds\right] \\
&\leq e^{-\sigma B(t)} \left[\frac{1}{X_0} e^{-(a-\frac{\sigma^2}{2})t} + b e^{\sigma \max\limits_{0\leq s\leq t} B(s)} \int_0^t e^{-(a-\frac{\sigma^2}{2})(t-s)} ds\right] \\
&\leq e^{\sigma\left[\max\limits_{0\leq s\leq t} B(s)-B(t)\right]} \left[\frac{1}{X_0} e^{-(a-\frac{\sigma^2}{2})t} + b\int_0^t e^{-(a-\frac{\sigma^2}{2})(t-s)} ds\right],
\end{aligned}$$

the last inequality is based on the property of Brown motion $B(0) = 0$. Similarly, we obtain

$$\frac{1}{X(t)} \geq e^{\sigma\left[\min\limits_{0\leq s\leq t} B(s)-B(t)\right]} \left[\frac{1}{X_0} e^{-(a-\frac{\sigma^2}{2})t} + b\int_0^t e^{-(a-\frac{\sigma^2}{2})(t-s)} ds\right].$$

Since

$$Z(t) = \frac{1}{\frac{1}{X_0} e^{-(a-\frac{\sigma^2}{2})t} + b\int_0^t e^{-(a-\frac{\sigma^2}{2})(t-s)} ds},$$

is the solution of equation

$$\begin{cases} \dot{Z}(t) = Z(t)\left[a - \frac{\sigma^2}{2} - bZ(t)\right], \\ Z(0) = X_0. \end{cases}$$

it follows that

$$e^{\sigma\left[\min\limits_{0\leq s\leq t} B(s)-B(t)\right]} \frac{1}{Z(t)} \leq \frac{1}{X(t)} \leq e^{\sigma\left[\max\limits_{0\leq s\leq t} B(s)-B(t)\right]} \frac{1}{Z(t)},$$

and

$$\sigma(B(t) - \max_{0\leq s\leq t} B(s)) \leq \ln X(t) - \ln Z(t) \leq \sigma(B(t) - \min_{0\leq s\leq t} B(s)). \tag{2.26}$$

The distribution of $\max\limits_{0\leq s\leq t} B(s)$ is the same as $|B(t)|$, when B is a standard Brownian motion, and $\min\limits_{0\leq s\leq t} B(s)$ has the same distribution as $-\max\limits_{0\leq s\leq t} B(s)$. Besides, from the representation of $Z(t)$, we can easily know

$$\lim_{t\to\infty} \frac{\log Z(t)}{t} = 0. \tag{2.27}$$

Therefore, by (2.27) and the large number theorem, (2.26) implies

$$\lim_{t\to\infty} \frac{\ln X(t)}{t} = 0 \quad \text{a.s.},$$

as the equality (2.24) is obtained. Next, we prove (2.25). Let $U(t) = \ln X(t)$, and by Itô's formula, we get

$$dU(t) = \left(a - \frac{\sigma^2}{2} - bX(t)\right) dt + \sigma dB(t). \tag{2.28}$$

Integrating it from 0 to t, and then dividing t on both sides, yields

$$\frac{U(t)}{t} - \frac{U(0)}{t} = \left(a - \frac{\sigma^2}{2}\right) - b\frac{1}{t}\int_0^t X(s)ds + \sigma\frac{B(t)}{t}.$$

By the property of Brownian motion, we know that $\frac{1}{t}\int_0^t X(s)ds \to \frac{a - \frac{\sigma^2}{2}}{b}$ is an easy consequence of

$$\frac{U(t)}{t} \to 0, \quad t \to \infty.$$

which is true by (2.24). □

Now, we prove Theorem (2.4)

Proof. Since the first component of the solution x to (2.21) is positive, we have

$$dx(t) \le x(t)\left(a - m\,u - \frac{a}{K}x(t)\right) dt + \alpha x(t) dB^1(t)$$

Let

$$\Phi(t) = \frac{e^{(a - m\,u - \frac{\alpha^2}{2})t + \alpha B^1(t)}}{\frac{1}{x_0} + \frac{a}{K}\int_0^t e^{(a - m\,u - \frac{\alpha^2}{2})s + \alpha B^1(s)} ds}.$$

Then $\Phi(t)$ is the unique solution of equation

$$\begin{cases} d\Phi(t) = \Phi(t)\left(a - m\,u - \frac{a}{K}\Phi(t)\right) dt + \alpha\Phi(t) dB^1(t), \\ \Phi(0) = x_0 \end{cases} \tag{2.29}$$

and

$$x(t) \leq \Phi(t) \quad \text{a.s.}$$

by the comparison theorem of stochastic equations. On the other hand, we have

$$dx(t) \geq x(t)\left(a - m\,u - \bar{b} - \frac{a}{K}x(t)\right)dt + \alpha x(t)dB^1(t).$$

Similarly

$$\phi(t) = \frac{e^{(a-m\,u-\bar{b}-\frac{\alpha^2}{2})t+\alpha B^1(t)}}{\frac{1}{x_0} + \frac{a}{K}\int_0^t e^{(a-m\,u-\bar{b}-\frac{\alpha^2}{2})s+\alpha B^1(s)}ds}$$

is the unique solution of equation

$$\begin{cases} d\phi(t) = \phi(t)\left(a - m\,u - \bar{b} - \dfrac{a}{K}\phi(t)\right)dt + \alpha\phi(t)dB^1(t), \\ \phi(0) = x_0. \end{cases} \tag{2.30}$$

and

$$x(t) \geq \phi(t) \quad \text{a.s.}$$

Consequently,

$$\phi(t) \leq x(t) \leq \Phi(t) \quad \text{a.s.} \tag{2.31}$$

Since $\varPhi$ and ϕ are solutions of systems (2.29) and (2.30), respectively. From Lemma (2.3) given above, when $a - \bar{b} - m\,u - \dfrac{\alpha^2}{2} > 0$, we get properties of solutions $\varPhi$ and ϕ:

$$\lim_{t\to\infty}\frac{\log\Phi(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi(s)ds = \frac{\left(a - m\,u - \frac{\alpha^2}{2}\right)K}{a} \quad \text{a.s,}$$

and

$$\lim_{t\to\infty}\frac{\log\phi(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t \phi(s)ds = \frac{\left(a - m\,u - \bar{b} - \frac{\alpha^2}{2}\right)K}{a} \quad \text{a.s.}$$

The inequalities above, together with (2.31), imply

$$\lim_{t\to\infty}\frac{\log x(t)}{t}=0\quad\text{a.s,}$$

and

$$\frac{\left(a-m\,u-\bar{b}-\frac{\alpha^2}{2}\right)K}{a}\le\liminf_{t\longrightarrow+\infty}\frac{1}{t}\int_0^t x(s)ds\le\limsup_{t\longrightarrow+\infty}\frac{1}{t}\int_0^t x(s)ds\le\frac{\left(a-m\,u-\frac{\alpha^2}{2}\right)K}{a}\quad\text{a.s.}$$

□

Theorem 2.5. *Suppose that Assumption 2 holds, and the second component y of the solution to* (2.21) *with initial value $y_0>0$, then*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{x(s)}{y(s)+D}ds=\frac{q+\frac{\beta^2}{2}}{e\,\bar{b}}\quad a.s. \tag{2.32}$$

Moreover, we can see

$$\liminf_{t\longrightarrow+\infty}\frac{1}{t}\int_0^t y(s)ds\ge\frac{e\,\bar{b}-D\left(q+\frac{\beta^2}{2}\right)}{e\,\bar{b}}.$$

Befor setting the proof of Theorem 2.5 we need the following lemma

Lemma 2.4. *Suppose Assumption 2 holds, the second component y of the solution to* (2.21) *and for any initial value $y_0>0$, then*

$$\lim_{t\to\infty}\frac{\log y(t)}{t}=0\quad a.s.$$

Proof.

$$dy(t)\le\left(-qy(t)+\frac{e\,\bar{b}}{D}\Phi(t)\right)dt-\beta y(t)dB^2(t). \tag{2.33}$$

Let $\Psi(t)$ be the solution of

$$\begin{cases}d\Psi(t)=\left(-q\Psi(t)+\frac{e\,\bar{b}}{D}\Phi(t)\right)dt-\beta\Psi(t)dB^2(t),\\ \Psi(0)=y_0.\end{cases} \tag{2.34}$$

Then, by the comparison theorem of stochastic equations, we have $y(t)\le\Psi(t)$ a.s. Integrating (2.34) from T to t ($t>T$), where T satisfies $\frac{1}{2}e^{(a-m\,u-\frac{\alpha^2}{2})T}\ge1$, yields

$$\Psi(t)=\Psi(T)e^{-(q+\frac{\beta^2}{2})(t-T)-\beta(B^2(t)-B^2(T))}+\frac{e\bar{b}}{D}\int_T^t\Phi(s)e^{-(q+\frac{\beta^2}{2})(t-s)-\beta(B^2(t)-B^2(s))}ds. \tag{2.35}$$

If $s \geq T$, then $\frac{1}{2}e^{(a-m\,u-\frac{\alpha^2}{2})s} \geq 1$ and

$$\begin{aligned}
\Phi(s) &= \frac{e^{(a-m\,u-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{\frac{1}{x_0}+\frac{a}{K}\int_0^s e^{(a-m\,u-\frac{\alpha^2}{2})v+\alpha B^1(v)}dv} \\
&\leq \frac{e^{(a-m\,u-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{\frac{a}{K}\int_0^s e^{(a-m\,u-\frac{\alpha^2}{2})v+\alpha B^1(v)}dv} \\
&\leq \frac{e^{(a-m\,u-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{\frac{a}{K}e^{\alpha \min\limits_{0\leq v\leq s} B^1(v)}\int_0^s e^{(a-m\,u-\frac{\alpha^2}{2})v}dv} \\
&= \frac{\left(a-m\,u-\frac{\alpha^2}{2}\right)K}{a}\frac{e^{(a-m\,u-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{e^{\alpha \min\limits_{0\leq v\leq s} B^1(v)}\left[e^{(a-m\,u-\frac{\alpha^2}{2})s}-1\right]} \\
&\leq \frac{2\left(a-m\,u-\frac{\alpha^2}{2}\right)K}{a}\frac{e^{(a-m\,u-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{e^{\alpha \min\limits_{0\leq v\leq s} B^1(v)}e^{(a-m\,u-\frac{\alpha^2}{2})s}} \\
&= \frac{2\left(a-m\,u-\frac{\alpha^2}{2}\right)K}{a}e^{\alpha(B^1(s)-\min\limits_{0\leq v\leq s} B^1(v))}.
\end{aligned}$$

Replacing this in (2.35), we obtain

$$\begin{aligned}
\Psi(t) &\leq \Psi(T)e^{-(q+\frac{\beta^2}{2})(t-T)-\beta(B^2(t)-B^2(T))} + \frac{2e\,\bar{b}\left(a-m\,u-\frac{\alpha^2}{2}\right)K}{aD}\int_T^t e^{\alpha(B^1(s)-\min\limits_{0\leq v\leq s}B^1(v))}e^{-(q+\frac{\beta^2}{2})(t-s)-\beta(B^2(t)-B^2(s))}ds \\
&\leq \Psi(T)e^{-(q+\frac{\beta^2}{2})(t-T)-\beta(B^2(t)-B^2(T))} + \frac{2e\,\bar{b}\left(a-m\,u-\frac{\alpha^2}{2}\right)K}{a\,D}e^{\alpha(\max\limits_{0\leq s\leq t}B^1(s)-\min\limits_{0\leq s\leq t}B^1(s))+\beta(\max\limits_{0\leq s\leq t}B^2(s)-B^2(t))} \\
&\times\int_T^t e^{-(q+\frac{\beta^2}{2})(t-s)}ds \\
&= \Psi(T)e^{-(q+\frac{\beta^2}{2})(t-T)-\beta(B^2(t)-B^2(T))} \\
&+\frac{2e\,\bar{b}\left(a-m\,u-\frac{\alpha^2}{2}\right)K}{a\,D\left(q+\frac{\beta^2}{2}\right)}e^{\alpha(\max\limits_{0\leq s\leq t}B^1(s)-\min\limits_{0\leq s\leq t}B^1(s))+\beta(\max\limits_{0\leq s\leq t}B^2(s)-B^2(t))}\left[1-e^{-(q+\frac{\beta^2}{2})(t-T)}\right] \\
&\leq e^{\alpha(\max\limits_{0\leq s\leq t}B^1(s)-\min\limits_{0\leq s\leq t}B^1(s))+\beta(\max\limits_{0\leq s\leq t}B^2(s)-B^2(t))}\left[\Psi(T)+\frac{2e\bar{b}\left(a-m\,u-\frac{\alpha^2}{2}\right)K}{a\,D\left(q+\frac{\beta^2}{2}\right)}\right] \\
&:= K_1 e^{\alpha(\max\limits_{0\leq s\leq t}B^1(s)-\min\limits_{0\leq s\leq t}B^1(s))+\beta(\max\limits_{0\leq s\leq t}B^2(s)-B^2(t))},
\end{aligned}$$

where $K_1 = \Psi(T) + \dfrac{2e\,\bar{b}\left(a-m\,u-\frac{\alpha^2}{2}\right)K}{a\,D\left(q+\frac{\beta^2}{2}\right)}$ is a constant. Therefore, we get

$$\frac{\log\Psi(t)}{t} \leq \alpha\frac{\max\limits_{0\leq s\leq t}B^1(s)}{t} - \alpha\frac{\min\limits_{0\leq s\leq t}B^1(s)}{t} + \beta\frac{\max\limits_{0\leq s\leq t}B^2(s)}{t} - \beta\frac{B^2(t)}{t} + \frac{\log K_1}{t}. \tag{2.36}$$

The distributions of $\max\limits_{0\leq s\leq t}B^1(s)$, $\max\limits_{0\leq s\leq t}B^2(s)$ are same as $|B^1(t)|$ and $|B^2(t)|$ respectively, and

$\min_{0\leq s\leq t} B^1(s)$ has the same distribution as $-\max_{0\leq s\leq t} B^1(s)$. Letting $t\to\infty$, by the strong law of large numbers, (2.36) implies

$$\limsup_{t\to\infty}\frac{\log\Psi(t)}{t}\leq 0 \quad \text{a.s},$$

as a result,

$$\limsup_{t\to\infty}\frac{\log y(t)}{t}\leq 0 \quad \text{a.s.} \tag{2.37}$$

On the other hand, we have,

$$\frac{y(s)}{y(s)+D}=1-\frac{D}{y(s)+D},$$

and,

$$1-\frac{D}{y(s)+D}\geq 1-\frac{Dx(s)}{y(s)+D},$$

as a result,

$$y(s)\geq\frac{y(s)}{y(s)+D}>1-\frac{Dx(s)}{y(s)+D}, \tag{2.38}$$

from this, we conclude that

$$\begin{aligned} dy(t)&=y(t)\left(-q+e\,\bar{b}\frac{x(t)}{y(t)+D}\right)dt-\beta y(t)dB^2(t)\\ &\geq y(t)\left(-q+\frac{e\,\bar{b}}{D}-\frac{e\,\bar{b}}{D}y(t)\right)dt-\beta y(t)dB^2(t). \end{aligned}$$

Let
$$\begin{cases} d\psi(t)=\psi(t)\left(-q+\frac{e\,\bar{b}}{D}-\frac{e\,\bar{b}}{D}\psi(t)\right)dt-\beta\psi(t)dB^2(t),\\ \psi(0)=y_0. \end{cases}$$

According to Lemma (2.3), Assumption 2 and suppose that $\beta<0$ we have,

$$\lim_{t\to\infty}\frac{\log\psi(t)}{t}=0 \quad a.s. \tag{2.39}$$

Therefore, (2.37) and (2.39) imply

$$\lim_{t\to\infty}\frac{\log y(t)}{t}=0 \quad a.s. \tag{2.40}$$

□

Now, we can prove Theorem 2.5

Proof. Denote $V(y) = \log y$. By Itô's formula, we obtain

$$d \log y(t) = \left(-q - \frac{\beta^2}{2} + \frac{e\,\bar{b}x(t)}{y(t)+D} \right) dt - \beta dB^2(t).$$

Integrating it from 0 to t, yields

$$\log y(t) - \log y(0) = \left(-q - \frac{\beta^2}{2} \right) t + \int_0^t \frac{e\,\bar{b}\,x(s)}{y(s)+D} ds - \beta B^2(t).$$

and,

$$\frac{\log y(t) - \log y(0)}{t} = -q - \frac{\beta^2}{2} + \frac{1}{t} \int_0^t \frac{e\,\bar{b}\,x(s)}{y(s)+D} ds - \beta \frac{B^2(t)}{t}. \tag{2.41}$$

Letting $t \to \infty$, (2.39) and (2.41) imply

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{x(s)}{y(s)+D} ds = \frac{q + \frac{\beta^2}{2}}{e\,\bar{b}} \quad a.s. \tag{2.42}$$

From (2.38) we can see

$$\liminf_{t\longrightarrow+\infty} \frac{1}{t} \int_0^t y(s) ds \geq 1 - \lim_{t\longrightarrow+\infty} D \int_0^t \frac{x(s)}{y(s)+D} ds \geq 1 - D \frac{q + \frac{\beta^2}{2}}{e\,\bar{b}}.$$

So, finally we have

$$\liminf_{t\longrightarrow+\infty} \frac{1}{t} \int_0^t y(s) ds \geq \frac{e\,\bar{b} - D\left(q + \frac{\beta^2}{2}\right)}{e\,\bar{b}} > 0.$$

□

Remark 4. *Under Assumptions1, 2 and by Theorem 2.4 and 2.5, we obtain*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t x(s) ds \geq \frac{K\left(a - \bar{b} - \frac{\alpha^2}{2} - m\,u\right)}{a} > 0 \quad a.s,$$

and

$$\liminf_{t\longrightarrow+\infty} \frac{1}{t} \int_0^t y(s) ds \geq 1 - D\,\frac{q + \frac{\beta^2}{2}}{e\,\bar{b}} > 0 \quad a.s.$$

According to Definition (1.10)*, we obtain the stochastic system* (2.21) *is persistent in mean.*

2.2.5 Extinction

In ecological models, extinction is when one or both species in the system tend to vanish in the long term with probability one. Extinction conditions are crucial to know in stochastic predator-prey models, as random perturbations can play a vital role in species survival. In our model, extinction occurs when the prey or predator population dwindles to zero in the long term, independent of initial population sizes. For this, we have sufficient conditions in which at least one of the species becomes extinct, considering the effect of environmental noise and model parameters.

In this section, we assume that Assumptions 1 and 2 are not satisfied and examine the case where $a - m\,u - \bar{b} - \frac{\alpha^2}{2} < 0$.

Theorem 2.6. *Let* $(x(t), y(t))_{t\geq 0}$ *be the solution of system* (2.21) *with any initial value* $x_0 > 0, y_0 > 0$. *If* $a - m\,u - \bar{b} - \frac{\alpha^2}{2} < 0$, *then*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0 \quad a.s.$$

Before moving to the proof of extinction in the model, we need the following lemma, which can be found in [21, 22].

Lemma 2.5. *Consider the following stochastic equation*

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t). \tag{2.43}$$

Assume X *is the solution of system* (2.43). *If* $S(-\infty) > -\infty$ *and* $S(+\infty) = +\infty$, *then*

$$\lim_{t\to\infty} X(t) = -\infty,$$

where the scale function

$$S(u) = \int_0^u e^{-\int_0^v \frac{2\mu(y)}{\sigma^2(y)}dy} dv.$$

Now we can move to prove Theorem 2.6

Proof. Define $U(t) = \log x(t)$ and $V(t) = \log y(t)$. Applying Itô's lemma, we obtain

$$dU(t) = \left(a - m\,u - \frac{\alpha^2}{2} - \frac{a}{K}e^{U(t)} - \frac{\bar{b}e^{V(t)}}{e^{V(t)} + D}\right) dt + \alpha dB^1(t) \leq \left(a - \frac{\alpha^2}{2} - m\,u\right) dt + \alpha dB^1(t).$$

By utilizing the stochastic comparison theorem and the properties of diffusion processes (refer to Lemma 2.5), for $\mu(t) = a - mu - \frac{\alpha^2}{2}$ and $\sigma(t) = \alpha$, it is easy to determine that $S(-\infty) > -\infty$ and $S(+\infty) = +\infty$, leading to

$$\lim_{t\to\infty} U(t) = -\infty \quad \text{a.s.}$$

which implies that

$$\lim_{t\to\infty} x(t) = 0 \quad \text{a.s.}$$

In a such case, it follows that

$$\lim_{t\to\infty} y(t) = 0 \quad \text{a.s.} \tag{2.44}$$

If this were not the case, then we must have

$$\limsup_{t\to\infty} y(t) := k > 0 \quad \text{a.s.}$$

Consequently, for any arbitrarily small $\varepsilon > 0$, there exist t_0 and a subset Ω_ε such that $\mathbb{P}(\Omega_\varepsilon) \geq 1 - \varepsilon$ and

$$\frac{e\,\bar{b}\,x(t)}{y(t)+D} \leq e\,b\,\varepsilon \quad \text{for } t \geq t_0 \text{ and } \omega \in \Omega_\varepsilon.$$

Thus, we derive

$$y(t)(-q\,dt - \beta dB^2(t)) \leq dy(t) \leq y(t)((-q + e\,\bar{b}\,\varepsilon)dt - \beta dB^2(t)),$$

and

$$-\left(q + \frac{\beta^2}{2}\right)dt - \beta dB^2(t) \leq dV(t) \leq \left(-\left(q + \frac{\beta^2}{2}\right) + e\,\bar{b}\,\varepsilon\right)dt - \beta dB^2(t).$$

By the same reasoning as above and the arbitrariness of ϵ, we can get

$$\lim_{t\to\infty} V(t) = -\infty \quad a.s.,$$

i.e.,

$$\lim_{t\to\infty} y(t) = 0 \quad a.s.$$

There is a contradiction, hence (2.44) is true. ☐

Remark 5. *Given the condition* $a - m\,u - \bar{b} - \dfrac{\alpha^2}{2} < 0$ *and the proof of Theorem 2.6, if the prey goes extinct, the predator will also die out.*

What about the other case? In response to this question, we make use of the following result.

Theorem 2.7. *Let* $(x(t), y(t))_{t\geq 0}$ *be the solution of system* (2.21) *with any initial value* $x_0 > 0, y_0 > 0$. *If* $a - m\,u - \bar{b} - \dfrac{\alpha^2}{2} > 0$ *and* $e\,\bar{b} - D\left(q + \dfrac{\beta^2}{2}\right) < 0$, *then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds = \frac{K\left(a - m\,u - \frac{\alpha^2}{2}\right)}{a}, \quad \lim_{t\to\infty} y(t) = 0 \quad a.s.$$

Proof. Consider the predator population $y(t)$, we can easily see that

$$y(t)\left(-qdt-\beta dB^2(t)\right)\leq dy(t)\leq y(t)\left(\left(-q+\frac{e\bar{b}}{D}y(t)\right)dt-\beta dB^2(t)\right),$$

by the comparison theorem of stochastic equations and if $e\bar{b}-D\left(q+\frac{\beta^2}{2}\right)<0$, then

$$\lim_{t\to\infty} y(t)=0 \quad a.s. \tag{2.45}$$

Now, let us analyze the behavior of the prey population $x(t)$. Clearly, we have:

$$dx(t)=x(t)\left(a-mu-\frac{a}{K}x(t)-\frac{\bar{b}y(t)}{y(t)+D}\right)dt+\alpha x(t)dB^1(t),$$

which can be rewritten as

$$dx(t)\geq x(t)\left(a-mu-\bar{b}-\frac{a}{K}x(t)\right)dt+\alpha x(t)dB_1(t),$$

if $a-mu-\bar{b}-\frac{\alpha^2}{2}>0$, then by Lemma 2.3, we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds\geq\frac{a-mu-\bar{b}-\frac{\sigma^2}{2}}{b}>0 \quad a.s.$$

This implies that there exists $T_0>0$ and a positive constant k_0 such that $x(t)>k_0$ almost surely for $t\geq T_0$. Moreover, Equation (2.45) tells us that for any $\varepsilon>0$, there exist $T>T_0$ and a set Ω_ε such that $\mathbb{P}(\Omega_\varepsilon)\geq 1-\varepsilon$ and

$$\frac{y(t)}{y(t)+D}\leq b\varepsilon, \quad \text{for } \omega\in\Omega_\varepsilon, t\geq T.$$

Thus, when $\omega\in\Omega_\varepsilon, t\geq T$, we get

$$dx(t)\geq x(t)\left(a-mu-\bar{b}\varepsilon-\frac{a}{K}x(t)\right)dt+\alpha x(t)dB^1(t),$$

as a result, we obtain:

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds\geq\frac{K\left(a-mu-\bar{b}\varepsilon-\frac{\alpha^2}{2}\right)}{a}>0.$$

On the other hand, we have the inequality:

$$dx(t) \le x(t)(a - mu - \frac{\alpha^2}{2})dt + \alpha x(t) dB^1(t).$$

Thus, we can deduce then

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le \frac{K\left(a - mu - \frac{\alpha^2}{2}\right)}{a} \quad a.s.$$

Therefore, by the arbitrariness of ε, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds = \frac{K\left(a - mu - \frac{\alpha^2}{2}\right)}{a} \quad a.s.$$

□

Numerical Simulation

We use Milstein method mentioned in [20] to get the simulation of the stochastic system (2.21) in order to confirm the results set above.

Consider the discretization of system (2.21)

$$x_{i+1} = x_i + \left(a x_i\left(1 - \frac{x_i}{K}\right) - \frac{\bar{b} x_i y_i}{y_i + D} - m x_i u\right)\Delta t + \alpha x_i\, \Delta B^1_{i+1} + \frac{1}{2}\alpha^2 x_i\left(\left(\Delta B^1_{i+1}\right)^2 - \Delta t\right),$$

$$y_{i+1} = y_i + \left(-q y_i + \frac{e\bar{b} x_i y_i}{y_i + D}\right)\Delta t - \beta y_i\, \Delta B^2_{i+1} - \frac{1}{2}\beta^2 y_i\left(\left(\Delta B^2_{i+1}\right)^2 - \Delta t\right),$$

where $\Delta B_{i+1} = B_{t_{i+1}} - B_{t_i} \sim \mathcal{N}(0, \Delta t)$ and $\Delta t = t_{i+1} - t_i$ is the time step.

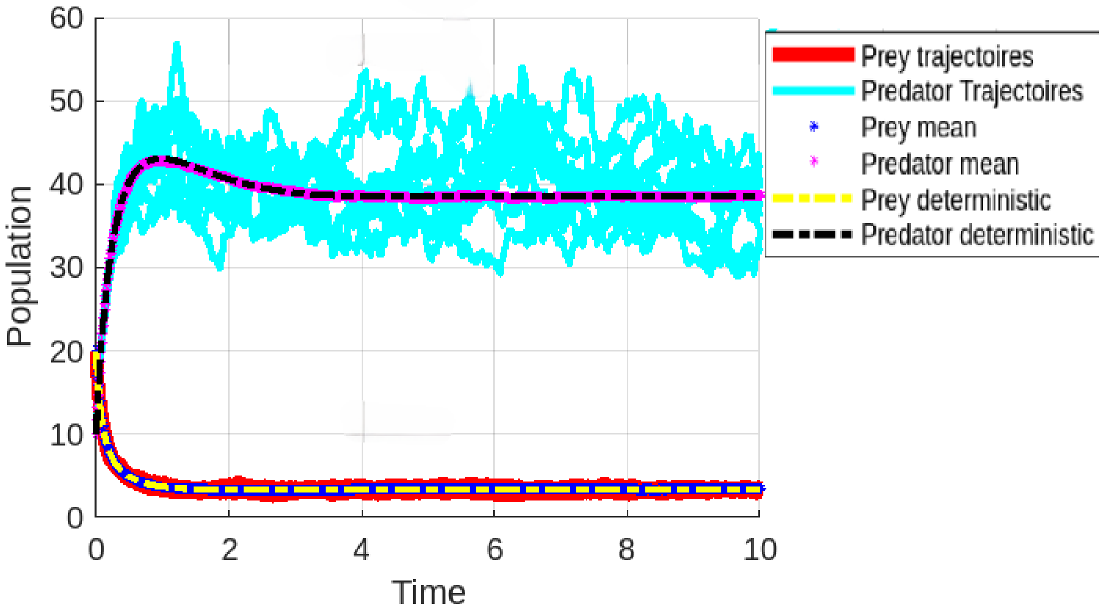

Figure 2.1: Solutions of systems (2.21).

For $(x_0, y_0) = (20, 10)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 0.1$ and $\beta = -0.1$. As mentioned in [5], the deterministic equivalent system possesses a coexisting equilibrium point $P^* = (x^*, y^*)$, which is globally asymptotically stable. Figure 2.1 represents the simulation of 10 trajectories of the solutions of the stochastic system (2.21). The trajectories in cyan represent the evolution of the predator population, while the red ones represent the dynamics of the prey population, and this what makes the stochastic system more realistic than the deterministic one is that in the studied lake we can't predict what exactly will happen due to the huge amount of natural changes. The stars in magenta and those in blue represent the means of x and y, respectively. We can see that the stochastic system converges in mean to the coexisting equilibrium, so the numerical simulation confirms the persistence of both species under assumptions 1 and 2 because we have here $a - \bar{b} - mu - \dfrac{\alpha^2}{2} \approx 1.0888 > 0$ and $e\bar{b} - D\left(q + \dfrac{\beta^2}{2}\right) \approx 8.5450 > 0$, so Theorems 2.4 and 2.5 are verified numerically.

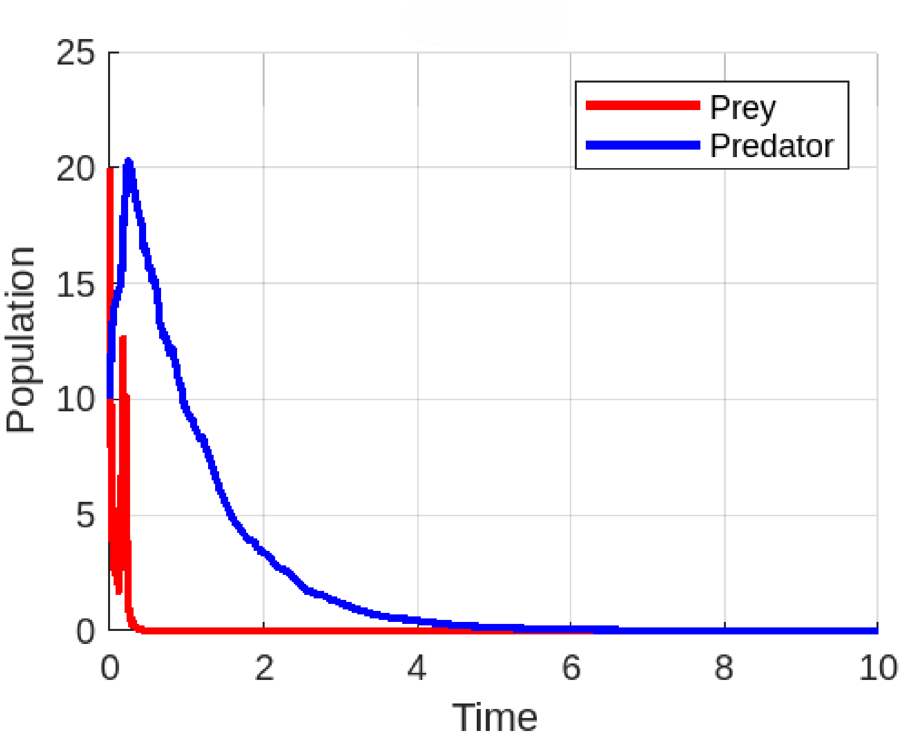

Figure 2.2: Extinction of both populations.

For $(x_0, y_0) = (20, 10)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 5$ and $\beta = -0.1$ Here in Figure 2.2 we verify numerically Theorem 2.6 and we get the extinction of both spaces since Assumption 1 is not satisfied, we have here $a - \bar{b} - mu - \dfrac{\alpha^2}{2} \approx -11.4000 < 0$.

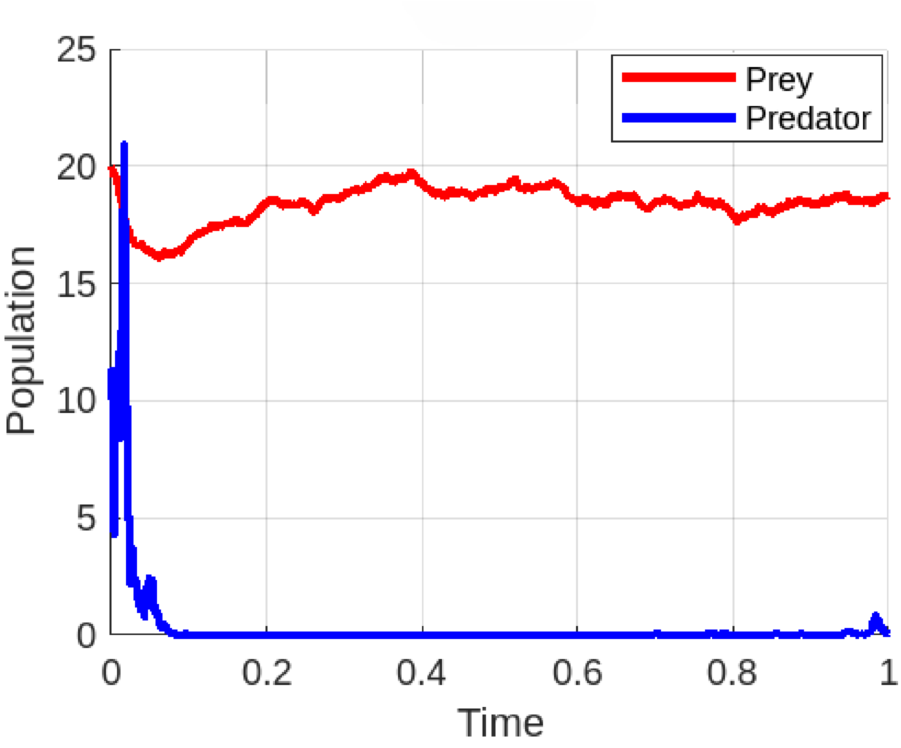

Figure 2.3: Extinction of predator population.

For $(x_0, y_0) = (20, 10)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 5$ and $\beta = -10$. Figure 2.3 represents the extinction of predator population when Assumption 2 is not satisfied because $e\bar{b} - D\left(q + \dfrac{\beta^2}{2}\right) \approx -191.5000 < 0$ we have here and the stability in average of the prey population.

2.2.6 The stability of the solution

Stability with Respect to Initial Values.

In stochastic dynamic systems, small variations in initial conditions can influence long-term behavior due to random fluctuations. In this context, stability ensures that the trajectories of initial values that are close to each other are not irretrievably lost in the long term. More specifically, a good stochastic system must have bounded fluctuations, meaning that the solutions remain within a finite region with a high probability.

For our predator-prey model, the stability of initial values translates into small perturbations of the initial number of prey and predators not leading to radically different long-term behavior. On the contrary, the system exhibits bounded oscillations, preventing extinction or explosive growth based solely on minor initial perturbations.

First, we define the norm used to assess stability, considering two solutions z and $\hat{z}$ of the same stochastic differential equation, corresponding to different initial conditions z_0 and $\hat{z}_0$ as:

$$||z-\hat{z}||^2 := \mathbb{E}\left[\sup_{t\in[0,T]} |z(t)-\hat{z}(t)|^2 e^{\lambda t}\right],$$

where $\lambda > 0$ is a given constant and this norm is equivalent to the standard supremum norm for $\lambda = 0$.

Let (x, y) and $(\hat{x}, \hat{y})$ are two solutions of the stochastic system (2.7) with different initial conditions (x_0, y_0) and $(\hat{x}_0, \hat{y}_0)$. Such that $x_0 \longrightarrow \hat{x}_0$ and $y_0 \longrightarrow \hat{y}_0$.

- Let $x(t)$ and $\hat{x}(t)$ are two first components of the solution of (2.21) with different initial conditions x_0 and $\hat{x}_0$. Such that $x_0 \longrightarrow \hat{x}_0$.
- Let $y(t)$ and $\hat{y}(t)$ are two second components of the solution of (2.21) with different initial conditions x_0 and $\hat{y}_0$. Such that $y_0 \longrightarrow \hat{y}_0$.

Based on the integral representation of $x(t)$ and $\hat{x}(t)$, we obtain

$$x(t) = x_0 + \int_0^t \left[ax(s)\left(1-\frac{x(s)}{K}\right) - \frac{\bar{b}x(s)}{y(s)+D}y(s) - m\,ux(s)\right]ds + \alpha\int_0^t x(s)dB^1(s).$$

$$\hat{x}(t) = \hat{x}_0 + \int_0^t \left[a\hat{x}(s)\left(1-\frac{\hat{x}(s)}{K}\right) - \frac{\bar{b}\hat{x}(s)}{\hat{y}(s)+D}\hat{y}(s) - m\,u\hat{x}(s)\right]ds + \alpha\int_0^t \hat{x}(s)dB^1(s).$$

By taking the difference between the integral forms of $x(t)$ and $\hat{x}(t)$, we get

$$\begin{aligned} x(t)-\hat{x}(s) = x_0 - \hat{x}_0 + \int_0^t &\left[a(x(s)-\hat{x}(s)) - \frac{a}{K}(x(s)-\hat{x}(s)) - \bar{b}\left(\frac{x(s)y(s)}{y(s)+D} - \frac{\hat{x}(s)\hat{y}(s)}{\hat{y}(s)+D}\right) - m\,u(x(s)-\hat{x})\right]ds \\ &+ \alpha\int_0^t (x(s)-\hat{x}(s))dB^1(s), \end{aligned}$$

applying the absolute value to both sides, we get

$$\begin{aligned} |x(t)-\hat{x}(t)| \leq |x_0-\hat{x}_0| &+ \int_0^t a|x(s)-\hat{x}(s)|ds + \frac{a}{K}\sup_{s\in[0,T]}|x(s)-\hat{x}(s)|\int_0^t a|x(s)-\hat{x}(s)|ds| \\ &+ \bar{b}\left(\frac{\max_{s\in[0,T]}\left(\sup_{s\in[0,T]} y(s), \sup_{s\in[0,T]}\hat{y}(s)\right)}{\min_{s\in[0,T]}\left(\min_{s\in[0,T]} y(s)+D, \min_{s\in[0,T]}\hat{y}(s)+D\right)}\right)\int_0^t |x(s)-\hat{x}(s)|ds \\ &+ m\,u\int_0^t |x(s)-\hat{x}(s)|ds + \alpha\int_0^t |x(s)-\hat{x}(s)|dB_1(s), \end{aligned}$$

now, we apply Itô's formula to the function $f(t,x) = (x-\hat{x})^2 e^{\lambda t}$

$$\mathbb{E}\left[\sup_{t\in[0,T]}|x(t)-\hat{x}(t)|^2 e^{\lambda t}\right] \leq |x_0-\hat{x}_0| + \lambda\mathbb{E}\left[\int_0^t e^{\lambda s}|x(s)-\hat{x}(s)|^2 ds\right] + \mathit{\Gamma}\,\mathbb{E}\left[\int_0^t e^{\lambda s}|x(s)-\hat{x}(s)|^2 ds\right],$$

with $\mathit{\Gamma} = \mathbb{E}(\Lambda)$ and

$$\Lambda = 2a + \frac{2a}{K}\sup_{s\in[0,T]}|x(s)-\hat{x}(s)| + 2\bar{b}\left(\frac{\max_{s\in[0,T]}\left(\sup_{s\in[0,T]} y(s), \sup_{s\in[0,T]} \hat{y}(s)\right)}{\min_{s\in[0,T]}\left(\min_{s\in[0,T]} y(s)+D, \min_{s\in[0,T]} \hat{y}(s)+D\right)}\right) + 2mu + \alpha^2,$$

and if we take $\lambda = -\mathit{\Gamma}$, we obtain $\mathbb{E}\left[\sup_{t\in[0,T]}|x(t)-\hat{x}(t)|^2 e^{\lambda t}\right] \leq |x_0 - \hat{x}_0|$ which means that $\|x-\hat{x}\| \longrightarrow 0$ when $x_0 \longrightarrow \hat{x}_0$.

Now, let us consider the integral representation of $y(t)$ and $\hat{y}(t)$

$$y(t) = y_0 + \int_0^t \left[-qy(s) + e\frac{\bar{b}x(s)}{y(s)+D}y(s)\right] ds + \beta\int_0^t y(s)dB^2(s).$$

$$\hat{y}(t) = \hat{y}_0 + \int_0^t \left[-q\hat{y}(s) + e\frac{b\hat{x}(s)}{\hat{y}(s)+D}\hat{y}(s)\right] ds + \beta\int_0^t \hat{y}(s)dB^2(s).$$

By taking the difference between the integral forms of $y(t)$ and $\hat{y}(t)$, we get

$$y(t)-\hat{y}(s) = y_0 - \hat{y}_0 + \int_0^t \left[-q(y(s)-\hat{y}(s)) + e\bar{b}\left(\frac{x(s)y(s)}{y(s)+D} - \frac{\hat{x}(s)\hat{y}(s)}{\hat{y}(s)+D}\right)\right] ds + \beta\int_0^t (u(s)-\hat{y}(s))dB^2(s),$$

applying the absolute value to both sides, we get

$$|y(t)-\hat{y}(t)| \leq |y_0-\hat{y}_0| + q\int_0^t |y(s)-\hat{y}(s)|ds + e\bar{b}\left(\frac{\max_{s\in[0,T]}\left(\sup_{s\in[0,T]} x(s), \sup_{s\in[0,T]} \hat{x}(s)\right)}{\min_{s\in[0,T]}\left(\min_{s\in[0,T]} y(s)+D, \min_{s\in[0,T]} \hat{y}(s)+D\right)}\right)$$
$$\times \int_0^t |y(s)-\hat{y}(s)|ds + \beta\int_0^t |y(s)-\hat{y}(s)|dB^2(s),$$

now, we apply Itô's formula to the function $f(t,y) = (y-\hat{y})^2 e^{\theta t}$

$$\mathbb{E}\left[\sup_{t\in[0,T]}|y(t)-\hat{y}(t)|^2 e^{\theta t}\right] \leq |y_0-\hat{y}_0| + \theta\mathbb{E}\left[\int_0^t e^{\lambda s}|y(s)-\hat{y}(s)|^2 ds\right] + \xi\ \mathbb{E}\left[\int_0^t e^{\theta s}|y(s)-\hat{y}(s)|^2 ds\right],$$

with $\xi = \mathbb{E}(\gamma)$ and

$$\gamma = 2q + 2b\left(\frac{\max_{s\in[0,T]}\left(\sup_{s\in[0,T]} x(s), \sup_{s\in[0,T]} \hat{x}(s)\right)}{\min_{s\in[0,T]}\left(\min_{s\in[0,T]} y(s)+D, \min_{s\in[0,T]} \hat{y}(s)+D\right)}\right) + \beta^2,$$

and if we take $\theta = -\xi$, we obtain $\mathbb{E}\left[\sup_{t\in[0,T]}|y(t)-\hat{y}(t)|^2 e^{\theta t}\right] \leq |y_0-\hat{y}_0|$ which means that $\|y-\hat{y}\| \longrightarrow 0$ when $y_0 \longrightarrow \hat{y}_0$. □

Remark 6. *We established the stability of the system in the sense that the trajectories remain close whenever the initial conditions satisfy* $x_0 \longrightarrow \hat{x}_0$ *and* $y_0 \longrightarrow \hat{y}_0$

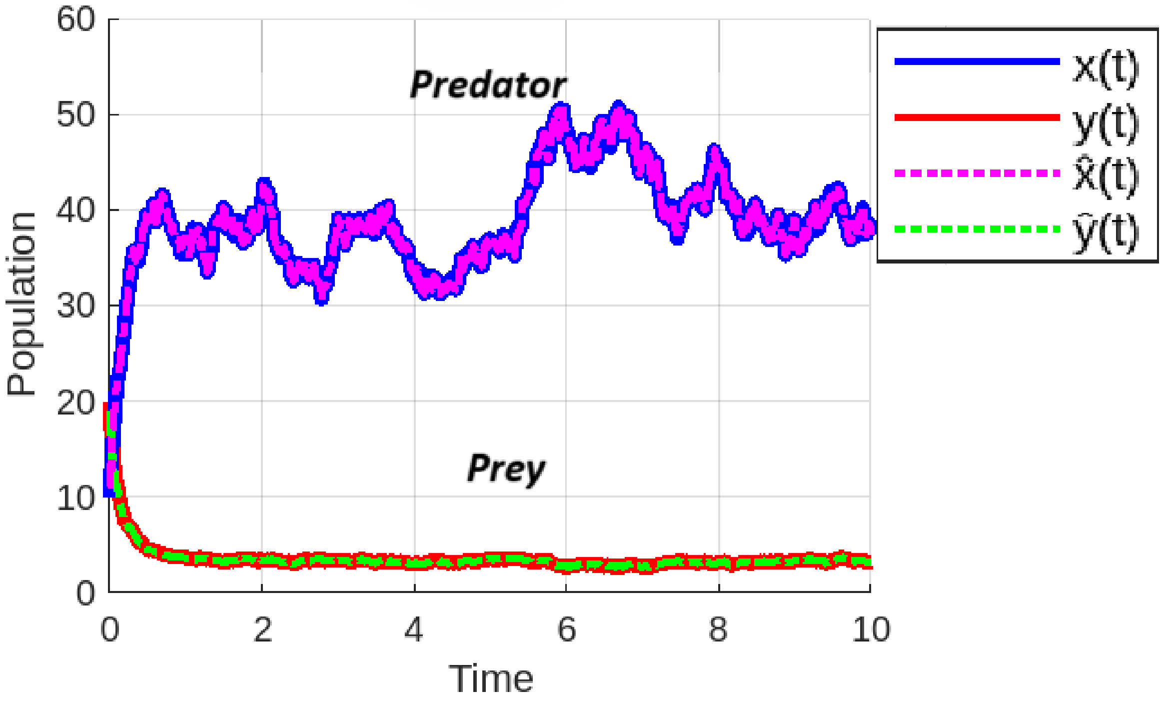

Figure 2.4: Stability of the Solution (2.21).

For $(x_0, y_0) = (20, 10)$ and $(\hat{x}_0, \hat{y}_0) = (19, 11)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 0.1$ and $\beta = -0.1$. Figure 2.4 confirms the stability of the stochastic system (2.21). Given closed initial conditions, the trajectories are almost the same for both populations, and after calculation of the mean square error, we get $MSE_x = 0.0041$ and $MSE_y = 0.0097$, which means that a small variation in the initial population of prey and predator doesn't influence the behavior of the system.

Chapter 3

Stochastic optimal control on the harvesting effort

In realistic ecological models, the dynamics of species populations and harvesting activities are often influenced by environmental perturbations. As we saw in Subsection 2.1.6, the optimal control problem, with its deterministic flavor, tries to optimally manage resource utilization, but it fails to capture the stochastic nature which is inherent in these systems.

To improve on this foundation, we now consider problems in which the time evolution of $x(s)$ is actively influenced by another stochastic process $u(s)$ [18], where the dynamics are governed by stochastic differential equations (SDEs). In particular, we describe the stochastic control problem as an extension of the deterministic one discussed in Subsection 2.1.6.

In this chapter we consider the following system:

$$\begin{cases} dx(t) = x(t)\left[a\left(1-\dfrac{x(t)}{K}\right) - \dfrac{\bar{b}y(t)}{y(t)+D} - m\,u(t)\right]dt + \alpha x(t)dB^1(t), \\ \\ dy(t) = y(t)\left[-q + e\,\dfrac{\bar{b}x(t)}{y(t)+D}\right]dt - \beta y(t)dB^2(t). \end{cases} \tag{3.1}$$

We wish to define an appropriate cost functional to be minimized, which leads to characterizing the value function that satisfies the Hamilton-Jacobi-Bellman equation (HJB) associated with the problem. Then, the optimal control is obtained by solving the HJB equation. We use the probabilistic approach based on Forward-Backward Stochastic Differential Equations (FBSDEs), which provides an efficient method for problems related to optimal control. Finally, we implement a numerical simulation using Least Squares Monte Carlo (LSMC) as an approximation of the value function to get an optimal harvesting policy.

3.1 Setup and preliminaries

The definitions and concepts introduced in this section are based on standard results, which can be found in [15, 34].

In this section, we assume that the filtration $\mathbb{F}$ is the canonical filtration of the Brownian motion B. We will also denote by

$$S := [t, T] \times \mathbb{R}^d \text{ where } T \in [0, \infty],$$

this set is called the parabolic interior of the state set.

Definition 3.1. *(Control processes) Given a subset U of $\mathbb{R}^k$, we denote by $\mathcal{U}$ the set of all progressivelly measurable processes $u = \{u(s), s \in [t, T]\}$ valued in U. The elements of $\mathcal{U}$ are called control processes.*

Remark 7. *To ensure the existence of an optimal control, it is standard to assume that the set of control values U is a nonempty compact metric space.*

Definition 3.2. *(Controlled process)*

let

$$b : (s, x, v) \in S \times U \longrightarrow b(s, x, v) \in \mathbb{R}^d$$

and

$$\sigma : (s, x, v) \in S \times U \longrightarrow \sigma(s, x, v) \in \mathcal{M}_{\mathbb{R}}(n, d)$$

be continuous functions satisfying the conditions

$$|b(s, x, v) - b(s, y, v)| + |\sigma(s, x, v) - \sigma(s, y, v)| \le K|x - y|$$
$$|b(s, x, v) - \sigma(s, x, v)| \le K(1 + |x| + |v|),$$

for some K independent of (s, x, y, v). For each control process $u \in \mathcal{U}$, we consider the controlles stochastic differential equation:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))\, dB(s), & s \in [t, T], \\ X(t) = x. \end{cases} \tag{3.2}$$

If the above equation has a unique solution X, for a given initial data x, then the process X is called the controlled process, as its dynamics is driven by the action of the control process u.

we shall be working with the following subclass of control processes:

$$\mathcal{U}_0 := \mathcal{U} \cap \mathbb{H}^2,$$

where $\mathbb{H}^2$ is the collection of all progressively measurable processes with finite $\mathbb{L}^2(\omega \times [0,T))$ norm. Then, for every finite maturity $T' \leq T$, it follows from the above uniform Lipschitz condition on the coefficients b and σ that

$$\mathbb{E}\left[\int_t^{T'} (|b| + |\sigma|^2)(s, X(s), u(s))\, ds\right] \leq \infty \quad \text{for all } u \in \mathcal{U}_0,\ X \in \mathbb{R}^d,$$

which guarantees the existence of a controlled process on the time interval $\left[0, T'\right]$ for each given initial condition and control.

We want to control a process given by a stochastic differential equation (3.2). In a most typical optimal control problem we want to find a control $u(.)$ with values in some metric space U which satisfy certain measurability properties, called optimal, which minimizes a cost functional

$$J(t,x;u) = \mathbb{E}\left[\int_t^T l(s, X(s), u(s))ds + g(X(T))\middle| X(t) = x\right],$$

among all admissible controls for some functions $l : [t,T] \times \mathbb{R}^d \times U \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^d \longrightarrow \mathbb{R}$. The solution is obtained by solving the associated Hamilton–Jacobi–Bellman (HJB) equation for the so-called value function

$$V(t,x) = \inf_u J(t,x;u),$$

which is characterizing as a solution of a fully second-order semilinear partial differential equation (PDE). The link between the value function V and the HJB equation is established by appliying the stochastic version of Bellman's principle of optimality known as the dynamic programming principle (DPP). It is shown (Fleming and Soner, 2006; Yong and Zhou, 1999) that if the optimal cost function V is in $C^{1,2}([t,T] \times \mathbb{R}^d)$, then it is a solution the following terminal value problem of a second order PDE, known as HJB equation which takes the form

$$\begin{cases} \dfrac{\partial V(s,x)}{\partial s} + \inf\limits_u \left\{ \dfrac{1}{2} \mathrm{Tr}\left[\sigma(s,x,u)\sigma(s,x,u)^\top \nabla^2 V(s,x)\right] + \langle b(s,x,u), \nabla V(s,x)\rangle + l(s,x,u) \right\} = 0, \\ V(T,x) = g(x), \end{cases} \tag{3.3}$$

where ∇V and $\nabla^2 V$ denote the gradient and the Hessian of V, respectively. The term inside the brackets is the Hamiltonian. The optimal control $u^*(t,x)$ minimizes the Hamiltonian:

$$\mathcal{H}(s,x,u) = \frac{1}{2}\text{Tr}[\sigma(s,x,u)\sigma(s,x,u)^\top \nabla^2 V(s,x)] + \langle b(s,x,u), \nabla V(s,x)\rangle + l(s,x,u).$$

3.2 A Feynman–Kac type representation through FBSDEs

There is an innate relation between stochastic differential equations and second-order partial differential equations (PDEs) of parabolic or elliptic type. Specifically, solutions to a certain class of nonlinear PDEs can be represented by solutions to forward– backward stochastic differential equations (FBSDEs), in the same spirit as demonstrated by the well-known Feynman–Kac formulas (Karatzas and Shreve, 1991) for linear PDEs [14].

We begin by stating the definitions of forward and backward processes, and then proceed to link their solution with the solution of PDEs, in the framework of a nonlinear Feynman–Kac formula. As a forward process is the square-integrable, $\{\mathcal{F}_s\}_{s\in[t,T]}$ adapted, which for any given initial condition $(s,x) \in [t,T] \times \mathbb{R}^d$, satisfies the forward SDE (3.2).

The forward process X is also called the state process. We denote the solution to the forward SDE (3.2) by $X^{t,x}$, where (t,x) are the initial conditions parameters. Unlike the forward process, the associated backward process is the square-integrable, $\{\mathcal{F}_s\}_{s\in[t,T]}$ adapted pair (Y,Z), defined by a BSDE with a given terminal condition.

$$\begin{cases} dY(s) = -h(s, X^{t,x}(s), Y(s), Z(s))ds + Z^\top(s)\, dB(s), \quad s \in [t,T] \\ Y(T) = g(X(T)). \end{cases} \tag{3.4}$$

The function $h(.)$ is referred to as the generator or driver of the equation. The solution is implicitly determined by the initial condition parameters (t,x) of the forward SDE, as it satisfies the terminal condition $g(X^{t,x}(T))$ Accordingly, we will denote the solution components by $Y^{t,x}$ and $Z^{t,x}$. While FSDEs have a fairly straightforward definition, in the sense that both the SDE and the filtration evolve forward in time, this is not the case for BSDEs. Indeed, since solutions to BSDEs need to satisfy a terminal condition, integration needs to be performed backwards in time in some sense, yet the filtration still evolves forward in time. It turns out that a terminal value problem involving BSDEs admits an adapted solution if we back-propagate the conditional expectation of the process, that is, if we set

$$Y(s) \triangleq E[Y(s)|\mathcal{F}_s]$$

(Ma and Yong, 1999) [14]. Moreover, if the functions b, σ, h and g are deterministic meaning they do not explicitly depend on $\omega \in \Omega$ then the adapted solution possesses the Markov property it can be expressed as deterministic functions depending only on time and the state process (El Karoui, Peng, and Quenez, 1997).

Lemma 3.1. *[14](The Markovian property)*
There exist deterministic functions $V : [t, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ and $d : [t, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, such that the solution $(Y^{t,x}, Z^{t,x})$ of the BSDE (3.4) *is*
$Y^{t,x}(s) = V(s, X^{t,x}(s))$, $Z^{t,x}(s) = \sigma^\top(s, X^{t,x}(s))d(s, X^{t,x}(s))$, *for all $s \in [t, T]$.*

We now present the nonlinear Feynman–Kac-type formula, which links the solution of a class of PDEs to that of FBSDEs. Indeed, the following lemma can be proven by an application of Itô's formula (see El Karoui et al., 1997; Ma and Yong, 1999; Yong and Zhou, 1999):

Lemma 3.2. *(Nonlinear Feynman–Kac)*
Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial V}{\partial s} + \dfrac{1}{2}\mathrm{Tr}(\nabla^2 V \sigma(s,x)\sigma^\top(s,x)) + \nabla V^\top b(s,x) + h(s,x,V,\sigma^\top(s,x)\nabla V) = 0, \quad (s,x) \in [t,T) \times \mathbb{R}^d, \\ V(T,x) = g(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{3.5}$$

wherein the functions σ, b, h, and g satisfy mild regularity conditions. Then (3.5) *admits a unique (viscosity) solution $V : [t, T] \times \mathbb{R}^d \to \mathbb{R}$, which has the following probabilistic representation:*

$$V(t,x) = Y^{t,x}(t), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \tag{3.6}$$

where (X, Y, Z) is the unique adapted solution of the FBSDE system (3.2), (3.4). *Furthermore,*

$$(Y^{t,x}(s), Z^{t,x}(s)) = \left(V(s, X^{t,x}(s)),\, \sigma^\top(s, X^{t,x}(s))\nabla V(s, X^{t,x}(s))\right), \tag{3.7}$$

for all $s \in [t, T]$, and if (3.6) *admits a classical solution, then* (3.7) *provides that classical solution.*

3.3 Approximating the solution of FBSDEs

The solution of FBSDEs has been studied, to a great extent, independently from its connection to PDEs, mainly within the field of mathematical finance. Though several generic schemes exist (Bender and Denk, 2007; Bouchard and Touzi, 2004) [14].

3.3.1 The forward and backward process discretization

We start by defining a time grid $\{t = t_0 < \cdots < t_N = T\}$ for the interval $[t, T]$. Let $\Delta t_i \triangleq t_{i+1} - t_i$ be the $(i+1)$ th interval of the grid, which may be chosen to be uniform. Similarly, we denote $\Delta B_{i+1} \triangleq B(t_{i+1}) - B(t_i)$ the $(i+1)$ th Brownian motion increment. We also denote $X_i \triangleq X(t_i)$. The most basic discretization method for the forward stochastic process is the Euler–Maruyama scheme:

$$X_{i+1} \approx X_i + b(t_i, X_i)\Delta t_i + \sigma(t_i, X_i)\Delta B_{i+1}, \tag{3.8}$$

for $i = 0, \cdots, N-1$ and $X_0 = x$. To descretize the backward process we use the notation $Y_i \triangleq Y(t_i)$ and $Z_i \triangleq Z(t_i)$. Recall that, for adapted solutions to backward stochastic differential equations (BSDEs), it holds that $Y(s) \triangleq \mathbb{E}[Y(s)|\mathcal{F}_s]$ and $Z(s) \triangleq \mathbb{E}[Z(s)|\mathcal{F}_s]$ (i.e., a back-propagation of the conditional expectations). First we introduce the shorthand

$$\mathbb{E}(.|X_i) = \mathbb{E}(.|\mathcal{F}_{t_i}),$$

for the conditional expectation with respect to the σ - algebra $\mathcal{F}_{t_i} = \sigma(\hat{B}_k : t_0 \leq k \leq t_i)$ that is generated by the discrete Brownian motion $\hat{B}_i := \sum_{k \leq i} \Delta B_k$. So (3.4) can be approximate as follows:

$$Y_i = \mathbb{E}[Y_i|\mathcal{F}_{t_i}] \approx \mathbb{E}[Y_{i+1} + h(t_i, X_i, Y_i, Z_i)\Delta t_i | X_i], \tag{3.9}$$

In order to compute Y_i from Y_{i+1}, it is convenient to replace (Y_i, Z_i) on the right hand side by (Y_{i+1}, Z_{i+1}), so that we end up with the time stepping scheme

$$Y_i = \mathbb{E}[Y_i|\mathcal{F}_{t_i}] \approx \mathbb{E}[Y_{i+1} + h(t_i, X_i, Y_{i+1}, Z_{i+1})\Delta t_i | X_i], \tag{3.10}$$

for $i = T-1, \cdots, t$. Note that in the last equality, the term $Z_i^\top \Delta B_{i+1}$ in equation (3.4) is eliminated under the conditional expectation, as ΔB_{i+1} has zero mean. Additionally, $\mathcal{F}_{t_i}$ is replaced by X_i, by the Markov property. According to Equation (3.7), the Z process in Equation (3.4) corresponds to the term $\sigma^\top(s, X^{t,x}(s))\nabla V(s, X^{t,x}(s))$. Therefore we can express $Z_i = \mathbb{E}[Z_i|\mathcal{F}_{t_i}] = \mathbb{E}[\sigma(t_i, X_i)^\top \nabla V(t_i, X_i)|X_i] = \sigma^\top(t_i, X_i)\nabla V(t_i, X_i)$. The backward propagation is initialized with $Y(t) = g(X(T))$ and $Z(t) = \sigma^\top(T, X(T))\nabla V g(X(T))$, assuming $g(.)$ is differentiable almost everywhere. While there are several approaches to approximate the conditional expectation in (3.10), in this work we adopt the Least Squares Monte Carlo (LSMC) method, which we briefly review in what follows.

3.3.2 Conditional expectation estimation using LSMC

The Least Squares Monte Carlo (LSMC) method for approximating conditional expectations (Longstaff and Schwartz, 2001) addresses the general problem of numerically estimating conditional expectations of the form $\mathbb{E}(Y|X)$ for the square integrable random variables X and Y, if one is able to sample M copies of pairs (X, Y) [14]. This method is based on the idea that conditional expectation of a random variable can be represented as a function of the variable on which it is conditioned. That is:

$$\mathbb{E}(Y|X) = \phi^*(X),$$

where ϕ^* solves the infinite dimensional minimisation problem:

$$\phi^* = \arg\min_{\phi} \mathbb{E}\left[|\phi(X) - Y|^2\right],$$

with ϕ extends over all measurable functions with $\mathbb{E}\left[|\phi(X)|^2\right] < \infty$.
A finite-dimensional approximation of this problem can be obtained by expressing $\varphi(.)$ as a linear combination of basis functions:

$$\phi(.) = \sum_{i=1}^{K} \varphi_i(.)\alpha_i = \varphi(.)\alpha,$$

where $\varphi(.)$ is a row vector of predefined basis functions, we assume that the ϕ_i are continuously differentiable and α is a column vector of coefficients, and k is the dimension of the basis. Finally, the problem can be reduced to a linear least-squares problem if the expectation operator is replaced by its empirical estimator. This leads to:

$$\alpha^* = \arg\min_{\alpha\in\mathbb{R}^k} \frac{1}{M}\sum_{j=1}^{M} \left|\varphi(X^j)\alpha - Y^j\right|^2,$$

with (X^j, Y^j) , $j = 1, ..., M$ are independent samples of pairs (X, Y). The LSMC estimator assumes then the form

$$\mathbb{E}(Y|X = x) = \phi^*(x) \approx \varphi(x)\alpha$$

To approximate the conditional expectation in Equation (3.10) at each time step, we may apply the LSMC method using Monte Carlo simulation with generating M independent trajectories $\{X_i^m\}_{i=1,...,N}$ and the index $m = 1, ..., M$. The backward numerical scheme starts from the final time T and proceeds backward through the time grid until reaching the initial time. At each time t_i, we collect M sample pairs (Y_i^m, X_i^m) to perform a linear regression that estimates the conditional expectation of Y_i as a function of X_i. It gives an approximation of the value function V at time t_i. Since $V(t_i, x) = \mathbb{E}\left(Y_i | X_i = x\right) \approx \varphi(x)\alpha_i$ we can approximate:

$$Y_i^m = \mathbb{E}\left(Y_i^m | X_i^m\right) \approx \varphi(X_i^m)\alpha_i.$$

The approximation of the conditional expectation of Z_i is then obtained by taking the gradient of $V(t_i, x)$ with respect to x, evaluating it at X_i^m, and scaling it by σ, yielding:

$$Z_i^m \approx \sigma(t_i, X_i^m)^\top \nabla\varphi(X_i^m)\alpha_i$$

This process is repeated backward for $t_{i-1}, ..., t_0$. The algorithm is summarized as follows:

- Initialize $Y(T) = g(X(T))$, $Z(T) = \sigma(T, X(T))^\top \nabla g(X(T))$.
- Compute
$$\alpha_i^* = \arg\min_{\alpha_i} \left|\phi(X_i)\alpha_i - (Y_{i+1} + \Delta t_i h(t_{i+1}, X_{i+1}, Z_{i+1}))\right|^2 \tag{3.11}$$
- Then $Y_i = \phi(X_i)\alpha_i^*$, $Z_i^m = \sigma(t_i, X_i^m)^\top \nabla\varphi(X_i^m)\alpha_i^*$,

here $m = 1, ..., M$. The minimizer in Equation (3.11) can be obtained either by directly solving the normal equations or using gradient descent. The output of this algorithm is the set of coefficients α_i^*, which are the basis function approximations of the value function instant.
At time $T := N\Delta t$, the data are determined by the terminal cost:

$$\hat{Y}_N^{(m)} = g\left(X_N^{(m)}\right), \quad \hat{Z}_N^{(m)} = \sigma\left(\hat{X}_N^{(m)}\right)^\top \nabla g\left(X_N^{(m)}\right) \tag{49}$$

The unknowns coefficients α_k should be computed in every time step, which makes them functions of time, i.e. $\alpha_k = \alpha_k(t_i)$. We call $\hat{\alpha} = (\alpha_1, \ldots, \alpha_K)$, so the least-squares problem that has to be solved in the i-th step of the backward iteration is

$$\hat{\alpha}(t_i) = \arg\min_{\alpha\in\mathbb{R}^K} \|\mathcal{A}_i\alpha - b_i\|^2, \tag{3.12}$$

with coefficients

$$\mathcal{A}_i = \left(\phi_k\left(\hat{X}_i^{(m)}\right)\right)_{m=1,\ldots,M;\, k=1,\ldots,K} \tag{3.13}$$

and data

$$b_i = \left(Y_{i+1}^{(m)} + \Delta t\, h\left(t_i, X_i^{(m)}, Y_{i+1}^{(m)}, Z_{i+1}^{(m)}\right)\right)_{m=1,\ldots,M}. \tag{3.14}$$

Assuming that the coefficient matrix $\mathcal{A}_i \in \mathbb{R}^{M\times K}$, $K \leq M$ defined by (3.13) has maximum rank K, then the solution to (3.12) is given by

$$\hat{\alpha}(t_i) = \left(\mathcal{A}_i^\top \mathcal{A}_i\right)^{-1} \mathcal{A}_i^\top b_i. \tag{3.15}$$

3.4 Stochastic optimal harvesting policy

As for the optimal control in the deterministic case we are interested in maximize the net benefits and minimizing the energy of the effort devoted for harvesting so the generel form of the cost functional is

$$J(t,x,y;u) = \mathbb{E}\left[\int_t^T \left(-e^{-\delta s}(p\, m\, x(s) - c)u(s) + \frac{1}{2}u(s)^2\right) ds \middle| w(t) = (x,y)\right], \tag{3.16}$$

where $(w(s))_{s\in[t,T]} = (x(s),y(s))_{s\in[t,T]}$. The associated optimal cost is

$$V(t,x,y) = \inf_u J(t,x,y;u),$$

where V solves the following dynamic programming equation

$$\begin{cases} \inf_u \left\{ \frac{1}{2}\mathrm{Tr}\left[\sigma(x,y)\sigma(x,y)^\top \nabla^2 V(s,x,y)\right] \right. \\ \left. + \langle B(x,y,u), \nabla V(s,x,y)\rangle + f(s,x,u) - \dfrac{\partial V(s,x,y)}{\partial s} \right\} = 0, \\ V(T,x,y) = 0, \end{cases} \tag{3.17}$$

where, ∇V and $\nabla^2 V$ are the gradient and the Hessian of V, and

$$f(s,x,u) = e^{-\delta s}(c - p\, m\, x)u + \frac{1}{2}u^2, \quad \sigma(x,y) = \begin{pmatrix} \alpha\, x & 0 \\ 0 & -\beta\, y \end{pmatrix}$$

$$B(x,y,u) = \begin{pmatrix} a\, x\left(1 - \dfrac{x}{K}\right) - \bar{b}\dfrac{x\, y}{y+D} - m\, u\, x \\ -q\, y + e\, \bar{b}\dfrac{x\, y}{y+D} \end{pmatrix}.$$

We obtain the optimal control

$$u^* = \frac{\partial V(s,w(s))}{\partial x} m\, x(s) - e^{-\delta s}(c - p\, m\, x(s)),$$

and by replacing u^* in (3.17) we get

$$\begin{cases} \frac{1}{2}\mathrm{Tr}\left[\sigma(x,y)\sigma(x,y)^\top \nabla^2 V(s,x,y)\right] + A(x)^\top (\nabla V(s,x,y))^2 \\ + \langle \tilde{B}(s,x,y), \nabla V(s,x,y)\rangle + \tilde{f}(s,x) - \dfrac{\partial V(s,x,y)}{\partial s} = 0, \\ V(T,x,y) = 0. \end{cases} \tag{3.18}$$

in which

$$\tilde{f}(s,x) = -\frac{1}{2}e^{-2\delta s}(c - p\,m\,x)^2, \quad A(x) = \begin{pmatrix} -\frac{1}{2}m^2x^2 \\ 0 \end{pmatrix}$$

$$\tilde{B}(s,x,y) = \begin{pmatrix} a\,x\left(1 - \dfrac{x}{K}\right) - \bar{b}\dfrac{x\,y}{y+D} + m\,x\,e^{-\delta s}(c - p\,m\,x) \\ -q\,y + e\,\bar{b}\dfrac{x\,y}{y+D} \end{pmatrix}.$$

• FBSDE representation of the dynamic programming equation

Define the processes

$$Y(s) = V(w(s),s)\ ,\ Z(s) = \sigma(w(s))^\top \nabla_x V(w(s),s),$$

wherein $B(s) = \left(B^1(s), B^2(s)\right)$ and $Z(s) = \left(z^1(s), z^2(s)\right)$.

Applying Ito's formula to V, using that it is a classical solution to (3.18), we obtain

$$dY(s) = \left[-\tilde{f}(s,x) - A(x)^\top \left(\frac{\partial V(s,w(s))}{\partial x}\right)^2\right] ds + Z(s)dB(s)\ ,\ Y(T) = 0,$$

the associated BSDE is given by

$$dY(s) = -h\left(s, w(s), Z(t)\right) ds + Z(s)dB(s)\ ,\ Y(T) = 0, \tag{3.19}$$

using the fact that $Z(s) = \sigma(X(s))^\top \nabla_x V(s, X(s))$ and given that

$$h\left(s, w(s), Z(s)\right) = \left(\tilde{f}(s, x(s)) - \frac{1}{2\,\alpha^2} m^2 (z^1(s))^2\right), \tag{3.20}$$

for the pair (Y,Z). Note that, by definition, Y is continuous and adapted to X, and Z is predictable and a.s. square integrable, i.e,

$$\int_0^T ||Z(t)||^2 dt < \infty.\ \text{ a.s.}$$

According to the interpretation of $Z(s)$ as a control variable. By definition $Y(s)$ at time $t \leq s < T$ is measurable with respect to the filtration generated by the Brownian motion $(B_s)_{t \leq s \leq T}$, While the time-reversed version of $Y(s)$ should depend on B_T via the terminal condition $Y_T = g(X_T)$, Which requires greater filtration.

Using the BSDE (3.19) and by exploiting the fact that the forward process X is independent of (Y, Z), we obtain the following representation of the solution to the dynamic programming equation (3.18)

$$\begin{cases} dw(s) = \tilde{B}\left(s, w(s)\right) ds + \sigma\left(w(s)\right) dB(s), \quad w(t) = (x, y) \\ dY(s) = -h\left(s, w(s), Z(s)\right) ds + Z(s) dB(s) \quad Y_T = 0. \end{cases} \tag{3.21}$$

Note that the solution to (3.21) now is a triplet (X, Y, Z), and since Y is adapted, it follows that $Y(t)$ is a deterministic function of the initial value (x, t) only. Since g is a constant, the results in [24] require existence and uniqueness of (3.21)

$$Y(t) = V(x, t) \text{ a.s.}$$

- **Least-squares Monte Carlo**

We now discuss the numerical discretization of (3.21) and using the fact that FBSDE is decoupled implies that it can be discretized using an explicit time-stepping scheme. The least-squares Monte Carlo scheme is based on the Euler discretisation. Thus, consider the Euler–Maruyama discretization of (3.21)

$$\begin{cases} X_{i+1} = X_i + \tilde{B}\left(t_i, w_i\right) \Delta t + \sigma(w_i) \Delta B_{i+1}, \\ Y_{i+1} = Y_i - h(t_i, w_i, Z_i) \Delta t + Z_i \Delta B_{i+1}. \end{cases}$$

Where (X_i, Y_i, Z_i) denotes the numerical discretisation of the joint process (X, Y, Z). adopting the LSMC method on given in the subsection 3.3.2 we have the following algorithm proposed in [23].

Algorithm 1 Least-squares Monte Carlo

```
Define $K, M, N$ and $\Delta t = T/M$.
Set initial condition $x \in \mathbb{R}^d$.
Choose radial basis functions $\{\phi_k \in C^1(\mathbb{R}^d, \mathbb{R}) : k = 1, \ldots, K\}$.
Generate $M$ independent realisations $w^{(1)}, \ldots, w^{(M)}$ of length $N$ from
    $\hat{X}_{i+1} = \hat{X}_i + \tilde{B}\left(\hat{X}_i\right)\Delta t + \sigma\left(\hat{X}_i\right)\Delta B_{i+1}, \quad \hat{X}_0 = x.$
Initialise BSDE by
    $\hat{Y}_N^{(m)} = g(\hat{X}_N^{(m)}), \quad \hat{Z}_N^{(m)} = \sigma(\hat{X}_N^{(m)})^\top \nabla g(\hat{X}_N^{(m)}).$
for $n = N - 1 : 1$ do
    Assemble linear system $\mathcal{A}_n \hat{\alpha}(t_n) = b_n$ according to (3.12)–(3.14).
    Evaluate $\hat{Y}_n^{(m)}$ and $\hat{Z}_n^{(m)}$ according to
        $\hat{Y}_n^{(m)} = \sum_{k=1}^{K} \alpha_k(t_n)\phi_k(\hat{X}_n^{(m)}), \quad \hat{Z}_n^{(m)} = \sigma(\hat{X}_n^{(m)})^T \sum_{k=1}^{K} \alpha_k(t_n)\nabla\phi_k(\hat{X}_n^{(k)}).$
    If necessary, adapt basis functions $\phi_k$.
end for
```

For the basis we have chosen is a radial Gaussian basis of functions (RBF)

$$\phi(x) = \frac{1}{\sqrt{2\,\pi\,\sigma}} esp\left(\frac{-1}{2\,\sigma}||x - C||^2\right),$$

where C is the mean of the basis and σ is the variance. The basis is constructed such that at each point in time i and each dimension k we have 3 centers spaced around the average $\bar{w}(k, i)$ with an amplitude equal to half the width of the interval covered by the data $w(k, i, :)$. The centers are symmetric around $\bar{w}(k, i)$ the empirical mean of the trajectories at instant i.

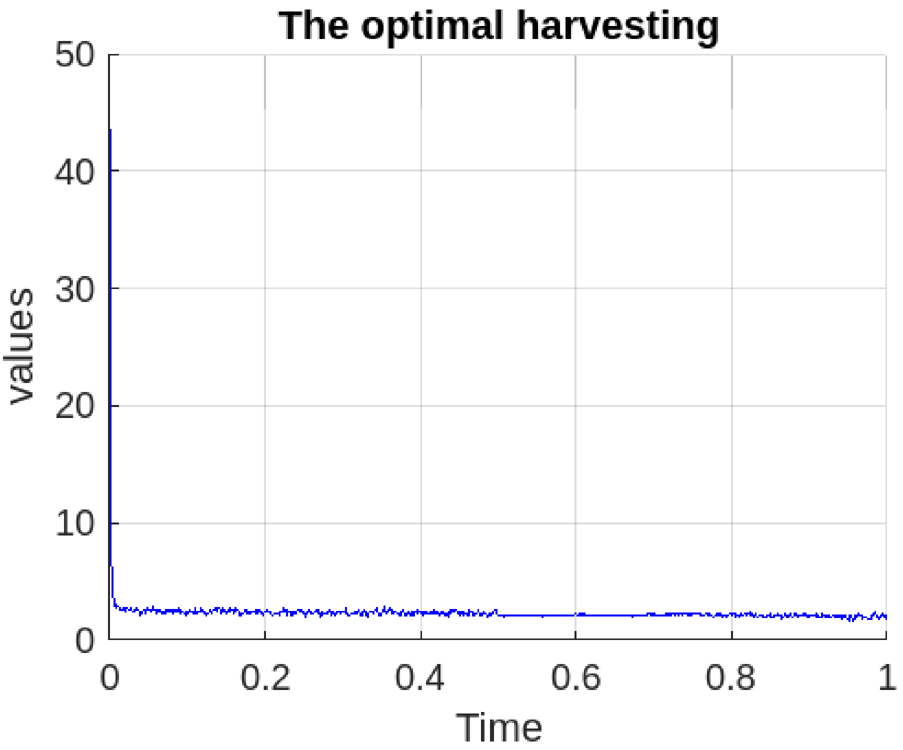

Figure 3.1: The optimal harvesting.

Figure 3.1 shows a plot of one realization of the optimal harvesting policy for the dynamic system (2.21) and the cost function (3.16), using the parameters $\delta = 0.01$, $c = 77$, $p = 1$, and $\sigma = 5$. We observe that the optimal control is high at $t = 0$ because, at the beginning, a significant effort is required to steer the system. Subsequently, the control fluctuates as it adjusts the system upward and downward in an attempt to reach an optimal state.

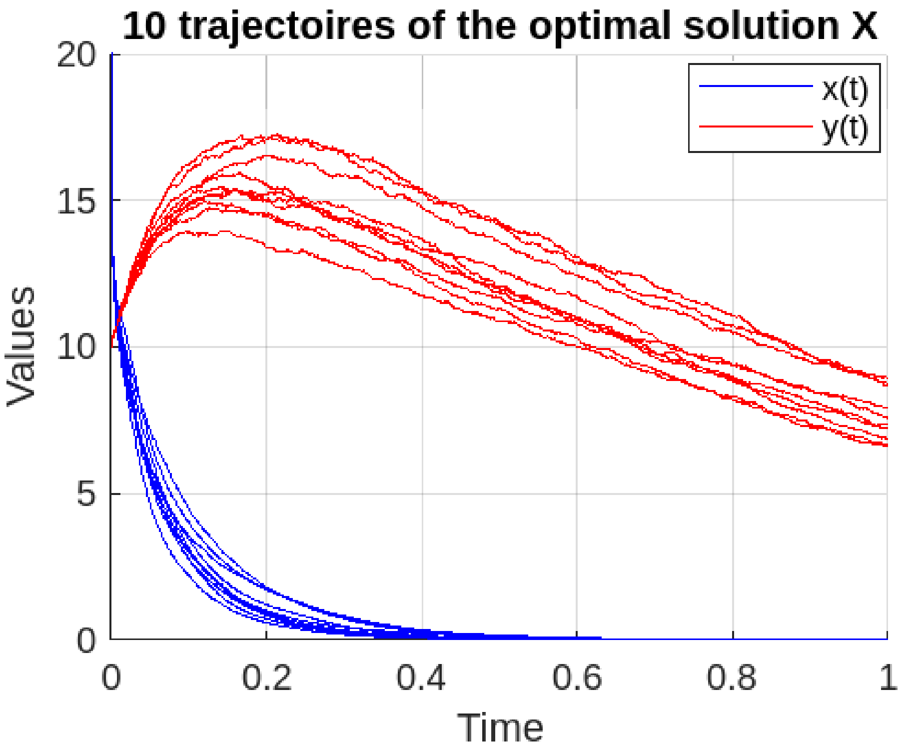

Figure 3.2: Trajectories of the optimal solution

Figure 3.2 presents a graphical representation of 10 trajectories of the optimal solution of the FSDE (3.21).

In the optimal control analysis of the stochastic predator-prey harvesting model, the results show that the optimal state trajectory $x(t)$ tends to extinction, even when the harvesting effort is minimal. This suggests that, under the current formulation, the harvesting strategy leads to over-exploitation of the resource. Therefore, it becomes necessary to either introduce a harvesting threshold (to ensure population survival), or to reformulate the optimal control problem with additional constraints, such as state constraints (e.g., maintaining the prey population above a critical level) or bounded control constraints to preserve sustainability.

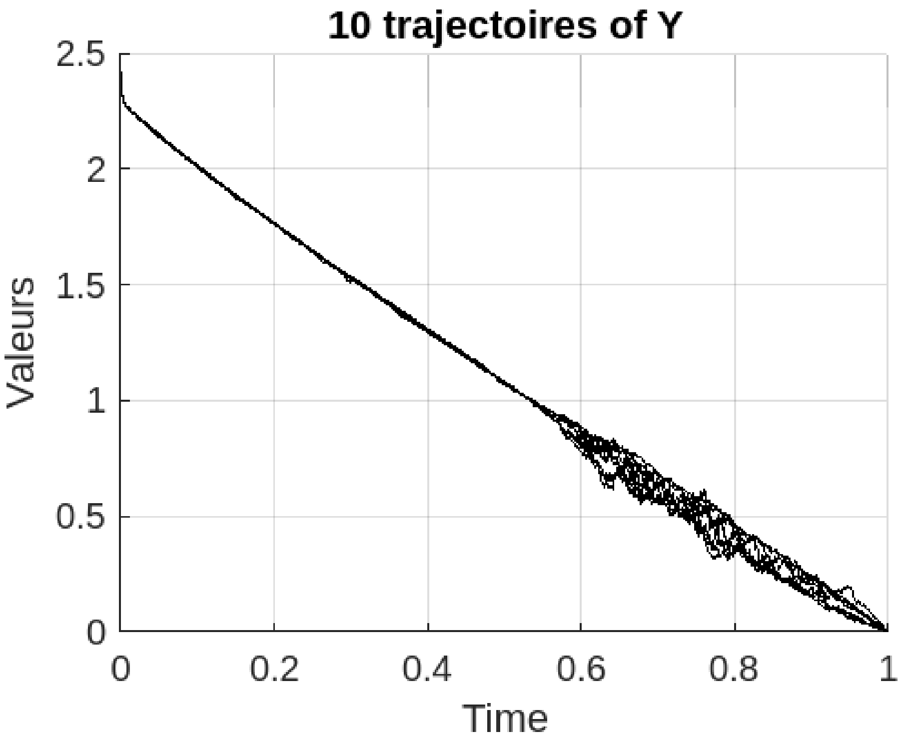

Figure 3.3: Trajectories of the optimal cost Y

Figure 3.2 presents a plot of 10 trajectories of the optimal cost of the BSDE (3.21). The optimal cost of our stochastic optimal control is

$$V(0, [20, 10]) = Y_0 = 2.4145.$$

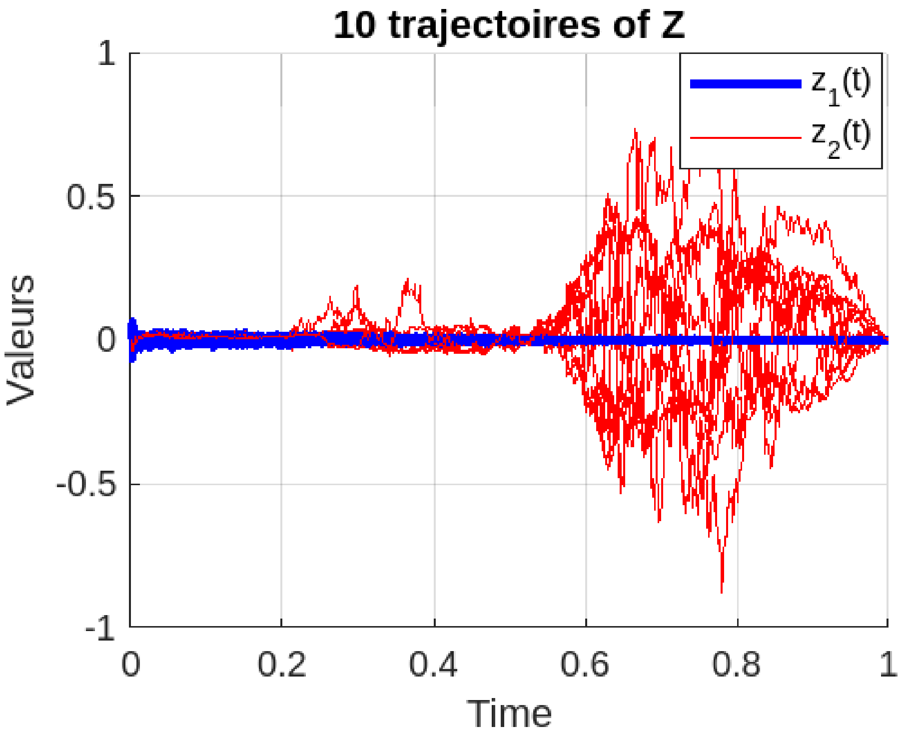

Figure 3.4: Trajectories of the control process Z

Figure 3.4 shows a graphical representation of 10 realizations of the control process Z. We can see that both z^1 and z^2 fluctuate around zero, but z^2 requires a greater effort because it depends on the predator population, which is larger in size compared to the prey population.

Conclusion

In this work, we have analyzed a fish predator-prey model with harvesting, incorporating the influence of water level fluctuations on the amount of food available to the predator. This study can be looked as an extension of previous work in the deterministic case.

The main modifications were the introduction of stochastic perturbations affecting the system's dynamic. The objective was to analyze the dynamical properties of the system such as the persistence, extinction and identify an optimal harvesting strategy that maximizes profit while minimizing the energy expended of harvesting.

Our findings confirm that the system's dynamics are highly sensitive to water level fluctuations. The stochastic case proves more realistic than the deterministic one, as it captures the unpredictability inherent in real-world ecosystems, where environmental variability plays an important role.

Both theoretical and numerical analyses were carried out. According to our results, particularly Theorems 2.4 and 2.5, when environmental noise (represented by the parameters α and β) has an impact, such that assumptions 1 and 2 are not satisfied, both populations will be extinct.

To address the harvesting strategy, we employed dynamic programming and a forward-backward stochastic differential equation (FBSDE). Finally, we used the Least Squares Monte Carlo (LSMC) method to approximate the expectations involved and to generate numerical simulations of both the optimal control and the corresponding optimal solution of the system. For our future work, we plan to define another optimal control with constraint that maximizes net revenue without leading to extinction, resulting in a more realistic objective.

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