République Algérienne Démocratique et Populaire

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université de Saida - Dr Moulay Tahar

Faculté des Sciences

Département de Mathématiques

Mémoire présenté en vue de l'obtention du diplôme de

Master Académique

Filiére : MATHEMATIQUES

Spécialité: Analyse Stochastique Statistique des Processus et Applications

par

Leboukh Younes Abdenour[1]

Sous la direction de

Dr. F. Mokhtari

Théme:

Integral Functional of Semi-Markov Process in Reliability Problems

Soutenue le 13/06/2024 devant le jury composé de

Pr. S.Rahmani	Université de Saida Dr. Moulay Tahar	Présidente
Dr. F.Mokhtari	Université de Saida Dr. Moulay Tahar	Encadreur
Dr. F.Benziadi	Université de Saida Dr. Moulay Tahar	Examinatrice

Année univ: 2023/2024

[1]e-mail: younesleboukh@gmail.com

Acknowledgments

In the Name of Allah, the Most Benevolent, the Most Cruel. I Praise be to Allah, and blessings and peace be upon His prophet and servant, Mohammed.

I must first and foremost express my sincere gratitude to Allah, the Ever-Magnificent, the Ever-Thankful, for all of His favors and assistance. Without His direction, I have no doubt that this work would never have come to pass.

I would like to thank **Dr. Fatiha Mokhtari**, my supervisor, from the bottom of my heart for her invaluable assistance during this master's thesis. Throughout this process, I have been so grateful for her support, helpful suggestions, invaluable guidance, encouragement, and patience.

I would like to extend my gratitude to all my teachers, including **F. Benziadi**, **Dr. S. Rahmani**, **A. kandouci**, and others, for their teachings and mentorship, which have greatly contributed to my academic and personal growth.

Special thanks are due to **Tayeb Hamlet**, a PhD student, for his invaluable assistance, support, and collaboration during this research endeavor.

May Allah bless and reward all those who have helped and supported me throughout this journey.

Dedication

In the name of Allah, the Most Gracious, the Most Merciful

I dedicate this thesis to the memory of my beloved **parents**, whose love and guidance continue to inspire me every day.

To my dear **mother** and **father**, who have returned to the embrace of Allah, I owe everything I am and everything I strive to be. Your endless sacrifices, unwavering faith, and constant encouragement have shaped my journey and inspired me to pursue my dreams. May Allah grant you the highest place in Jannah, and may your souls find eternal peace. This work is a testament to your everlasting love and the profound impact you have had on my life.

To **Dr. Fatiha Mokhtari**, for her invaluable support and guidance throughout this journey. To my brothers, **Islam** and **Farouk**, for their constant encouragement and unwavering support. To my **friends**, for always being there for me. To all my **teachers**, whose wisdom and dedication have been instrumental in my academic journey. Thank you all for being my pillars of strength.

Contents

Notations

$\mathbb{N}$	Set of positive natural numbers.
$\mathbb{R}_+$	Set of nonnegative real numbers.
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space.
$\mathbb{E}$	Expectation with respect to $\mathbb{P}$.
$\mathbf{E} = \{1, ..., s\}$	Finite state space.
$\mathcal{M}_E$	Set of real matrix on $\mathbb{E} \times \mathbb{E}$.
$\mathcal{M}_E(\mathbb{N})$	Matrix-valued functions defined on $\mathbb{N}$ **,** with values in $\mathcal{M}_E$.
$Z := (Z_k)_{k\in\mathbb{N}}$	Semi-Markov chain.
$Z := (Z_t)_{t\in\mathbb{R}^+}$	Semi-Markov process (SMP).
$(J, S) := (J_n, S_n)_{n\in\mathbb{N}}$	Markov renewal chain (MRC).
$J := (J_n)_{n\in\mathbb{N}}$	Visited states, embedded Markov chain (EMC).
$S := (S_n)_{n\in\mathbb{N}}$	Jump times.
$X := (X_n)_{n\in\mathbb{N}}$	Sojourn times.
M	Fixed censoring time.
$N(M)$	Number of jumps of Z in the time interval $[1, M]$.
$N_i(M)$	Number of visits to state i of the EMC, up to time M.
$N_{ij}(M)$	Number of transitions from state i to state j. of the EMC, up to time M.
$N_{ij}(k, M)$	Number of transitions from state i to state j of the EMC, up to time M, with sojourn time in state i equal to k.

$\mathbf{p} := (p_{ij})_{i,j\in E}$	Transition matrix of the EMC J.
$\mathbf{q} := (q_{ij}(k))_{i,j\in E,k\in\mathbb{N}}$	Semi-Markov kernel.
$\mathbf{Q} := (Q_{ij}(k))_{i,j\in E,k\in\mathbb{N}}$	Cumulated semi-Markov kernel.
$\mathbf{Q} := (Q_{ij}(t))_{i,j\in E,t\in\mathbb{R}^+}$	First derivative of semi Markov kernel.
$\mathbf{f} := (f_{ij}(k))_{i,j\in E,k\in\mathbb{N}}$	Conditional sojourn time in state i, before visiting state j.
$\mathbf{F} := (F_{ij}(k))_{i,j\in E,k\in\mathbb{N}}$	Conditional cumulative sojourn time distribution in state i, before visiting state j.
$\mathbf{F} := (F_{ij}(t))_{i,j\in E,t\in\mathbb{R}^+}$	Sojourn time distribution in state i, before visiting state j.
$\mathbf{h} := (h_i(k))_{i\in E,k\in\mathbb{N}}$	Sojourn time distribution in state i.
$\mathbf{H} := (H_i(k))_{i\in E,k\in\mathbb{N}}$	Cumulative distribution of sojourn time in state i.
$\mathbf{H} := (H_i(t))_{i\in E,t\in\mathbb{R}^+}$	Sojourn time distribution in state i.
$\overline{\mathbf{H}} := (H_i(k))_{i\in E,k\in\mathbb{N}}$	Survival function in state i.
$\mathbf{P} := (P_{ij})(k)_{i,j\in E,k\in\mathbb{N}}$	Transition function of the semi-Markov chain Z.
$\mathbf{P} := (P_{ij})(t)_{i,j\in E,t\in\mathbb{R}^+}$	Transition function of the semi-Markov process Z.
$\Psi(t) = (\Psi_{ij}(t))_{i,j\in E,t\in\mathbb{R}^+}$	Markov renewal matrix.
μ_{ij}	Mean first passage time from state i to state j, for the semi-Markov process Z.
μ^*_{ij}	Mean first passage time from state i to state j for the embedded Markov chain J .
$\nu = (\nu(j))_{j\in E}$	Stationary distribution of the EMC J.
$\alpha = (\alpha_i)_{i\in E}$	Initial distribution of semi-Markov process Z.
$A * B$	Discrete-time matrix convolution product of A, B.
$Q * \phi$	Stieltjes convolution of ϕ, Q.

$A^{(n)}$	n-fold convolution of $A \in \mathcal{M}_{\mathbf{E}}(\mathbb{N})$.
$\xrightarrow{\text{a.s}}$	Almost sure convergence (strong consistency).
$\xrightarrow{\mathcal{P}}$	Convergence in probability.
$\xrightarrow{D}$	Convergence in distribution.
δ_{ij}	Symbol of Kronecker.
$\mathbf{1}_A$	Indicatrice function of A.
$\mathcal{N}(0,\sigma^2)$	Standard normal random variable.
$DTMP$	Discrete-time Markov Process.
$CTMP$	Continuous-time Markov Process.
$DTSMP$	Discrete-time Semi-Markov Process.
SMP	Semi-Markov Process.
$CTSMP$	Continuous-time Semi-Markov Process.
$DTMRP$	Discrete-time Markov Renewal Process.
EMC	Embedded Markov Chain.
MLE	Maximum-Likelihood Estimator.
$SLLN$	Strong Law of Large Numbers.
CLT	Central Limit Theorem.

Introduction

The Semi-Markov Process (SMP) is a generalization of the Markov process that was introduced independently by Lévy (1954) [27] and Smith (1955) [37]. A Markov chain (MC) with a random change in the time scale can be considered as an SMP with a finite state space (Pyke [34], [35]). This means that the sojourn periods in each state can have arbitrary distributions that may be influenced by the next state visited. This generalization makes semi-Markov processes a powerful tool for modeling a wide range of stochastic systems where the assumption of exponentially or geometrically distributed sojourn times is too restrictive.

Semi-Markov processes play a crucial role in probability and statistical modeling, with applications in a variety of domains such as survival analysis, biology, reliability, DNA analysis, insurance and finance, earthquake modeling, meteorology studies, and so on; see, for example, Heutte and Huber-Carol (2002) [23], Ouhbi and Limnios (2003) [30], Chryssaphinou et al. (2008) [10], Janssen and Manca (2006) [24], and Votsi et al [41].

It's worth noting that semi-Markov theory is primarily researched in a continuous-time framework, with very few papers addressing the discrete-time scenario. For a continuous-time framework, see Limnios and Oprisan (2001)[28], and for a discrete-time framework, see Barbu and Limnios (2008)[3] and the references therein.

Several research papers have been published on the development of estimators and the investigation of their asymptotic qualities, namely convergence (or consistency) and normality asymptotic.

By integrating over sets of states or time intervals within the semi-Markov processes, integral functionals play a crucial role in characterizing various properties of the system, such as transition probabilities, state occupation times, and expected rewards. These functionals provided a systematic framework for analyzing the temporal evolution of the process and extracting meaningful information about its dynamic behavior.

The concept of functional integrals, also known as path integrals, was initially introduced by R. P. Feynman (1948) [15] and has since become a cornerstone in quantum mechanics and statistical mechanics. in the context of semi Markov process, they were presented by D. Silvestrov (1980) [36] and N. Limnios & G. Oprisan (2001) [28].

Over the years, integral functionals in semi-Markov processes have undergone continuous refinement and enhancement, propelled by advancements in probability theory, functional analysis, and stochastic modeling. Today, they stand as essential tools for understanding and modeling complex stochastic systems and have found applications in diverse fields such as finance, telecommunications, and reliability engineering.

The integration of integral functionals of semi-Markov processes with reliability analysis represents a powerful approach to studying the dynamic behavior and performance of complex systems over time. In the context of reliability, integral functionals can be used to model and analyze the reliability characteristics of systems subject to different types of failures, repairs, maintenance activities, and other operational events.

There are four chapters in this master memory.

In the first chapter, we provide some background and introduce basic concepts and properties related to the discrete time homogeneous Markov and continuous-time homogeneous Markov process. Additionally, we consider a homogeneous discrete-time finite state space semi-Markov model. We present its basic probabilistic properties and introduce their empirical estimators for the main characteristics (such as semi-Markov kernel, sojourn time distributions, transition probabilities, etc.). These estimations are derived by considering a sample path of the discrete-time semi-Markov process (DTSMP) in the time interval $[0, M]$ with M an arbitrarily chosen positive integer. At the end of this chapter, we provide the asymptotic properties of the estimators, including strong consistency and asymptotic normality.

In Chapter 2, we delve into continuous-time semi-Markov processes, presenting their fundamental probability properties and giving the empirical estimators for the semi-Markov kernel, renewal function, and the transition function. We also examine their asymptotic properties, focusing on convergence and asymptotic normality.

In Chapter 3, we begin by defining the integral functionals of semi-Markov processes. We discuss several concepts and theorems as covered in [21], [30], [31] , describing the reliability function, and addressing the asymptotic normality and convergence of the $R(t)$ estimation, along with its confidence interval.

In the final chapter, we will explore the utilization of the R programming language (`smmR and SemiMarkov` packages) for simulating and estimating semi-Markov processes (SMPs). This chapter will provide detailed guidance and practical examples to illustrate how SMPs can be simulated and assessed in R. It will delve into the estimation of the reliability function, a key measure for assessing the long-term reliability of systems. Additionally, the computation of the integral functional of SMPs will be explored, providing additional insights into the system's behavior and performance.

Chapter 1

Introduction to Markov process and discret time semi-Markov process

Markov processes are an important class of stochastic processes characterised by the property of having no memory, which means that the future evolution of the process depends only on its current state and not on its past history. This property makes Markov processes, essential for modelling real random processes, particularly in the fields of reliability and maintenance.

Discrete-time semi Markov processes (DTSMP) and discrete-time Markov renewal processes (DTMRP) generalise discrete-time Markov processes and renewal processes. In a discrete-time Markov process, the sejourn time in each state is geometrically distributed. However, semi-Markov processes allow the sejourn time to follow any distribution over $\mathbb{N}^*$, which provides greater flexibility and makes them more suitable for a variety of applications, This chapter covers the fundamental concepts, properties and theorems related to these processes.

1.1 Discrete state space Markov process

1.1.1 Definitions and examples

Definition 1.1.1 (Markov Chain). *Let $J = \{J_n; n \geq 0\}$ be a sequence of random variables defined on the same probability space $(\Omega, A, \mathbb{P})$ with values in $\mathbf{E}$ (state space), which is a finite or countable space. We say that J is a Markov Chain (MC) if, for all $i_1, i_2, \ldots, i_{n+1} \in \mathbf{E}$, we have:*

$$\mathbb{P}(\underbrace{J_{n+1} = i_{n+1}}_{\text{The future}} \mid \underbrace{J_1 = i_1, \ldots, J_n = i_n}_{\text{The past and the present}}) = \mathbb{P}(\underbrace{J_{n+1} = i_{n+1}}_{\text{The future}} \mid \underbrace{J_n = i_n}_{\text{The present}}).$$

Definition 1.1.2 (Homogeneous Markov chain). *A Markov chain is homogeneous if, for all $n \geq 0$, i and j in $\mathbf{E}$:*

$$\mathbb{P}(J_{n+1} = i \mid J_n = j) = \mathbb{P}(J_1 = i \mid J_0 = j).$$

In this case, we define

$p_{ij} = \mathbb{P}(J_1 = i \mid J_0 = j)$ for $i, j \in \mathbf{E}$, and p_{ij} is called the transition probability.

Definition 1.1.3 (Transition Matrix). *The matrix $\boldsymbol{p} = (p_{ij})_{i,j \in \mathbf{E}}$ is a stochastic matrix, i.e., for all $i, j \in \mathbf{E}$, $\boldsymbol{p} \geq 0$, and for all $i \in \mathbf{E}$, $\sum_{j \in \mathbf{E}} p_{ij} = 1$.*

Definition 1.1.4. *The distribution of J_0, denoted by $\alpha = (\alpha_1, ..., \alpha_s)$, is called the initial distribution of the Markov Chain.*

$$\alpha_i = \mathbb{P}\left(J_0 = i\right) \text{ for any state } i \in \mathbf{E}.$$

Theorem 1.1.1. *For all $m \geq 0$, the probability of transitioning from i to j in m steps is equal to the element (i, j) of the matrix $\boldsymbol{p}^m = p_{ij}^{(m)}$ (matrix product of $\boldsymbol{p}$ repeated m times).*

Example 1.1.1 (Homogeneous Markov Chain). *A frog climbs a ladder. Every minute, it can move up a rung with probability $1/2$, or go down a rung with probability $1/2$. The ladder has 5 rungs. If the frog reaches the top it immediately jumps down the ladder and starts again.*

We denote J_n as the position of the frog on the ladder. The state space is therefore $\mathbf{E} = \{0, 1, 2, ..5\}$. If at time n the frog is at level $x \in \{1, 2, 3, 4\}$ on the ladder, then at time $n+1$ it will be on rung $x + 1$ with a probability of $1/2$, or on rung $x - 1$ with the same probability, which is expressed as:

$$\mathbb{P}\left(J_{n+1} = x + 1 \mid J_n = x\right) = \frac{1}{2} \quad \left(= \mathbb{P}\left(J_1 = x + 1 \mid J_0 = x\right)\right)$$
$$\mathbb{P}\left(J_{n+1} = x - 1 \mid J_n = x\right) = \frac{1}{2} \quad \left(= \mathbb{P}\left(J_1 = x - 1 \mid J_0 = x\right)\right)$$

The probabilities do not depend on n, and the transition matrix is given by:

$$\mathrm{p_{ij}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \end{pmatrix}$$

If the frog is at state 5, then it can either transition to 4 or transition to state 0. The last row of the matrix is thus $(1/2, 0, 0, 0, 1/2, 0)$ *(again, this does not depend on time n).*
If the frog is at state 0, it can only transition to state 1. The first row of the matrix is therefore $(0, 1, 0, 0, 0, 0)$.

J_n *is indeed a homogeneous Markov chain, with the transition matrix* $\mathbf{p}$.

1.1.2 Graph associated with a transition matrix

Definition 1.1.5 (Graph Associated with a Transition Matrix). *To visualize the evolution of a homogeneous Markov chain, it is often useful to represent the transition matrix* $\mathbf{p}$ *of the Markov chain by a directed graph: The nodes of the graph are the possible states of the Markov chain, An arrow pointing from state i to state j indicates that there is a strictly positive probability that the next state in the Markov chain will be state j if it is currently in state i. We put the weight p_{ij} on the arrow going from state i to state j (Figure 1.1).*

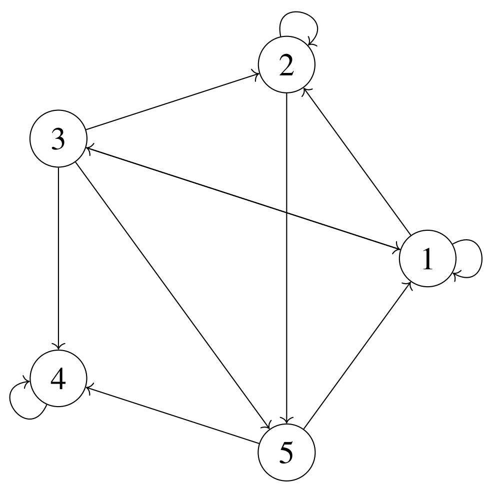

Figure 1.1: 5-state graph.

1.1.3 Characterization of a homogeneous Markov chain

Let $(J_n)_{n\in\mathbb{N}}$ be a homogeneous Markov chain. Then we have the following definitions:

Definition 1.1.6 (Sojourn Time). *Let (J_n) be a homogeneous Markov chain with transition matrix $p_{i,j}$. We denote by R_i the random variable equal to the sojourn time in state i. For every $k \in \mathbb{N}^*$,*

$$\mathbb{P}\left(R_i = k\right) = \mathbb{P}\left(J_{n+1} = i, \ldots, J_{n+k} = i, J_{n+k+1} \neq i \mid J_n = i\right).$$

Proposition 1.1.1. *[3] Let $(J_n)_{n\geq 0}$ be a Markov chain with transition function p. Then:*

$$\mathbb{P}\left(R_i = k\right) = (1 - p_{ii})p_{ii}^{(k)} \text{ (geometric distribution)}$$

and on the other hand, if $p_{ij} \neq 1$, we have, for $j \neq i$:

$$\mathbb{P}\left(J_{n+1} = j \mid J_n = i, J_{n+1} \neq i\right) = \frac{p_{ij}}{1 - p_{ii}}.$$

Suppose $p_{ii} \neq 1$. If the chain is in state i at time n, it stays there for an unspecified duration that follows a geometric distribution with parameter $1 - p_{ii}$. Note that n is a fixed and non-random time. However, it's worth noting that we can only deduce the sojourn time in state i as a geometric distribution with parameter $1 - p_{ii}$.

1.1.4 Classification of states

Definition 1.1.7 (reachable state). *We say that state j is reachable from state i, written as $i \to j$ if $p_{ij}^{(n)} > 0$. We assume every state is reachable from itself since $p_{ii}^{(0)} = 1$.*

Definition 1.1.8 (Communicate state). *Two states i and j are said to communicate, written as $i \leftrightarrow j$ if they are accessible from each other. In other words,*

$$i \leftrightarrow j \text{ means } i \to j \text{ and } j \to i.$$

Definition 1.1.9 (Absorbing Markov Chain). *A state $i \in E$ is said to be absorbing if $p_{ii} = 1$ (and therefore necessarily $p_{ij} = 0$ for any $j \neq i$; if the chain enters this state, it remains there with probability 1). A Markov chain is said to be absorbing if there exists, for any state of $\mathbb{E}$, an absorbing state accessible from this state. In an absorbing chain, any non-absorbing state is called transient.*

Definition 1.1.10 (irreducible Markov Chain). *A Markov chain is said to be irreducible if all states communicate with each other.*

Definition 1.1.11 (Recurrent, positive recurrent state). *A state is said to be recurrent if, any time that we leave that state, we will return to that state in the future with probability one. On the other hand, if the probability of returning is less than one, the state is called transient. A state that has a finite predicted time to return to from its current state is known as a positive recurrent state. Here, we provide a formal definition: For any state i, we define*

$$G_{ii} = \mathbb{P}\left(J_n = i, \textit{ for some } \quad n \geq 1 \mid J_0 = i\right).$$

We have that:

- *State i is **recurrent** if $G_{ii} = 1$.*
- *State i is **transient** if $G_{ii} < 1$.*

*A state i is called **positive recurrent** if it is recurrent ($G_{ii} = 1$) and the expected return time to state i is finite. More formally, let T_i be the return time to state i, defined as the smallest $n > 0$ such that $J_n = i$. The state i is positive recurrent if:*

$$\mathbb{E}[T_i \mid J_0 = i] < \infty.$$

Definition 1.1.12 (Periodic, aperiodic state). *A state $i \in \mathbf{E}$ is said to be periodic of period $d > 1$, or d-periodic, if d is equal to the greatest common divisor of all n such that $\mathbb{P}\left(J_{n+1} = i \mid J_1 = i\right) > 0$. If $d = 1$, then the state i is said to be aperiodic.*

Definition 1.1.13 (Ergodic state). *An aperiodic recurrent state is called ergodic. An irreducible Markov chain with one state ergodic (and then all states ergodic) is called ergodic.*

1.2 Continuous-time Markov process

Definition 1.2.1 (Continuous-time Markov process). *Let $(J(t))_{t\in\mathbb{R}_+}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in a measurable space $(\mathbf{E}, \varepsilon)$. Unless otherwise stated, we assume that $\mathbf{E} = \{1, 2, \ldots, s\}$ or $\mathbf{E} = \{1, 2, \ldots\}$.*

1. *A stochastic process $(J(t))_{t\in\mathbb{R}_+}$ is called continuous-time Markov process with the state space $\mathbf{E}$ if, for any $h, t \geq 0$ and $j \in \mathbf{E}$ we have:*

   $$\mathbb{P}\left(J(h+t) = j \mid J\left(h_1\right) = i_1, \ldots, J\left(h_n\right) = i_n, J(h) = i\right) = \mathbb{P}(J(h+t) = j \mid J(h) = i)$$

   $$0 \leq h_1 < \ldots < h_n < h, n \in \mathbb{N}, i_1, \ldots, i_n, i, j \in \mathbf{E}.$$

2. *If* $\mathbb{P}(J(h+t) = j \mid J(h) = i)$ *does not depend on* h*, then* $(J(t))_{t\in\mathbf{R}_+}$*is said to be homogeneous with respect to time.*

Definition 1.2.2 (Transition matrix). *Let* $(J(t))_{t\in\mathbb{R}_+}$ *be a homogeneous continuous-time Markov process with state space* E*. The functions defined on* $\mathbb{R}$ *by*

$$t \to p_{ij}(t) := \mathbb{P}(J(h+t) = j \mid J(h) = i), \quad i, j \in \mathbf{E}$$

are called transition functions of the process. The matrix $\mathbf{p}(t) = (p_{ij}(t))_{i,j\in\mathbf{E}}$ *is called the transition matrix (possibly infinite), and* $(p(t))_{t\in\mathbb{R}_+}$ *is called the transition semigroup of the continuous-time Markov process.*

Proposition 1.2.1. *[16] Let* T_i *be the waiting time in state* i*. The Chapman-Kolmogorov equation allows that* T_i *always has an exponential distribution with a parameter* $\lambda_i > 0$*,*

$$G_i(t) = \mathbb{P}(T_i \leq t) = 1 - e^{-\lambda_i t}, \quad t \geq 0, \quad i \in \mathbf{E}.$$

1.3 Markov renewal chain and semi-Markov chain

Let us consider :

- $\mathbf{E}$ the state space. We suppose $\mathbf{E}$ to be finite, with $|\mathbf{E}| = s$.
- The stochastic process $J = (J_n)_{n>0}$ with state space $\mathbf{E}$ for the system state at the n-th jump.
- The stochastic process $S = (S_n)_{n>0}$ with state space $\mathbb{N}$ for the n-th jump. We suppose $S_0 = 0$ and $0 < S_1 < S_2 < \ldots < S_n < S_{n+1} < \ldots$
- The stochastic process $X = (X_n)_{n>0}$ with state space $\mathbb{N}^\star$ for the sojourn time X_n in state J_{n-1} before the n-th jump. Thus, $X_n = S_n - S_{n-1}$, for all $n \in \mathbb{N}^*$.

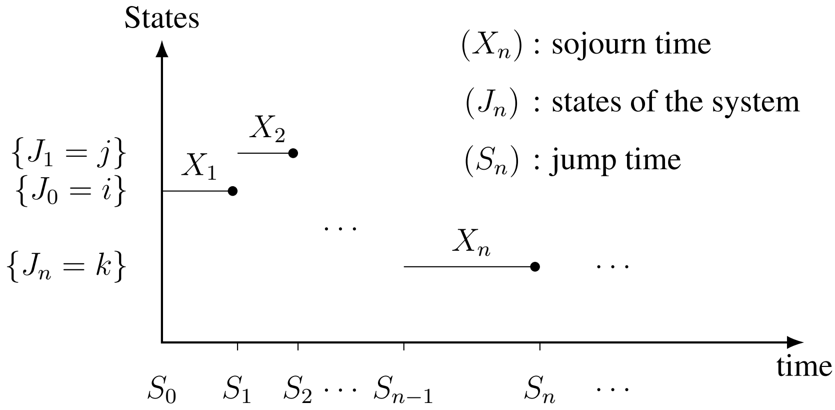

Figure 1.2: A typical sample path of a discrete time semi-Markov process.

Definition 1.3.1. *A matrix-valued function* $\mathbf{q} \in \mathcal{M}_{\mathbf{E}}(\mathbb{N})$ *is said to be a discrete-time semi-Markov kernel if it satisfies the following three properties:*

1. $0 \leq q_{ij}(\gamma) \leq 1$, $i, j \in \mathbf{E}$, $\gamma \in \mathbb{N}$
2. $q_{ij}(0) = 0$ *and* $\sum_{\gamma=0}^{\infty} q_{ij}(\gamma) \leq 1$, $i, j \in \mathbf{E}$
3. $\sum_{\gamma=0}^{\infty} \sum_{j\in\mathbf{E}} q_{ij}(\gamma) = 1$, $i \in \mathbf{E}$.

Definition 1.3.2 (Markov renewal chain). *The stochastic process* $(J, S) = ((J_n, S_n); n \in \mathbb{N})$ *is said to be a discrete-time Markov renewal process (DTMRP), for all* $n \in \mathbb{N}$*, for all* $i, j \in \mathbf{E}$*, and for all* $\gamma \in \mathbb{N}$*, it satisfies almost surely*

$$
\begin{aligned}
&\mathbb{P}\left(J_{n+1} = j, S_{n+1} - S_n = \gamma \mid J_0, \ldots, J_n; S_0, \ldots, S_n\right) \\
=&\mathbb{P}\left(J_{n+1} = j, S_{n+1} - S_n = \gamma \mid J_n\right).
\end{aligned}
\tag{1.1}
$$

Where $J = (J_n)_{n\in\mathbb{N}}$ *is a Markov chain with state space* $\mathbf{E}$*, called the embedded Markov chain of the MRC* (J, S)*.*

Furthermore, (J, S) *is considered homogeneous if Equation 1.1 is independent of* n*, with the discrete semi-Markov kernel* q *defined by*

$$q_{ij}(\gamma) = \mathbb{P}\left(J_{n+1} = j, X_{n+1} = \gamma \mid J_n = i\right).$$

Proposition 1.3.1.

(i) j *is recurrent for the MRP if and only if* j *is recurrent, necessarily positive in the embedded MC.*

(ii) j being transient for the MRP implies that j is also transient for the embedded MC.

The operation that will be commonly used when working on the space $\mathcal{M}_{\mathbf{E}}(\mathbb{N})$ of matrix-valued functions will be the discrete-time matrix convolution product. In the sequel, we recall its definition, and we define recursively the n-fold convolution.

Definition 1.3.3 (discrete time matrix convolution product). *Let $A, B \in \mathcal{M}_{\mathbf{E}}(\mathbb{N})$ two matrix-valued functions. The matrix convolution product $A*B$ is the matrix-valued function $C \in \mathcal{M}_{\mathbf{E}}(\mathbb{N})$ defined by*

$$C_{ij}(\gamma) := \sum_{k\in E} \sum_{l=0}^{\gamma} A_{ik}(\gamma - l) B_{kj}(l), \quad i, j \in \mathbf{E}, \quad \gamma \in \mathbb{N}.$$

Definition 1.3.4 (discrete time n-fold convolution). *Let $A \in \mathcal{M}_E(\mathbb{N})$ be a matrix-valued function and $n \in \mathbb{N}$. The n-fold convolution of A is the matrix function $A^{(n)} \in \mathcal{M}_{\mathbf{E}}(\mathbb{N})$ defined recursively by:*

$$\begin{aligned}
A_{ij}^{(0)}(\gamma) &:= \begin{cases} 1 & \textit{if} \quad \gamma = 0 \quad \textit{and} \quad i = j, \\ 0 & \textit{elsewhere} \end{cases}, \\
A_{ij}^{(1)}(\gamma) &:= A_{ij}(\gamma), \\
A_{ij}^{(n)}(\gamma) &:= \underbrace{(A * A * \ldots * A)_{ij}}_{n-\ \textit{times}} \\
&= \sum_{k\in\mathbf{E}} \sum_{l=0}^{\gamma} A_{ik}(l) A_{kj}^{(n-1)}(\gamma - l), \quad n \geq 2, \gamma \in \mathbb{N}.
\end{aligned}$$

Lemme 1.1. *Let $\delta I = (d_{ij}(\gamma); i, j \in \mathbf{E}) \in \mathcal{M}_{\mathbf{E}}(\mathbb{N})$ be the matrix - valued function defined by*

$$d_{ij}(\gamma) := \begin{cases} 1 & \textit{if} \quad i = j \quad \textit{and} \quad \gamma = 0 \\ 0 & \textit{elsewhere.} \end{cases}$$

Then, δI satisfies

$$\delta I * A = A * \delta I = A, \quad A \in \mathcal{M}_{\mathbf{E}}(\mathbb{N}), \ \textit{i.e.,}$$

δI is the neutral element for the discrete time matrix convolution product.

For a DTMRP (J, S), the n-fold convolution of the semi-Markov kernel q can be expressed as follows.

Proposition 1.3.2. *[3] For all $i, j \in \mathbf{E}$, for all n and $\gamma \in \mathbb{N}$, we have*

$$q_{ij}^{(n)}(\gamma) = \mathbb{P}\left(J_n = j, S_n = \gamma \mid J_0 = i\right).$$

Proof. For $n = 0$, we have

$$q_{ij}^{(0)}(\gamma) = \mathbb{P}\left(J_0 = j, S_0 = \gamma \mid J_0 = i\right).$$

Thus the result follows. For $n = 1$, we have

$$q_{ij}^{(1)}(\gamma) = \mathbb{P}\left(J_1 = j, S_1 = \gamma \mid J_0 = i\right).$$

Since $S_1 = X_1$, we have $q_{ij}^{(1)}(\gamma) = q_{ij}(\gamma)$.

For $n \geq 2$:

$$\begin{aligned}
&\mathbb{P}\left(J_n = j, S_n = \gamma \mid J_0 = i\right) \\
&= \sum_{k\in\mathbf{E}}\sum_{l=1}^{\gamma-1} \mathbb{P}\left(J_n = j, S_n = \gamma, J_1 = k, S_1 = l \mid J_0 = i\right) \\
&= \sum_{k\in\mathbf{E}}\sum_{l=1}^{\gamma-1} \mathbb{P}\left(J_n = j, S_n = \gamma \mid J_1 = k, S_1 = l, J_0 = i\right) \\
&\quad \times \mathbb{P}\left(J_1 = k, S_1 = l \mid J_0 = i\right) \\
&= \sum_{k\in\mathbf{E}}\sum_{l=1}^{\gamma-1} \mathbb{P}\left(J_{n-1} = j, S_{n-1} = \gamma - l \mid J_0 = k\right) \\
&\quad \times \mathbb{P}\left(J_1 = k, X_1 = l \mid J_0 = i\right) \\
&= \sum_{k\in\mathbf{E}}\sum_{l=1}^{\gamma-1} q_{kj}^{(n-1)}(\gamma - l) q_{ik}(l) = q_{ij}^{(n)}(\gamma),
\end{aligned}$$

thus the result follows. □

Definition 1.3.5 (Discrete-time semi-Markov chain). *The stochastic process $Z = (Z_\gamma; \gamma \in \mathbb{N})$ is said to be a discrete time semi-Markov process associated with the DTMRP (J, S), if*

$$Z_\gamma = J_{N_\gamma}, \gamma \in \mathbb{N},$$

where $N_\gamma := \sup\{n \geq 0; S_n \leq \gamma\}$ is the discrete time counting process of the number of jumps in $[1, \gamma] \subset \mathbb{N}$. Thus, Z_γ gives the system state at time γ. We have also $J_n = Z_{S_n}, n \in \mathbb{N}$.

Example 1.3.1. *Let $\mathcal{F}$ be a textile factory. Prior to being disposed of in river $\mathcal{R}$, its waste is processed by treatment unit $\mathcal{U}$. In order to avoid having to halt production in the event of a failure in $\mathcal{U}$, a waste storage depot $\mathcal{D}$ has been created. A trash storage depot $\mathcal{D}$ has been created in order to act as a buffer between $\mathcal{F}$ and $\mathcal{U}$. If there is a breakdown in $\mathcal{U}$ and it is fixed before $\mathcal{D}$ fills up, then $\mathcal{D}$ continues to run on a regular basis. Thus, $\mathcal{D}$ continues to run on a* *regular basis, and $\mathcal{D}$ is assumed to empty immediately upon fixing $\mathcal{U}$. instantaneously, once $\mathcal{U}$ is determined. If $\mathcal{F}$ production is not corrected, then it must end.*

Let $\mathbf{E} = \{1, 2, 3\}$ be the set of possible states of the system:

- 1 *: all is well ($\mathcal{F}$ and $\mathcal{U}$ are operational, $\mathcal{D}$ is empty).*
- 2 *: failure of $\mathcal{U}$, but $\mathcal{D}$ is not yet full ($\mathcal{F}$ is therefore still operational).*
- 3 *: failure of $\mathcal{U}$, $\mathcal{D}$ is full, and $\mathcal{F}$ is not operational.*

We observe that :

- $1 \rightsquigarrow 2$ *(if $\mathcal{U}$ fails).*
- $2 \rightsquigarrow 3$ *(if $\mathcal{D}$ is full).*
- $2 \rightsquigarrow 1$ *(If the failure of $\mathcal{U}$ is resolved before $\mathcal{D}$ becomes full).*
- $3 \rightsquigarrow 1$ *(When $\mathcal{U}$ is repaired).*

We thus have a process J_n, which represents the successive states of the system, and which is defined by:

- *an initial distribution $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where $0 < \alpha_1, \alpha_2, \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$,*
- *a transition matrix (as 1 only communicates with 2, 2 with 1 and 3, and 3 with 1):*

$$\mathrm{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$$

However, to model the system of interest properly, state changes alone are not sufficient: we also need to consider the time X_n that the system spends in each state (from which, for example, a production estimate will result). Suppose, therefore, that we can discretize time (for example, by taking an hour as the time unit), and that state changes can only occur at these moments. Let's introduce, for all $k \in \mathbb{N}$:

$$\begin{pmatrix} 0 & f_{12}(k) & 0 \\ f_{21}(k) & 0 & f_{23}(k) \\ f_{31}(k) & 0 & 0 \end{pmatrix}$$

where:

- f_{12} *(.) is the distribution of the time it takes for* $\mathcal{U}$ *to break down.*

- f_{21} *(.) is the distribution of the time it takes* $\mathcal{U}$ *to be repaired.*

- f_{23} *(.) is the distribution of the filling time of* $\mathcal{D}$.

- f_{31}*(.) is the distribution for the time needed to repair* $\mathcal{U}$*, and therefore for* $\mathcal{F}$ *to work again, once* $\mathcal{D}$ *is full.*

The sojourn time in a state therefore depends on the state of the system. We can then choose the discrete laws (Poisson, geometric, etc.) that best fit the real experiment (using statistical estimates, for example).

We can then define $S_n = \sum_{k=0}^{n} X_k$*, which is the time of the* n*th change of state. The system is therefore defined by two pieces of data:*

- *A Markov process* J_n*, with initial distribution* π_0 *and transition matrix* P*, which represents the state of the system after the* n*th change of state.*
- *A process* S_n *which represents the time of the* n*th change of state.*

Under certain assumptions, which appears up above, the pair (J_n, S_n) *forms what is known as a MRC. The process* $Z_n = J_{N(n)}$*, where* $N(n) = \max\{k \in \mathbb{N}, S_k \leq n\}$*, which represents the state of the system at time* n*, will then be what we call a semi-Markov chain.*

Definition 1.3.6. *The transition matrix* $\boldsymbol{p} = (p_{ij}; i, j \in \mathbf{E}) \in \mathcal{M}_{\mathbf{E}}$ *of* (J_n) *is defined by*

$$p_{ij} = \mathbb{P}\left(J_{n+1} = j \mid J_n = i\right), i, j \in \mathbf{E}, n \in \mathbb{N}.$$

Note that, for any $i, j \in \mathbf{E}$*,* p_{ij} *can be expressed in terms of the semi-Markov kernel by*

$$p_{ij} = \sum_{k=0}^{\infty} q_{ij}(k).$$

Example 1.3.2. *A Markov chain with the transition matrix* $(p_{ij}; i, j \in \mathbf{E})$ *is a particular case of a semi-Markov chain with semi-Markov kernel* $(q_{ij}; i, j \in \mathbf{E})$

$$q_{ij}(\gamma) = \begin{cases} p_{ij}(p_{ii})^{\gamma-1} & \text{if } i \neq j \text{ and } \gamma \in \mathbb{N}^* \\ 0 & \text{elsewhere} \end{cases}.$$

Here, p_{ij} represents the probability of directly transitioning from state i to state j in one time unit in the Markov chain, and $(p_{ii})^{\gamma-1}$ represents the probability of remaining in state i for $\gamma - 1$ time units, followed by a transition to state j in the γ-th time unit.

This definition of the semi-Markov kernel ensures that the transition probabilities $q_{ij}(\gamma)$ match the transition probabilities of the Markov chain for all $i, j \in \mathbf{E}$ and $\gamma \in \mathbb{N}^*$.

Therefore, a Markov chain with the given transition matrix can be considered a particular case of a Semi-Markov chain with the specified semi-Markov kernel.

Definition 1.3.7. *The cumulated semi-Markov kernel,* $\boldsymbol{Q} = (Q_{ij}(k), k \in \mathbb{N}) \in \mathcal{M}_{\mathbf{E}}(\mathbb{N})$ *defined, for all* $i, j \in \mathbf{E}$ *and for all* $k \in \mathbb{N}$*, by*

$$Q_{ij}(k) = \mathbb{P}(J_{n+1} = j, X_{n+1} \leq k | J_n = i) = \sum_{l=0}^{k} q_{ij}(l).$$

and the Markov renewal matrix $\psi := (\psi(\gamma); \gamma \in \mathbb{N}) \in \mathcal{M}_{\mathbf{E}}(\mathbb{N})$*, defined by*

$$\psi_{ij}(\gamma) := \sum_{n=0}^{\gamma} q_{ij}^{(n)}(\gamma), i, j \in \mathbf{E}, \gamma \in \mathbb{N}.$$

Let $\left(S_\gamma^j; \gamma \in \mathbb{N}^*\right)$ *be the successive passage times in a fixed state* $j \in \mathbf{E}$*. For an arbitrary state* $i \in \mathbf{E}$*, we consider the distribution of the first hitting time of state* j*, starting from state* i

$$g_{ij}(\gamma) = \mathbb{P}_i\left(S_1^j = \gamma\right), \gamma \geq 1.$$

Let us also denote by μ_{jj} *the mean recurrence time of state* j *for the associated semi-Markov process* $(Z_\gamma; \gamma \in \mathbb{N})$*, i.e.,* μ_{jj} *represents the mean of* g_{jj}*,* $\mu_{jj} = \mathbb{E}_j\left(S_1^j\right) = \mathbb{E}\left(S_2^j - S_1^j\right)$*.*

Definition 1.3.8. *The matrix renewal function* $\Psi = (\Psi(\gamma); \gamma \in \mathbb{N}) \in \mathcal{M}_E(\mathbb{N})$ *of the DTMRP is defined by*

$$\Psi_{ij}(\gamma) = \mathbb{E}_i\left[N_j(\gamma)\right], i, j \in E, \gamma \in \mathbb{N},$$

where $N_j(\gamma)$ *is the number of visits to state* j *before time* γ*. The matrix renewal function can be expressed in the following form:*

$$\Psi_{ij}(\gamma) = \sum_{n=0}^{\gamma} Q_{ij}^{(n)}(\gamma), i, j \in E, \gamma \in \mathbb{N}.$$

Indeed,

$$\mathbb{E}_i\left[N_j(\gamma)\right] = \sum_{n=0}^{\gamma} \mathbb{P}\left(J_n = j; S_n \leq \gamma \mid J_0 = i\right) = \sum_{n=0}^{\gamma} Q_{ij}^{(n)}(\gamma).$$

We have the following relation between the matrix-valued functions Ψ *and* ψ*:*

$$\Psi_{ij}(\gamma) = \sum_{k=0}^{\gamma} \psi_{ij}(k).$$

1.3.1 Sojourn times

Definition 1.3.9 (conditional distributions of sojourn times). *For all $i, j \in \mathbf{E}$, let us define:*

1. $f_{ij}(.)$ *, the conditional distribution of* $X_{n+1}, n \in \mathbb{N}$

$$f_{ij}(\gamma) := \mathbb{P}\left(X_{n+1} = \gamma \mid J_n = i, J_{n+1} = j\right), \gamma \in \mathbb{N}.$$

2. $F_{ij}(.)$ *, the conditional cumulative distribution of* $X_{n+1}, n \in \mathbb{N}$

$$F_{ij}(\gamma) := \mathbb{P}\left(X_{n+1} \leq \gamma \mid J_n = i, J_{n+1} = j\right) = \sum_{l=0}^{k} f_{ij}(l), \quad \gamma \in \mathbb{N}.$$

Obviously, for all $i, j \in \mathbf{E}$ and for all $\gamma \in \mathbb{N}$, we have

$$f_{ij}(\gamma) = \begin{cases} q_{ij}(\gamma)/p_{ij} & \text{if } \; p_{ij} \neq 0. \\ \mathbf{1}_{\{\gamma = \infty\}} & \text{if } \; p_{ij} = 0. \end{cases}$$

Definition 1.3.10 (sojourn time distributions in a given state). *For all $i \in \mathbf{E}$, let us define:*

1. $h_i(\cdot)$*, the sojourn time distribution in state i:*

$$h_i(k) := \mathbb{P}\left(X_1 = k \mid J_0 = i\right) = \sum_{j \in E} q_{ij}(k), k \in \mathbb{N}^*.$$

2. $H_i(\cdot)$*, the sojourn time cumulative distribution function in state i:*

$$H_i(k) := \mathbb{P}\left(X_1 \leq k \mid J_0 = i\right) = \sum_{l=1}^{k} h_i(l), k \in \mathbb{N}^*.$$

For G the cumulative distribution of a certain random variable X, we denote the survival function by $\overline{G(n)} := 1 - G(n) = P(X > n)$, $n \in \mathbb{N}$. Thus for all states $i, j \in \mathbf{E}$ we establish $\overline{F}_{ij}$ and $\overline{H}_i$ as the corresponding survival functions.

Example 1.3.3. *Consider the following DNA sequence of HEV (hepatitis E virus):*

$AGGCAGACCACATATGTGGTCGATGCCATGGAGGCCCATCAGTTTATTA$

$AGGCTCCTGGCATCACTACTGCTATTGAGCAGGCTGCTCTAGCAGCGGC$

$CATCCGTCTGGACACCAGCTACGGTACCTCCGGGTAGTCAAATAATTCC$

$GAGGACCGTAGTGATGACGATAACTCGTCCGACGAGATCGTCGCCGGT$

Suppose that the bases $\{A, C, G, T\}$ are independent of each other and have the same probability of appearing in a location, which is equal to $1/4$. Thus the occurrences of one of them, say, C*, form a renewal chain.*

The common distribution of $(X_n)_{n\in\mathbb{N}^*}$ is called the waiting time distribution of the renewal chain. Denote it by $f = (f_n)_{n\in\mathbb{N}}$, $f_n := \mathbb{P}(X_1 = n)$, with $f_0 := 0$, and denote by F the cumulative distribution function of the waiting time, $F(n) := \mathbb{P}(X_1 \leq n)$.
Set $\bar{f} := \sum_{n>0} f_n \leq 1 = \mathbb{P}(X_1 < \infty)$ for the probability that a renewal will ever occur.

Definition 1.3.11. *The transition function of the semi-Markov process Z is the matrix-valued function $P \in \mathcal{M}_{\mathbf{E}}(\mathbb{N})$ defined by:*

$$P_{ij}(\gamma) = \mathbb{P}(Z_\gamma = j \mid Z_0 = i), i, j \in \mathbf{E}, \gamma \in \mathbb{N}.$$

Proposition 1.3.3. *For all $i, j \in \mathbf{E}$ and for all $\gamma \in \mathbb{N}$, we have:*

$$P_{ij}(\gamma) = \mathbf{1}_{\{i=j\}}(\gamma)\left[1 - H_i(\gamma)\right] + \sum_{k\in\mathbf{E}} \sum_{l=0}^{\gamma} q_{ik}(l) P_{kj}(\gamma - l), \tag{1.2}$$

where

$$\mathbf{1}_{\{i=j\}}(\gamma) = \begin{cases} 1 \ \textit{if} \ i = j \ \textit{and} \ \gamma \geq 0. \\ 0 \ \textit{elsewhere.} \end{cases}$$

Let us define for all $\gamma \in \mathbb{N}$:

- $I(\gamma) = \left(\mathbf{1}_{\{i=j\}}(\gamma); i, j \in \mathbf{E}\right), I = (I(\gamma); \gamma \in \mathbb{N})$,
- $H(\gamma) = \operatorname{diag}(H_i(\gamma); i \in \mathbf{E}), H = (H(\gamma); \gamma \in \mathbb{N})$.

In matrix-valued function notation, Equation 1.2 becomes

$$P = I - H + q * P.$$

Proof. For all $i, j \in \mathbf{E}$ and for all $\gamma \in \mathbb{N}$, we have

$$\begin{aligned}
&P_{ij}(\gamma) \\
&= \mathbb{P}\left(Z_\gamma = j, S_1 > \gamma \mid Z_0 = i\right) + \mathbb{P}\left(Z_\gamma = j, S_1 \leq \gamma \mid Z_0 = i\right) \\
&= \mathbf{1}_{\{i=j\}}(\gamma)\left(1 - H_i(\gamma)\right) \\
&+ \sum_{k \in \mathbf{E}} \sum_{l=0}^{\gamma} \mathbb{P}\left(Z_\gamma = j, Z_{S_1} = k, S_1 = l \mid Z_0 = i\right) \\
&= \mathbf{1}_{\{i=j\}}(\gamma)\left(1 - H_i(\gamma)\right) \\
&+ \sum_{k \in \mathbf{E}} \sum_{l=0}^{\gamma} \mathbb{P}\left(Z_\gamma = j \mid Z_{S_1} = k, S_1 = l, Z_0 = i\right) \\
&\quad \times \mathbb{P}\left(J_1 = k, S_1 = l \mid J_0 = i\right) \\
&= \mathbf{1}_{\{i=j\}}(\gamma)\left(1 - H_i(\gamma)\right) \\
&+ \sum_{k \in \mathbf{E}} \sum_{l=0}^{\gamma} \mathbb{P}\left(Z_{\gamma-l} = j \mid Z_0 = k\right) \mathbb{P}\left(J_1 = k, X_1 = l \mid J_0 = i\right) \\
&= \mathbf{1}_{\{i=j\}}(\gamma)\left(1 - H_i(\gamma)\right) + \sum_{k \in \mathbf{E}} \sum_{l=0}^{\gamma} P_{kj}(\gamma - l) q_{ik}(l).
\end{aligned}$$

□

The following assumptions concerning the Markov renewal chain will be needed in the rest of this work.

A1 The Markov chain $(J_n)_{n\in\mathbb{N}}$ is irreducible.

A2 The mean sojourn times are finite, i.e.,$\sum_{n\geq 0}(1 - H_i(n)) < \infty$, for any state $i \in \mathbf{E}$.

A3 The Markov renewal process $(J_n, S_n)_{n\in\mathbb{N}}$ is aperiodic.

Proposition 1.3.4. *If the Assumptions* ***A1*** *and* ***A3*** *is satisfied, then for a fixed $j \in \mathbf{E}$, we have*

$$\lim_{\gamma\to\infty} \psi_{jj}(\gamma) = \frac{1}{\mu_{jj}}.$$

Definition 1.3.12. *For a discrete time semi-Markov process $(Z_\gamma; \gamma \in \mathbb{N}$), the limit distribution $(\pi_j; j \in \mathbf{E})$ is defined by*

$$\pi_j = \lim_{\gamma\to\infty} P_{ij}(\gamma), i, j \in \mathbf{E}.$$

Proposition 1.3.5. *For an aperiodic DTMRP and under Assumption* ***A1*** *and* ***A2****, the limit distribution is given by*

$$\pi_j = \lim_{\gamma\to\infty} P_{ij}(\gamma) = \frac{1}{\mu_{jj}} m_j = \frac{\nu(j)m_j}{\sum_{i\in\mathbf{E}} \nu(i)m_i}, j \in \mathbf{E},$$

where ν is the stationary distribution of the embedded Markov chain $(J_n; n \in \mathbb{N})$, and

$$m_j = \sum_{n \geq 0} (1 - H_j(n)).$$

1.4 Elements of statistical estimation

This section builds nonparametric estimators for a discrete-time semi-Markov system's primary attributes, such as the semi-Markov kernel, the transition matrix of the embedded Markov chain, the conditional distributions of the sojourn times, or the semi-Markov transition function. We investigate the asymptotic properties of the estimators, namely, the strong consistency and the asymptotic normality.

1.4.1 Construction of the estimators

Let us consider a sample path of an ergodic Markov renewal chain $(J_n, S_n)_{n \in \mathbb{N}}$, censored at fixed arbitrary time $M \in \mathbb{N}^*$,

$$\mathcal{H}(M) := \left(J_0, X_1, \ldots, J_{N(M)-1}, X_{N(M)}, J_{N(M)}, u_M\right),$$

where $N(M)$ is the discrete-time counting process of the number of jumps in $[1, M] \subset \mathbb{N}$, and $u_M := M - S_{N(M)}$ is the censored sojourn time in the last visited state $J_{N(M)}$.

Empirical estimators

Starting from the sample path $\mathcal{H}(M)$, we will propose empirical estimators for the quantities of interest of the semi-Markov chain. For any states $i, j \in E$ and positive integer $k \in \mathbb{N}, k \leq M$, we define the empirical estimators of the transition matrix of the embedded Markov chain p_{ij}, of the conditional distribution of the sojourn times $f_{ij}(k)$, and of the discrete-time semi-Markov kernel $q_{ij}(k)$ by

$$\begin{aligned} \widehat{p}_{ij}(M) &:= \frac{N_{ij}(M)}{N_i(M)}, \text{ if } N_i(M) \neq 0. \\ \widehat{f}_{ij}(k, M) &:= \frac{N_{ij}(k, M)}{N_{ij}(M)}, \text{ if } N_{ij}(M) \neq 0. \\ \widehat{q}_{ij}(k, M) &:= \frac{N_{ij}(k, M)}{N_i(M)}, \text{ if } N_i(M) \neq 0, \end{aligned} \tag{1.3}$$

where $N_{ij}(k, M), N_i(M)$ and $N_{ij}(M)$ are given by :

- $N_i(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_n = i\}}$: the number of visits to state i, up to time M;

- $N_{ij}(M) := \sum_{n=1}^{N(M)} 1_{\{J_{n-1}=i, J_n=j\}}$: the number of transitions from i to j, up to time M;
- $N_{ij}(k, M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n=k\}}$: the number of transitions from i to j, up to time M, with sojourn time in state i equal to k, $1 \leq k \leq M$.

If $N_i(M) = 0$ we set $\widehat{p}_{ij}(M) := 0$ and $\widehat{q}_{ij}(k, M) := 0$ for any $k \in \mathbb{N}$, and if $N_{ij}(M) = 0$ we set $\widehat{f}_{ij}(k, M) := 0$ for any $k \in \mathbb{N}$.

The likelihood function corresponding to the history $\mathcal{H}(M)$ is

$$\mathrm{L}(M) = \alpha_{J_0} \prod_{k=1}^{N(M)} p_{J_{k-1}J_k} f_{J_{k-1}J_k}(X_k)\, \bar{H}_{J_{N(M)}}(u_M),$$

where $\bar{H}_{J_{N(M)}}$ is the survival function in state i and α_i is the initial distribution of state i.

Lemma 1.4.1. *[3] For a semi-Markov chain $(Z_n)_{n\in\mathbb{N}}$ we have*

$$u_M/M \xrightarrow[M\to\infty]{a.s.} 0.$$

The previous lemma tells us that, for large M, u_M does not add significant information to the likelihood function. For these reason, we will neglect the term $\bar{H}_{J_{N(M)}}(u_M)$ in the expression of the likelihood function $\mathrm{L}(M)$. On the other side, the sample path $\mathcal{H}(M)$ of the MRC $(J_n, S_n)_{n\in\mathbb{N}}$ contains only one observation of the initial distribution α of $(J_n)_{n\in\mathbb{N}}$, so the information on α_{J_0} does not increase with M. As we are interested in large-sample estimation of semi-Markov chains, the term α_{J_0} will be equally neglected in the expression of the likelihood function.

Consequently, we will be concerned with the maximization of the approached likelihood function defined by

$$\mathrm{L}_1(M) = \prod_{k=1}^{N(M)} p_{J_{k-1},J_k} f_{J_{k-1}J_k}(X_k). \tag{1.4}$$

Proposition 1.4.1. *[3] For a sample path of a DTMRP $(J_n, S_n)_{n\in\mathbb{N}}$, censored at time $M \in \mathbb{N}$, the empirical estimators $\widehat{p}_{ij}(M)$, $\widehat{f}_{ij}(k, M)$ and $\widehat{q}_{ij}(k, M)$, proposed in Equations 1.3, are approached non-parametric maximum likelihood estimators i.e. they maximize the approached likelihood function L_1, given in Equation 1.4 .*

Proof. We consider the approached likelihood function $\mathrm{L}_1(M)$ given by Equation 1.4. Using the equality

$$\sum_{j=1}^{s} p_{ij} = 1. \tag{1.5}$$

the approached log-likelihood function can be written in the form

$$log\left(\mathrm{L}_1(M)\right) = \sum_{k=1}^{M}\sum_{i,j=1}^{s}\left[N_{ij}(M)log\left(p_{ij}\right) + N_{ij}(k,M)log\left(f_{ij}(k)\right) + \lambda_i\left(1-\sum_{j=1}^{s}p_{ij}\right)\right]. \tag{1.6}$$

Where the Lagrange multipliers λ_i are arbitrarily chosen constants. In order to obtain the approached MLE of p_{ij} we maximize Equation 1.6 with respect to p_{ij}, and get $p_{ij} = N_{ij}(M)/\lambda_i$. Equation 1.5 becomes

$$1 = \sum_{j=1}^{s} p_{ij} = \sum_{j=1}^{s}\frac{N_{ij}(M)}{\lambda_i} = \frac{N_i(M)}{\lambda_i}.$$

Finally, we infer that the values λ_i which maximize Equation 1.6 with respect to p_{ij} are given by $\lambda_i = N_i(M)$ and we obtain

$$\widehat{p}_{ij}(M) := \frac{N_{ij}(M)}{N_i(M)}.$$

The expression of $\widehat{f_{ij}}(k,M)$ can be obtained by the same method. Indeed, using the equality

$$\sum_{k=1}^{\infty} f_{ij}(k) = 1, \tag{1.7}$$

we write the approached log-likelihood function in the form

$$log\left(\mathrm{L}_1(M)\right) = \sum_{k=1}^{M}\sum_{i,j=1}^{s}\left[N_{ij}(M)log\left(p_{ij}\right) + N_{ij}(k,M)log\left(f_{ij}(k)\right) + \lambda_{ij}\left(1-\sum_{k=1}^{\infty}f_{ij}(k)\right)\right]. \tag{1.8}$$

Where λ_{ij} are arbitrarily chosen constants. Maximizing 1.8 with respect to $f_{ij}(k)$ we obtain $\widehat{f_{ij}}(k,M) := N_{ij}(k,M)/\lambda_{ij}$.

From Equation 1.7 we obtain $\lambda_{ij}(M) = N_{ij}(M)$. Thus $\widehat{f_{ij}}(k,M) := N_{ij}(k,M)/N_{ij}(M)$.

In an analogous way we can prove that the expression of the approached MLE of the kernel $q_{ij}(k)$ is given by equation 1.3. $\square$

Lemma 1.4.2. *[3] For a MRC that satisfies Assumptions **A1** and **A2**, we have:*

1. $\lim_{M\to\infty} S_M = \infty$ *a.s .*
2. $\lim_{M\to\infty} N(M) = \infty$ *a.s .*

Lemma 1.4.3. *For the DTMRP $(J_n, S_n)_{n\in\mathbb{N}}$. We have*

$$\frac{N_i(M)}{M} \xrightarrow[M\to\infty]{a.s.} \frac{1}{\mu_{ii}},\ \frac{N_{ij}(M)}{M} \xrightarrow[M\to\infty]{a.s.} \frac{p_{ij}}{\mu_{ii}},\ \frac{N(M)}{M} \xrightarrow[M\to\infty]{a.s.} \frac{1}{\nu(l)\mu_{ll}}.$$

Where μ_{ii} is the mean recurrence time of state i for the semi-Markov process $(Z_n)_{n\in\mathbb{N}}$,$(\nu(l); l\in\mathbf{E})$ the stationary distribution and l is an arbitrary fixed state.

1.4.2 Asymptotic properties of the estimators

In this part, we study the asymptotic properties (consistency and asymptotic normality) of the proposed estimators $\widehat{P}_{ij}, \widehat{F}_{ij}, \widehat{\psi}_{ij}$ and $\widehat{Q}_{ij}$, with :

$$\begin{aligned} \widehat{F}_{ij}(k, M) &:= \frac{1}{N_{ij}(M)} \sum_{l=1}^{N_{ij}(M)} \mathbf{1}_{\{X_{n_l} \leq k\}}, \\ \widehat{Q}_{ij}(k, M) &:= \frac{1}{N_i(M)} \sum_{m=1}^{N_i(M)} \mathbf{1}_{\{J_{m-1}=i, J_m=j, X_m \leq k\}}, \\ \widehat{\psi}_{ij}(k, M) &:= \sum_{n=0}^{k} \widehat{q}_{ij}^{(n)}(k, M), \\ \widehat{P}(k, M) &:= (\delta \mathbf{I} - \widehat{q}(., M))^{-1} * (\mathbf{I} - \operatorname{diag}(\widehat{Q}(., M)\mathbf{1}))(k). \end{aligned} \tag{1.9}$$

Theorem 1.4.1. *[3] For any fixed arbitrary states $i, j \in \mathbf{E}$,*

(a) ***(Strong consistency)*** $\max\limits_{i,j \in \mathbf{E}} \max\limits_{0 \leq k \leq M} \left| \widehat{F}_{ij}(k, M) - F_{ij}(k) \right| \xrightarrow[M \longrightarrow \infty]{a.s.} 0.$

(b) ***(Asymptotic normality)*** $\sqrt{M} \left[\widehat{F}_{ij}(k, M) - F_{ij}(k) \right] \xrightarrow[M \to \infty]{\mathcal{D}} \mathcal{N} \left(0, \sigma_F^2(i, j, k)\right),$

with the asymptotic variance

$$\sigma_F^2(i, j, k) = \sum_{l \in \mathbf{E}} \sum_{r \in \mathbf{E}} \delta_{il} \delta_{jr} \frac{\mu_{ii}}{p_{ij}} F_{ij}(k) \left(1 - F_{ij}(k)\right),$$

and μ_{ii} is the mean recurrence time of state i for the SMP $(Z_k)_{k \in \mathbb{N}}$.

Theorem 1.4.2. *[3] For any fixed $k \in \mathbb{N}$ and $i, j \in \mathbf{E}$, we have*

(a) ***(Strong consistency)*** $\max\limits_{i,j \in \mathbf{E}} \max\limits_{0 \leq k \leq M} \left| \widehat{Q}_{ij}(k, M) - Q_{ij}(k) \right| \xrightarrow[M \longrightarrow \infty]{a.s.} 0.$

(b) ***(Asymptotic normality)*** $\sqrt{M} \left[\widehat{Q}_{ij}(k, M) - Q_{ij}(k) \right] \xrightarrow[M \to \infty]{\mathcal{D}} \mathcal{N} \left(0, \sigma_Q^2(i, j, k)\right),$

where

$$\sigma_Q^2(i, j, k) = \sum_{l \in \mathbf{E}} \sum_{r \in \mathbf{E}} \delta_{il} \mu_{ii} Q_{ij}(k) \left(\delta_{jr} - Q_{ir}(k)\right).$$

Theorem 1.4.3. *[3] For any fixed $k \in \mathbb{N}$ and $i, j \in \mathbf{E}$, we have*

(a) ***(Strong consistency)*** $\max\limits_{i,j \in \mathbf{E}} \max\limits_{0 \leq k \leq M} \left| \widehat{\psi}_{ij}(k, M) - \psi_{ij}(k) \right| \xrightarrow[M \to \infty]{a.s.} 0.$

(b) ***(Asymptotic normality)*** $\sqrt{M} \left[\widehat{\psi}_{ij}(k, M) - \psi_{ij}(k) \right] \xrightarrow[M \to \infty]{\mathcal{D}} \mathcal{N} \left(0, \sigma_\psi^2(i, j, k)\right),$

where

$$\sigma_{\psi}^{2}(i,j,k)=\sum_{m=1}^{s}\mu_{mm}\left\{\sum_{r=1}^{s}\left[\left(\psi_{im}*\psi_{rj}\right)^{2}*q_{mr}\right](k)\right.$$
$$\left.-\left[\sum_{r=1}^{s}\left(\psi_{im}*q_{mr}*\psi_{rj}\right)\right]^{2}\right\}.$$

Theorem 1.4.4. *[3], [1] For any fixed $k\in\mathbb{N}$ and $i,j\in\mathbf{E}$, we have*

(a) ***(Strong consistency)*** $\max_{i,j\in\mathbf{E}}\max_{0\le k\le M}\left|\widehat{P}_{ij}(k,M)-P_{ij}(k)\right|\xrightarrow[M\to\infty]{a.s.}0.$

(b) ***(Asymptotic normality)***$\sqrt{M}\left[\widehat{P}_{ij}(k,M)-P_{ij}(k)\right]\xrightarrow[M\to\infty]{\mathcal{D}}\mathcal{N}\left(0,\sigma_P^2(i,j,k)\right),$

where

$$\sigma_{P}^{2}(i,j,k)=\sum_{m=1}^{s}\mu_{mm}\left\{\sum_{r=1}^{s}\left[\delta_{mj}\Psi_{ij}-(1-H_j)*\psi_{im}*\psi_{rj}\right]^{2}*q_{mr}(k)\right.$$
$$\left.-\left[\delta_{mj}\psi_{ij}*H_m(k)-\sum_{r=1}^{s}(1-H_j)*\psi_{im}*\psi_{rj}*q_{mr}\right]^{2}(k)\right\}.$$

Chapter 2

Continuous-time semi-Markov process

This chapter provides the definitions and basic properties related to Continuous-time semi-Markov process (CTSMP). The semi Markov process (SMP) is constructed by the so-called Markov renewal process (MRP) that is a special case of the two-dimensional Markov sequence. The MRP is defined by the transition probabilities matrix, called the renewal kernel and an initial distribution, or by other characteristics that are equivalent to the renewal kernel. The counting process corresponding to the SMP allows us to determine the concept of process regularity. The process is said to be regular if the corresponding counting process has a finite number of jumps in a finite period.

2.1 Definitions and properties

Definition 2.1.1 (Markov renewal process). *Let $\mathbf{E}$ be the state space. A Markov renewal process is a bivariate stochastic process (J_n, S_n) where J_n are the values of the state space $\mathbf{E}$ in the Markov chain and S_n are the jump times. We define $X_{n+1} = S_{n+1} - S_n$ to be the sojourn time in the state J_n. The process has to satisfy the following equality*

$$\begin{aligned} &\mathbb{P}\left(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_0, J_1, \ldots, J_n, S_0, S_1, \ldots, S_n\right) \\ =&\mathbb{P}\left(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_n\right), \end{aligned} \tag{2.1}$$

for all $j \in \mathbf{E}$, all $t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$.

Definition 2.1.2 (Renewal matrix, renewal kernel). *The matrix defined as*

$$\boldsymbol{Q}(t) = \{Q_{ij}(t) : i, j \in \mathrm{E}\},$$
$$Q_{ij}(t) := \mathbb{P}(J_{n+1} = j, X_{n+1} \leq t \mid J_n = i),$$

is called a renewal matrix. We identify the renewal matrix $\mathbf{Q}$ *as the renewal kernel.*

Proposition 2.1.1. *[16] The Markov renewal matrix* $\mathbf{Q}$ *satisfies the following conditions:*

(i) For all $t \geq 0$ *and* $i, j \in \mathbf{E}$*, it holds true that* $Q_{ij}(t) \geq 0$.

(ii) The functions $Q_{ij}(t)$ *are right-continuous.*

(iii) For all $i, j \in \mathbf{E}$*, it holds true that* $Q_{ij}(0) = 0$ *and* $Q_{ij}(t) \leq 1$ *for all* $t \geq 0$.

(iv) For all $i \in \mathbf{E}$*, it holds that* $\lim\limits_{i \to \infty} \sum\limits_{j \in \mathbf{E}} Q_{ij}(t) = 1$.

Definition 2.1.3. *The probabilities*

$$\begin{aligned} p_{ij} &= \lim_{t \to \infty} Q_{ij}(t) = Q_{ij}(\infty) \\ &= \mathbb{P}(J_{n+1} = j \mid J_n = i), \end{aligned}$$

are the transition probabilities from state i *to state* j *of the embedded Markov* chain $\{J_n; n \in \mathbb{N}\}$.

We assume that the transition probabilities do not depend on the time n.

Proposition 2.1.2. *[16] For a Markov renewal process with a renewal kernel* $\mathbf{Q}(t), t \geq 0$*, the following equality is satisfied*

$$\mathbb{P}(J_0 = i_0, J_1 = i_1, X_1 \leq t_1, \ldots, J_n = i_n, X_n \leq t_n) = \alpha_{i_0} Q_{i_0 i_1}(t_1) Q_{i_1 i_2}(t_2) \ldots Q_{i_{n-1} i_n}(t_n),$$

where $\alpha_{i_0} := \mathbb{P}(J_0 = i_0)$ *is the initial distribution of the Markov renewal process.*
For $t_1 \to \infty, \ldots, t_n \to \infty$*, we obtain*

$$\mathbb{P}(J_0 = i_0, J_1 = i_1, \ldots, J_n = i_n) = \alpha_{i_0} p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{n-1} i_n}.$$

Definition 2.1.4 (Continuous-time semi-Markov process). *Consider a Markov-renewal process* $\{(J_n, S_n) : n \in \mathbb{N}\}$ *defined on a complete probability space and with state space* $\mathbf{E}$*. The stochastic process* $\{Z_t; t \in \mathbb{R}_+\}$ *defined by*

$$Z_t = J_{N(t)}.$$

is called a semi-Markov process (SMP) where $N(t) = \max\{n \in \mathbb{N} : S_n \leq t\}$ *is the counting process of the semi-Markov process up to time* t*. we can also define the semi-Markov Process by*

$$Z_t = J_n \ \textit{for} \ \ t \in [S_n, S_{n+1}), n \in \mathbb{N}.$$

Definition 2.1.5. *We define the transition matrix of the process* $\{Z_t; t > 0\}$ *as*

$$\begin{aligned}
\mathbf{P}(t) &= \{P_{ij}(t) : i, j \in \mathbf{E}\}, \\
P_{ij}(t) &= \mathbb{P}(Z_t = j \mid Z_0 = i), \\
&= \mathbb{P}\left(J_{N(t)} = j \mid J_0 = i\right).
\end{aligned}$$

For all $i, j \in \mathbf{E}$. *Then the unconditional semi-Markov state probability is equal to*

$$\begin{aligned}
P_j(t) = \mathbb{P}(Z_t = j) &= \mathbb{P}\left(J_{N(t)} = j\right) \\
&= \sum_{i=1}^{s} \mathbb{P}\left(J_{N(t)} = j \mid J_0 = i\right) \mathbb{P}(J_0 = i) \\
&= \sum_{i=1}^{s} \alpha_i P_{ij}(t).
\end{aligned}$$

Where $\alpha_i = \mathbb{P}(J_0 = i)$.

Definition 2.1.6. *The matrix renewal function* $\Psi = (\Psi(t); t \in \mathbb{R}_+) \in \mathcal{M}_E(\mathbb{R}_+)$, *is defined by*

$$\Psi_{ij}(t) = \mathbb{E}_i[N_j(t)] = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t) =: (I - Q(t))^{(-1)}(i, j), \quad t \in \mathbb{R}.$$

Where the n*-fold Stieltjes convolution of* $Q_{ij}(t)$ *by itself is defined as,* $t \geq 0$,

$$Q_{ij}^{(n)}(t) = \begin{cases} \sum_{k \in \mathbf{E}} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t-s), & n \geq 2, \\ Q_{ij}(t), & n = 1, \\ \delta_{ij}, & n = 0. \end{cases}$$

and

$I(t) = \left(\mathbf{1}_{\{i=j\}}(t); i, j \in \mathbf{E}\right), I = (I(t); t \in \mathbb{R}^+)$.

Definition 2.1.7 (Regularity of SMP). *A semi-Markov process* $\{Z(t) : t \geq 0\}$ *is said to be regular if the corresponding counting process* $\{N(t) : t \geq 0\}$ *has a finite number of jumps on a finite period with probability* 1 *:*

$$\forall t \in \mathbb{R}_+, \quad \mathbb{P}(N(t) < \infty) = 1. \tag{2.2}$$

Proposition 2.1.3. *[17] A SMP* $\{Z(t) : t \geqslant 0\}$ *is regular if and only if*

$$\forall t \in \mathbb{R}_+, \lim_{n \to \infty} \mathbb{P}(N(t) \geqslant n) = \lim_{n \to \infty} \mathbb{P}(S_n \leqslant t) = 0.$$

Definition 2.1.8 (Distribution functions of sojourn time). *For all $i, j \in \mathbf{E}, \forall t \in \mathbb{R}_+$.*

1. $F_{ij}(.)$ *, the distribution function associated with the sojourn time in state i, before going to state j :*
$$F_{ij}(t) := \mathbb{P}\left(X_{n+1} \leq t \mid J_n = i, J_{n+1} = j\right).$$

2. $H_i(.)$ *, the distribution function of the sojourn time, also called the waiting time, in state i :*
$$H_i(t) := \mathbb{P}\left(X_{n+1} \leq t \mid J_n = i\right) = \sum_{j \in \mathbf{E}} Q_{ij}(t).$$

From the definition before we can derive the following result.

Proposition 2.1.4. *[16] It holds true that*
$$F_{ij}(t) = \frac{Q_{ij}(t)}{p_{ij}}.$$

For all $t \geq 0$ and $i, j \in \mathbf{E}$.

Proof. From the definition of conditional probabilities, it follows that
$$\begin{aligned}
F_{ij}(t) &= \mathbb{P}\left(X_{n+1} \leq t \mid J_n = i, J_{n+1} = j\right) \\
&= \frac{\mathbb{P}\left(X_{n+1} \leq t, J_n = i, J_{n+1} = j\right)}{\mathbb{P}\left(J_n = i, J_{n+1} = j\right)} \\
&= \frac{\mathbb{P}\left(X_{n+1} \leq t, J_n = i, J_{n+1} = j\right)}{\mathbb{P}\left(J_n = i\right)} \frac{\mathbb{P}\left(J_n = i\right)}{\mathbb{P}\left(J_n = i, J_{n+1} = j\right)} \\
&= \frac{\mathbb{P}\left(J_{n+1} = j, X_{n+1} \leq t \mid J_n = i\right)}{\mathbb{P}\left(J_{n+1} = j, J_n = i\right)} \\
&= \frac{Q_{ij}(t)}{p_{ij}}.
\end{aligned}$$

□

2.1.1 Connection between semi-Markov and Markov process

A discrete state space and continuous-time SMP is a generalization of that kind of Markov process. The Markov process can be treated as a special case of the SMP.

Theorem 2.1.1 (Korolyuk and Turbin). *[17] Every homogeneous Markov process $\{J(t) : t \geq 0\}$ with the discrete space $\mathbf{E}$ and the right-continuous trajectories keeping constant values on the half-intervals, given by the transition rate matrix*

$\Lambda = [\lambda_{ij} : i, j \in \mathbf{E}], 0 < -\lambda_{ii} = \lambda_i < \infty$ *is the SMP with the kernel*

$$\boldsymbol{Q}(t) = [Q_{ij}(t) : i, j \in \mathbf{E}],$$

where

$$Q_{ij}(t) = p_{ij}\left(1 - \mathrm{e}^{-\lambda_i t}\right), \quad t \geqslant 0,$$
$$p_{ij} = \frac{\lambda_{ij}}{\lambda_i} \quad \textit{for } i \neq j, p_{ii} = 0.$$

Proof. [19] [22] □

The length of interval $[S_n, S_{n+1})$ given states at instants S_n and S_{n+1} is a random variable having an exponential distribution with parameter independent of state at the moment S_{n+1} :

$$F_{ij}(t) = \mathbb{P}\left(S_{n+1} - S_n \leqslant t \mid J\left(S_n\right) = i, J\left(S_{n+1}\right) = j\right) = 1 - \mathrm{e}^{-\lambda_i t}, t \geqslant 0.$$

As we know, the function $F_{ij}(t)$ is a cumulative probability distribution of a holding time in the state i, if the next state is j. Recall that the function

$$H_i(t) = \sum_{j \in \mathbf{E}} Q_{ij}(t) = 1 - \mathrm{e}^{-\lambda_i t}, \quad t \geqslant 0.$$

is a CDF of a waiting time in the state i.

2.2 Elements of statistical estimation

2.2.1 Useful technical results

We introduce the following technical results which will be needed for the proofs for Q_{ij}, Ψ_{ij}, and P_{ij}.

Theorem 2.2.1 (Anscombe's theorem, [8]). *Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables and $(N_n)_{n\in\mathbb{N}}$ a positive integer-valued stochastic process. Suppose that*

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{n} Y_m \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \sigma^2\right) \quad \textit{and} \quad N_n/n \xrightarrow[n\to\infty]{P} \theta,$$

where θ is a constant, $0 < \theta < \infty$. Then,

$$\frac{1}{\sqrt{N_n}} \sum_{m=1}^{N_n} Y_m \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \sigma^2\right).$$

Theorem 2.2.2 (Slutsky's theorem). *Let $X, X_t, Y_t, t \in \mathbb{R}$, be random variables or vectors. If $X_t \xrightarrow[t\to\infty]{\mathcal{D}} X$ and $Y_t \xrightarrow[t\to\infty]{\mathcal{P}} c$, with c a constant, then*

1. $X_t + Y_t \xrightarrow[t\to\infty]{\mathcal{D}} X + c$
2. $Y_t X_t \xrightarrow[t\to\infty]{\mathcal{D}} cX$
3. $Y_t^{-1} X_t \xrightarrow[t\to\infty]{\mathcal{D}} c^{-1}X$, *for* $c \neq 0$.

We present the central limit theorem for additive functional of MRPs, given by Pyke and Schaufele [33] .

Notation 2.2.1. *For a real measurable function f, defined on $\mathbf{E} \times \mathbf{E} \times \mathbb{R}$, define, for each $M > 0$, the functional $W_f(M)$ as*

$$W_f(M) := \sum_{n=1}^{N(M)} f\left(J_{n-1}, J_n, X_n\right).$$

Set

$$A_{ij} := \int_0^\infty f(i,j,x)dQ_{ij}(x), \quad A_i := \sum_{j=1}^s A_{ij},$$
$$B_{ij} := \int_0^\infty (f(i,j,x))^2 dQ_{ij}(x), \quad B_i := \sum_{j=1}^s B_{ij}.$$

Let μ_{ij} and μ^*_{ij} denote the mean first passage times from state i to j in the MRP (J_n, S_n) and in the corresponding Markov chain $(J_n)_{n\in\mathbb{N}}$, respectively. Write

$$r_i := \sum_{u=1}^s A_u \frac{\mu^*_{ii}}{\mu^*_{uu}},$$
$$\sigma_i^2 := -r_i^2 + \sum_{u=1}^s B_u \frac{\mu^*_{ii}}{\mu^*_{uu}} + 2\sum_{u=1}^s \sum_{l\neq i} \sum_{j\neq i} A_{ul} A_j \mu^*_{ii} \frac{\mu^*_{li} + \mu^*_{ij} - \mu^*_{lj}}{\mu^*_{uu}\mu^*_{jj}}.$$

Finally, put

$$m_f := \frac{r_i}{\mu_{ii}}$$
$$B_f := \frac{\sigma_i^2}{\mu_{ii}}.$$

Theorem 2.2.3 (Central Limit Theorem). *For an irreducible recurrent MRPs that satisfies Assumptions **A1** and **A2**, we have*

$$M^{-1/2}\left[W_f(M) - M \cdot m_f\right] \xrightarrow[M\to\infty]{\mathcal{D}} \mathcal{N}\left(0, B_f\right).$$

Theorem 2.2.4 (Glivenko-Cantelli theorem, [7]**).** *Let $F_n(x) = \frac{1}{n}\sum_{k=1}^{n} 1_{\{X_k \le x\}}$ be the empirical distribution function of the i.i.d. random sample $X_1, \ldots, X_n$. Denote by F the common distribution function of $X_i, i = 1, \ldots, n$. Thus*

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \xrightarrow[n\to\infty]{a.s} 0.$$

2.2.2 Empirical estimators

Estimators for semi Markov kernel $Q_{ij}(t)$ are defined on sample functions of the MRP over $[0, M]$. These sample functions of the MRP are equivalent to the sample functions $\left(J_0, J_1, \ldots, J_{N(M)}, X_0, X_1, \ldots, X_{N(M)}\right)$.

For the semi-Markov kernel $Q_{ij}(t)$, we have the following empirical estimator

$$\widehat{Q}_{ij}(t, M) = \frac{1}{N_i(M)} \sum_{n=1}^{N(t)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n \le t\}},$$

where

$$N_i(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_n=i\}} = \sum_{n=1}^{\infty} \mathbf{1}_{\{J_n=i, S_n \le M\}}.$$

The empirical estimator of the transition matrix of the embedded Markov chain p_{ij} is defined by

$$\widehat{p}_{ij} := \frac{N_{ij}(M)}{N_i(M)},$$

where

$$N_{ij}(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j\}} = \sum_{n=1}^{\infty} \mathbf{1}_{\{J_{n-1}=i, J_n=i, S_n \le M\}}.$$

Because $F_{ij}(t) = Q_{ij}(t)/p_{ij}$, in a similar way we obtain that $\widehat{F}_{ij}(t, M) = \widehat{Q}_{ij}(t, M)/\widehat{p}_{ij}(M)$ with

$$\widehat{F}_{ij}(t, M) = \frac{1}{N_{ij}(M)} \sum_{n=1}^{N(t)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n \le t\}}.$$

The quantities $\widehat{F}_{ij}(t, M)$ and $\widehat{p}_{ij}$ are respectively the empirical estimators for the conditional transition functions and the transition probabilities.

We define the following empirical estimators of P_{ij}, and Ψ_{ij}:

$$\begin{aligned} \widehat{\Psi}_{ij}(t) &:= \sum_{n=0}^{\infty} \widehat{Q}_{ij}^{(n)}(t), \\ \widehat{P}(t, M) &:= \widehat{\Psi} \star (I - \operatorname{diag}(\widehat{Q}(t, M))). \end{aligned} \tag{2.3}$$

Theorem 2.2.5. *(**Barbu & Limnios**, [3]) The empirical estimator $\hat{p}_{ij}(M)$ of p_{ij} , for all $i, j \in \mathbf{E}$ is strongly consistent, i.e.*

$$\widehat{p}_{ij}(M) \xrightarrow{a.s.} p_{ij} \quad as \quad M \to \infty.$$

Theorem 2.2.6. *For any fixed $i, j \in \mathbf{E}$, as $M \to \infty$, we have:*

(a) ***(Strong consistency)*** $\max_{i,j\in\mathbf{E}} \sup_{t\in(0,M)} \left|\widehat{Q}_{ij}(t, M) - Q_{ij}(t)\right| \xrightarrow{a.s.} 0,$

(b) ***(Asymptotic normality)*** $M^{1/2}\left(\widehat{Q}_{ij}(t, M) - Q_{ij}(t)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_{ij}^2(t)\right)$*, where* $\sigma_{ij}^2(t) := \mu_{ii} Q_{ij}(t) \left[1 - Q_{ij}(t)\right]$.

Proof. (**Limnios** & **Oprisan**, [28])

(a) We define $\Delta Q_{ij} = \left(\hat{Q}_{ij}(t, M) - Q_{ij}(t)\right)$, it holds true that $Q_{ij}(t) = F_{ij}(t)p_{ij}$ and therefore $\widehat{Q}_{ij}(t, M) = \widehat{F}_{ij}(t, M)\widehat{p}_{ij}(M)$ as well. Then it follows that

$$\begin{aligned} \max_{i,j\in\mathbf{E}} \sup_{t\in[0,M)} |\Delta Q_{ij}| &= \max_{i,j\in\mathbf{E}} \sup_{t\in[0,M)} \left|\widehat{F}_{ij}(t, M)\widehat{p}_{ij}(M) - F_{ij}(t)p_{ij}\right| \\ &= \max_{i,j\in\mathbf{E}} \sup_{t\in[0,M)} \left|\widehat{F}_{ij}(t, M)\widehat{p}_{ij}(M) - \widehat{F}_{ij}(t, M)p_{ij} + \widehat{F}_{ij}(t, M)p_{ij} - F_{ij}(t)p_{ij}\right| \\ &\leq \max_{i,j\in\mathbf{E}} \sup_{t\in[0,M)} \left|\widehat{F}_{ij}(t, M)\widehat{p}_{ij}(M) - \widehat{F}_{ij}(t, M)p_{ij}\right| \\ &+ \max_{i,j\in\mathbf{E}} \sup_{t\in[0,M)} \left|\widehat{F}_{ij}(t, M)p_{ij} - F_{ij}(t)p_{ij}\right| \\ &= \max_{i,j\in\mathbf{E}} \sup_{t\in[0,M)} \left|\widehat{F}_{ij}(t, M)\left(\widehat{p}_{ij}(T) - p_{ij}\right)\right| \\ &+ \max_{i,j\in\mathbf{E}} \sup_{t\in[0,M)} \left|\left(\widehat{F}_{ij}(t, M) - F_{ij}(t)\right)p_{ij}\right| \\ &= \max_{i,j\in\mathbf{E}} \sup_{t\in[0,M)} \left|\widehat{p}_{ij}(M) - p_{ij}\right| \widehat{F}_{ij}(t, M) \\ &+ \max_{i,j\in\mathbf{E}} \sup_{t\in[0,M)} \left|\widehat{F}_{ij}(t, M) - F_{ij}(t)\right| p_{ij}. \end{aligned}$$

By theorem 2.2.5, the first term converges to 0 (a.s.), By theorem 2.2.4 (**Glivenko-Cantelli theorem**), the second converges to 0 (a.s.) as well.

(b) We have

$$M^{1/2}\left[\hat{Q}_{ij}(t,M)-Q_{ij}(t)\right]=\frac{M}{N_i(M)}M^{-1/2}\sum_{k=1}^{N(t)}\left(\mathbf{1}_{\{J_k=j,X_k\le t\}}-Q_{ij}(t)\right)\mathbf{1}_{\{J_{k-1}=i\}}.$$

Consider the function

$$f(m,\ell,u)=\left(\mathbf{1}_{\{\ell=j,u\le t\}}-Q_{ij}(t)\right)\mathbf{1}_{\{m=i\}}.$$

By the Pyke and Schaufele CLT (see Theorem 2.2.1 and 2.2.3), and since $N_i(M)/M$ converges to $1/\mu_{ii}$ (a.s.), we get the desired result. $\square$

Lemme 2.1. *Under the assumptions of Theorem 2.2.6, we have, for all $i,j\in E$ and $n\in\mathbb{N}$,*

$$\max_{i,j\in\mathbf{E}}\sup_{t\in[0,\infty[}\left|\hat{Q}_{ij}^{(n)}(t,M)-Q_{ij}^{(n)}(t)\right|\longrightarrow 0 \textit{ a.s.},\qquad \textit{as}\quad M\longrightarrow\infty.$$

Theorem 2.2.7. *[31] The estimator $\widehat{\Psi}_{ij}(t,M)$ of the Markov renewal function $\Psi_{ij}(t)$ satisfies the following two properties:*

(a) ***(Strong consistency)*** *it is uniformly strongly consistent, i.e., as $M\to\infty$,*

$$\max_{i,j\in\mathbf{E}}\sup_{t\in[0,M)}\left|\widehat{\Psi}_{ij}(t,M)-\Psi_{ij}(t)\right|\xrightarrow{a.s.}0.$$

(b) ***(Asymptotic normality)*** *For any fixed $t>0$, it converges in distribution, as $M\to\infty$, to a normal random variable, i.e.,*

$$M^{1/2}\left(\widehat{\Psi}_{ij}(t,M)-\Psi_{ij}(t)\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\sigma_{ij}^2(t)\right),$$

where

$$\sigma_{ij}^2(t)=\sum_{r\in\mathbf{E}}\sum_{k\in\mathbf{E}}\mu_{rr}\left\{(\psi_{ir}*\psi_{kj})^2*Q_{rk}-(\psi_{ir}*\psi_{kj}*Q_{rk})^2\right\}(t),$$

and ψ_{ij} is the density function of Ψ_{ij}.

Proof. (a) We have $\hat{\Psi}(t)=\sum_{n=0}^{\infty}\hat{Q}^{(n)}(t,M)$. To prove the uniform strong consistence of the estimator of the Markov renewal matrix, we need the lemma 2.1.

Let i and j be two states and $t\in[0,M]$ and $\epsilon>0$. Since $S_n\longrightarrow\infty$, (because the MRP which we have considered is positive recurrent), there exists a constant $k_0>0$ such that $\max_{i\in\mathbf{E}}\sum_{j=1}^{s}Q_{ij}^{(k_0)}(t)<1$.

Let Ω be a set of probability 1 , where the convergence of Lemma 2.1 is valid for all $n \geqslant 1$. Set $\epsilon = 1 - \max_{i\in\mathbf{E}} \sum_{j=1}^{s} Q_{ij}^{(k_0)}(t)$. For all $\omega \in \Omega$, there exists $T_0(\omega)$, such that for all $M \geqslant M_0(\omega)$

$$\max_{i\in\mathbf{E}} \sum_{j=1}^{s} \hat{Q}_{ij}^{(k_0)}(t,M) \leqslant \max_{i\in\mathbf{E}} \left| \sum_{j=1}^{s} \left[\hat{Q}_{ij}^{(k_0)}(t,M) - Q_{ij}^{(k_0)}(t) \right] \right| + \\ + \max_{i\in\mathbf{E}} \sum_{j=1}^{s} Q_{ij}^{(k_0)}(t) \leqslant 1 - \frac{\epsilon}{2}.$$

Moreover, for all $m \geqslant k_0$, there exists $(q,r) \in \mathbb{N}^* \times \mathbb{N}$ such that $m = qk_0 + r$ where $0 \leqslant r < k_0$ and we see that,

$$\begin{aligned}
\max_{i,j\in\mathbf{E}} \hat{Q}_{ij}^{(m)}(t,M) &= \max_{i,j\in\mathbf{E}} \sum_{n=1}^{s} \hat{Q}_{in}^{(r)} * \hat{Q}_{nj}^{(qk_0)}(t,M) \\
&\leqslant \max_{i,j\in\mathbf{E}} \sum_{n=1}^{s} \hat{Q}_{in}^{(r)}(t,M) \cdot \hat{Q}_{nj}^{(qk_0)}(t,M) \\
&\leqslant \max_{i,j\in\mathbf{E}} \hat{Q}_{ij}^{(qk_0)}(t,M).
\end{aligned}$$

Let us now prove that, for all $q \in \mathbb{N}^*$,

$$\max_{i\in\mathbf{E}} \sum_{j=1}^{s} \hat{Q}_{ij}^{(qk_0)}(t,M) \leqslant \left(1 - \frac{\epsilon}{2}\right)^q.$$

In fact, for $q = 1$, the result is true. Suppose that the result is valid until order q and prove it to order $q+1$. In fact, for $q = 1$, the result is true. Suppose that the result is valid until order q and prove it to order $q+1$.

$$\begin{aligned}
\max_{i\in\mathbf{E}} \sum_{j=1}^{s} \hat{Q}_{ij}^{((q+1)k_0)}(t,M) &= \max_{i\in\mathbf{E}} \sum_{j=1}^{s} \sum_{n=1}^{s} \hat{Q}_{in}^{(qk_0)} * Q_{nj}^{(k_0)}(t,M) \\
&\leqslant \max_{i\in\mathbf{E}} \sum_{n=1}^{s} \hat{Q}_{in}^{(qk_0)}(t,M) \cdot \max_{i\in\mathbf{E}} \sum_{j=1}^{s} Q_{ij}^{(k_0)}(t,M) \\
&\leqslant \left(1 - \frac{\epsilon}{2}\right)^q \cdot \left(1 - \frac{\epsilon}{2}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\hat{\Psi}_{ij}(t,M) &= \sum_{l=0}^{\infty} \hat{Q}_{ij}^{(n)}(t,M) \\
&= \sum_{n=0}^{k_0} \hat{Q}_{ij}^{(n)}(t,M) + \sum_{n=k_0+1}^{2k_0} \hat{Q}_{ij}^{(n)}(t,M) + \sum_{n=2k_0+1}^{3k_0} \hat{Q}_{ij}^{(n)}(t,M) + \cdots
\end{aligned}$$

Let $\beta_{ij}^{m}(t)$ be a sequence of functions defined by

$$\beta_{ij}^{m}(t) = \begin{cases} \hat{Q}_{ij}^{(m)}(t,M) & \text{if} \quad m < k_0, \\ k_0 \left(1 - \frac{\epsilon}{2}\right)^{[m/k_0]} & \text{otherwise.} \end{cases}$$

Where $[x]$ is the integer part of x. We have, $\hat{\Psi}_{ij}(t,M) - \Psi_{ij}(t) \leqslant \sum_{m=0}^{\infty} \beta_{ij}^m(t) < \infty$. Thus by Lemma 2.1 and the Lebesgue's dominated convergence theorem, we get

$$\hat{\Psi}_{ij}(t,M) \xrightarrow{\text{a.s.}} \Psi_{ij}(t) \quad \text{as} \quad M \longrightarrow \infty.$$

To prove that the estimator of the Markov renewal matrix is uniformly strongly consistent on compact $[0,M]$, for $M \in \mathbb{R}^+$, observe that $\Psi_{ij}(t)$ is monotone and continuous (since F is continuous), so, the convergence of $\hat{\Psi}_{ij}(t,M)$ to $\Psi_{ij}(t,T)$ is uniform for $t \in [0,M]$.

(b) First we define $\Delta\Psi_{ij} = \Big(\hat{\Psi}_{ij}(t,M) - \Psi_{ij}(t)\Big)$, By the Markov renewal equation we see that,

$$\begin{aligned}
M^{1/2}\Delta\Psi_{ij} &= M^{1/2}\left[\hat{\Psi}_{ij}(t) - (\hat{\Psi} * \Psi)_{ij}(t) + (\hat{\Psi} * \Psi)_{ij}(t) - \Psi_{ij}(t)\right] \\
&= M^{1/2}\left\{[\hat{\Psi} * (I - \Psi)]_{ij}(t) + [(\hat{\Psi} - I) * \Psi]_{ij}(t)\right\} \\
&= M^{1/2}\left\{-(\hat{\Psi} * Q * \Psi)_{ij}(t) + (\hat{\Psi} * \hat{Q} * \Psi)_{ij}(t)\right\} \\
&= M^{1/2}[\hat{\Psi} * \Delta Q * \Psi]_{ij}(t) \\
&= M^{1/2}[\hat{\Psi} * \Delta Q * \Psi * \Delta Q * \Psi]_{ij}(t) \\
&\quad + M^{1/2}[\Psi * \Delta Q * \Psi]_{ij}(t).
\end{aligned} \tag{2.4}$$

Since for all $i,k,l,r,v,w \in \mathbf{E}$,

$$\sup_{s \leqslant t}\left[\hat{\Psi}_{ik}(\cdot,M) * \Psi_{lr}(\cdot) * \Psi_{vw}(\cdot)\right](s) \leqslant \left[\sum_{m=1}^{\infty} \beta_{ij}^m(t)\right] \cdot \Psi_{lr}(t) \cdot \Psi_{vw}(t) < +\infty,$$

we conclude by Lemma 2.1, that the first term of 2.4 converges in probability to zero as M tends towards infinity.

The last term can be written as follows:

$$\begin{aligned}
&M^{1/2}[\Psi * \Delta Q * \Psi]_{ij}(t) \\
=&M^{1/2}\sum_{l=1}^{s}\sum_{r=1}^{s}\Big(\Psi_{il} * (\widehat{\mathbf{Q}}(\cdot,M) - \mathbf{Q})_{lr} * \Psi_{rj}\Big)(t) \\
=&M^{1/2}\sum_{l=1}^{s}\sum_{r=1}^{s}\Big(\Psi_{il} * \widehat{Q}(\cdot,M)_{lr} * \Psi_{rj}\Big)(t) - \sqrt{M}\sum_{l=1}^{s}\sum_{r=1}^{s}\left(\Psi_{il} * Q_{lr} * \Psi_{rj}\right)(t) \\
=&\frac{1}{M^{1/2}}\sum_{n=1}^{N(M)}\sum_{l=1}^{s}\frac{M}{N_l(M)}\sum_{r=1}^{s}\left[\left(\Psi_{il} * \mathbf{1}_{\{J_{n-1}=l, J_n=r, X_n=\cdot\}} * \Psi_{rj}\right)(t)\right. \\
&\left. - \left(\Psi_{il} * Q_{lr}\mathbf{1}_{\{J_{n-1}=l\}} * \Psi_{rj}\right)(t)\right].
\end{aligned}$$

Since $N_l(M)/M \xrightarrow[M\to\infty]{\text{a.s.}} 1/\mu_{ll}$, using Slutsky's Theorem 2.2.2 we obtain that $M^{1/2}\left[\widehat{\Psi}_{ij}(t,M)-\Psi_{ij}(t)\right]$ has the same limit in distribution as

$$\begin{aligned}
&\frac{1}{M^{1/2}}\sum_{n=1}^{N(M)}\sum_{l=1}^{s}\mu_{ll}\sum_{r=1}^{s}\left[\left(\Psi_{il}*\mathbf{1}_{\{J_{n-1}=l,J_n=r,X_n=\cdot\}}*\Psi_{rj}\right)(t)\right.\\
&\left.-\left(\Psi_{il}*Q_{lr}\mathbf{1}_{\{J_{n-1}=l\}}*\Psi_{rj}\right)(t)\right]\\
=&\sqrt{\frac{N(M)}{M}}\frac{1}{\sqrt{N(M)}}\sum_{n=1}^{N(M)}f(J_{n-1},J_n,X_n)
\end{aligned}$$

where the random variables $f(J_{n-1},J_n,X_n)$ are defined by

$$\begin{aligned}
f(J_{n-1},J_n,X_n):=&\sum_{l=1}^{s}\mu_{ll}\sum_{r=1}^{s}\left[\left(\Psi_{il}*\mathbf{1}_{\{J_{n-1}=l,J_n=r,X_n=\cdot\}}*\Psi_{rj}\right)(t)\right.\\
&\left.-\left(\Psi_{il}*Q_{lr}\mathbf{1}_{\{J_{n-1}=l\}}*\Psi_{rj}\right)(k)\right].
\end{aligned}$$

By theorem 2.2.3, we deduce that the second term of 2.4 converges in law to a normal random variable with mean zero and variance $\sigma_{ij}^2(t)$. □

Lemme 2.2. *For all $i,j,k,l\in\mathbf{E}$, $M^{1/2}\left|\left[\Delta Q_{ij}*\Delta Q_{kl}\right](t)\right|$ converges in probability to zero, when M tends to infinity.*

Lemme 2.3. *For all $i,j\in\{1,\dots,s\}$, $t\in[0,M]$ and $l\in\mathbb{N}^*$, the random variate $M^{1/2}\Delta Q_{ij}^{(l)}(t)$ has the same limit in law as $\sum_{r=1}^{l}\sum_{k=1}^{s}\sum_{n=1}^{s}M^{1/2}\left[Q_{ik}^{(r-1)}*\Delta Q_{kn}*\ Q_{nj}^{(l-r)}\right](t)$ as M tends to infinity.*

Theorem 2.2.8. *[32] The estimator $\widehat{P}_{ij}(t,M)$ of the transition function $P_{ij}(t)$, satisfies the following two properties:*

(a) ***(Strong consistency)*** *For any fixed $L>0$, we have, as $M\to\infty$,*

$$\max_{i,j\in\mathbf{E}}\sup_{t\in[0,L]}\left|\widehat{P}_{ij}(t,M)-P_{ij}(t)\right|\xrightarrow{a.s.}0,$$

(b) ***(Asymptotic normality)*** *For any fixed $t>0$, we have, as $M\to\infty$,*

$$M^{1/2}\left(\widehat{P}_{ij}(t,M)-P_{ij}(t)\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\sigma_{ij}^2(t)\right),$$

where

$$\begin{aligned}
\sigma_{ij}^2(t)=&\sum_{r\in\mathbf{E}}\sum_{k\in\mathbf{E}}\mu_{rr}\left[(1-H_i)*B_{irkj}-\Psi_{ij}\mathbf{1}_{\{r=j\}}\right]^2*Q_{rk}(t)\\
&-\left\{\left[(1-H_i)*B_{irkj}-\Psi_{ij}\mathbf{1}_{\{r=j\}}\right]*Q_{rk}(t)\right\}^2,
\end{aligned}$$

and

$$B_{irkj}(t)=\sum_{n=1}^{\infty}\sum_{\ell=1}^{n}Q_{ir}^{(\ell-1)}*Q_{kj}^{(n-\ell)}(t).$$

Proof. (a) Let us consider the matrices $B(t) = I - \operatorname{diag}(Q(t)\cdot\mathbf{1})$ and $\hat{B}(t,M) = I - \operatorname{diag}(\hat{Q}(t,M)\cdot\mathbf{1})$. then for $(i,j) \in \mathbf{E}\times\mathbf{E}$ be fixed then,

$$\begin{aligned}\sup_{t\in[0,L]}\left|\hat{P}_{ij}(t,M) - P_{ij}(t)\right| &= \sup_{t\in[0,L]}\left|(\hat{\Psi}*\hat{B})_{ij}(t,M) - (\Psi*B)_{ij}(t)\right| \\ &\le \sup_{t\in[0,L]}\left|\hat{\Psi}_{ij}(t,M) - \Psi_{ij}(t)\right| \\ &+ \sup_{t\in[0,L]}\left|\hat{\Psi}_{ij}(t,M) - \Psi_{ij}(t)\right|\cdot\operatorname{diag}(\hat{Q}(t,M)\cdot\mathbf{1}) \\ &+ \sup_{t\in[0,L]}\left|\operatorname{diag}((\hat{Q}-Q)\cdot\mathbf{1})_{jj}(t,M)\right|\Psi_{ij}(L).\end{aligned}$$

Since s, the number of states, is finite, the process is normal and therefore $\Psi_{ij}(t)$ is finite (cf. [33]). From Theorems 2.2.6 and 2.2.7 on the uniformly strong consistency of the estimators of the semi-Markov kernel and of the Markov renewal function in $[0,L]$, we see that $\operatorname{diag}((\hat{Q}-Q)\cdot\mathbf{1})_{jj}(t,M)$ and $\Delta\Psi_{ij}(t,M)$ on $[0,L]$ converge a.s. to zero as M tends to infinity.

(b) First we define $\Delta P_{ij} = \left(\hat{P}_{ij}(t,M) - P_{ij}(t)\right)$, $\Delta Q_{ij} = \left(\hat{Q}_{ij}(t,M) - Q_{ij}(t)\right)$, from 2.3 we see that,

$$\begin{aligned}M^{1/2}\Delta P_{ij} =& M^{1/2}\left[\hat{\Psi}_{ij}*(I-\operatorname{diag}(\hat{Q}\mathbf{1}))_{jj} - \Psi_{ij}*(I-\operatorname{diag}(Q\mathbf{1}))_{jj}\right. \\ =& M^{1/2}\left[\left(\hat{\Psi}_{ij}-\Psi_{ij}\right)*(I-\operatorname{diag}(Q\mathbf{1}))_{jj}-\right. \\ & \left.-\Psi_{ij}*\operatorname{diag}([\hat{Q}-Q]\mathbf{1})_{jj}\right] - \\ & -\left(\hat{\Psi}_{ij}-\Psi_{ij}\right)*\operatorname{diag}([\hat{Q}-Q]\mathbf{1})_{jj}.\end{aligned} \tag{2.5}$$

From Lemma 2.2, $M^{1/2}\left(\hat{\Psi}_{ij}-\Psi_{ij}\right)*\operatorname{diag}([\hat{Q}-Q]\mathbf{1})_{jj}$ converges, in probability, when M tends to infinity, to zero. So, $M^{1/2}\Delta P_{ij}(t,M)$ has the same limit as

$$M^{1/2}\left[\left(\hat{\Psi}_{ij}-\Psi_{ij}\right)*(I-\operatorname{diag}(Q\mathbf{1}))_{jj} - \Psi_{ij}*\operatorname{diag}([\hat{Q}-Q]\mathbf{1})_{jj}\right].$$

From Lemma 2.3 and Theorem 2.2.7, it has the same limit as,

$$\begin{aligned}&M^{1/2}\left[\left(1-\sum_{l=1}^{s}Q_{jl}\right)*\left(\sum_{n=1}^{s}\sum_{k=1}^{s}B_{inkj}*\Delta Q_{nk}\right) - \Psi_{ij}*\left(\sum_{k=1}^{s}\Delta Q_{jk}\right)\right] \\ &= \sum_{n=1}^{s}\sum_{k=1}^{s}M^{1/2}\left[\left(1-\sum_{l=1}^{s}Q_{jl}\right)*B_{inkj}*\Delta Q_{nk}\right] - \sum_{k=1}^{s}M^{1/2}\Psi_{ij}*\Delta Q_{jk}.\end{aligned}$$

Let f be a real function defined on $\mathbf{E}\times\mathbf{E}\times\mathbb{R}_+$by

$$\begin{aligned}f(r,m,x) = &\left[\left(1-\sum_{l=1}^{s}Q_{jl}\right)*B_{inkj} - \Psi_{ij}\mathbf{1}_{\{n=j\}}\right]\times \\ &\times\mathbf{1}_{\{r=n\}}\left(\mathbf{1}_{\{m=k,x\leqslant t\}} - Q_{nk}\right).\end{aligned}$$

So, $M^{1/2}\Delta P_{ij}(t,M) = M^{1/2}W_f(t)$, where

$$W_f(t) = \sum_{n=1}^{s}\sum_{k=1}^{s}\frac{\sum_{l=1}^{N_n} f\left(J_{l-1},J_l,X_l\right)}{N_n},$$

we obtain the desired findings using Pyke and Schaufele's ([33]) central limit theorem 2.2.3 . $\square$

Chapter 3

Integral functionals of semi-Markov processes in reliability problems

The wear on technological objects is the result of a variety of random harmful events. The wear is frequently the outcome of the repeated extortion's impacts. When the accumulated effects of extortion surpass a certain threshold of deterioration, the object is damaged. It is a difficult task to numerically simulate many real-world object wearing processes. Here, we only look at situations that result in quite straightforward models. The research examines the possibility of determining the parameters and reliability features of an item degradation based on stochastic process models. Stochastic models of breakdown have been constructed using the integral functional of semi-Markov processes.

3.1 Definition and basic properties

Definition 3.1.1 (**Integral Functionals**). *Let us consider a right-continuous semi-Markov process $Z_t, t \geqslant 0$, with state space $\mathbf{E}$ and a function $h : \mathbf{E} \to \mathbb{R}$. Define the following integral functional*

$$\mathrm{L}(t) = \int_0^t h(Z_s)\, ds, \quad t \geq 0, \tag{3.1}$$

is called an integral functional of the SM process or a cumulative process of the SM process $\{Z(t) : t \geqslant 0\}$. If $\{(J_n, S_n) : n \in \mathbb{N}_0\}$ is the MRP defining the process $\{\mathrm{L}(t) : t \geqslant 0\}$ then

$$\mathrm{L}(t) = h(J_0)\, S_1 + \ldots + h(J_{n-1})\, S_n + h(J_n)(t - Z_n) \quad \textit{for} \quad t \in [Z_n, Z_{n+1})\,. \tag{3.2}$$

This formula allows to generate the trajectories of the process $\{\mathrm{L}(t) : t \geqslant 0\}$. *Using the definition of the counting process we obtain an equivalent form of the formula 3.2*

$$\mathrm{L}(t) = \sum_{k=1}^{N(t)} h\left(J_{k-1}\right) S_k + \left(t - Z_{N(t)}\right) h\left(J_{N(t)}\right). \tag{3.3}$$

Example 3.1.1. *Let* $\{Z(t) : t \geqslant 0\}$ *be a semi-Markov process with a state space* $\mathbf{E} = \{0, 1, 2\}$ *which is generated by* MRP $\{(J_n, S_n) : n \in \mathbb{N}\}$. *Assume* $h(x) = 0.5x, \quad x \geqslant 0,\ 0 \leqslant t \leqslant 12$ *and*

$$\begin{gathered} \{(J_0 = 2, S_0 = 0), (J_1 = 1, S_1 = 4.2), (J_2 = 0, S_2 = 2.8), \\ (J_3 = 2, S_3 = 1.2), (J_4 = 2, S_4 = 3.1), \ldots\}. \end{gathered} \tag{3.4}$$

From 3.2 we obtain a piece of trajectory of the stochastic process $\{\mathrm{L}(t) : t \geqslant 0\}$:

$$\mathrm{L}(t) = \begin{cases} t & \textit{for} \quad t \in [0, 4.2). \\ 4.2 + 0,5(t - 4.2) & \textit{for} \quad t \in [4.2, 7). \\ 5.6 & \textit{for} \quad t \in [7, 8.2). \\ 5.6 + 0.5(t - 8.2) & \textit{for} \quad t \in [8.2, 11.3). \\ 7.15 + (t - 11.3) & \textit{for} \quad t \in [11.3, 12). \end{cases} \tag{3.5}$$

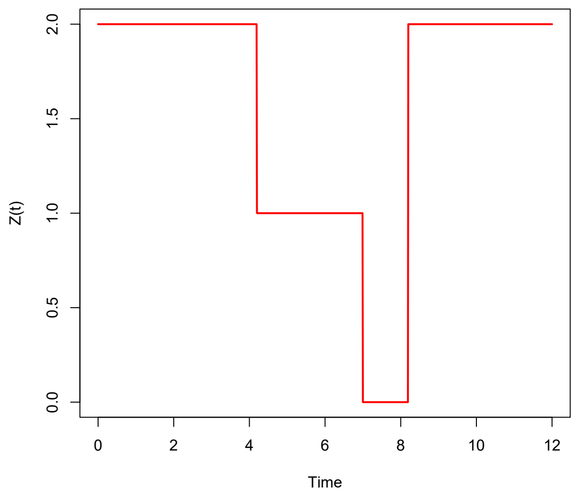

Figure 3.1: Trajectory of Z(t).

Figure 3.1 shows a trajectory of a semi-Markov process corresponding to a MRP.

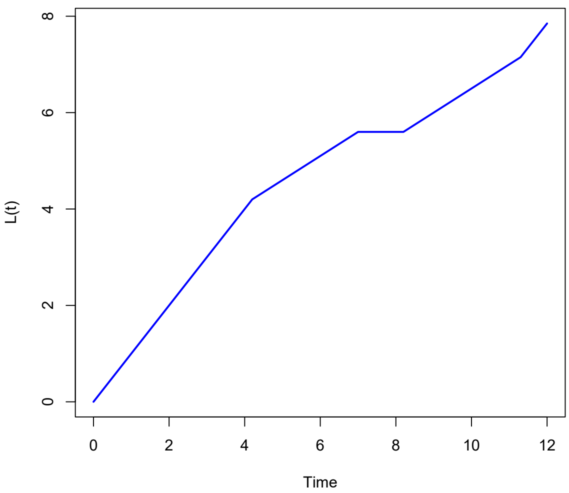

Figure 3.2: Trajectory of L(t).

Realization 3.4 and figure 3.2 illustrates the corresponding trajectory of the integral functional.

Consider a joint distribution of the process $\{\mathrm{Z}(t) : t \geqslant 0\}$ and $\{\mathrm{L}(t) : t \geqslant 0\}$. Let

$$U_{iA}(t, Z) = \mathbb{P}(Z(t) \in A, L(t) \leqslant x \mid Z(0) = i), \quad i \in \mathbf{E}. \tag{3.6}$$

Theorem 3.1.1. *[20] The functions $U_{iA}(t, x), i \in \mathbf{E}$ satisfy a system of integral equations*

$$\begin{aligned} U_{iA}(t, x) &= I_{A\times[0,x]}(i, h(i)t)\,[1 - H_i(t)] \\ &+ \sum_{j\in\mathbf{E}} \int_0^t U_{jA}(t - v, x - h(i)v) dQ_{ij}(v), \quad i \in \mathbf{E}. \end{aligned} \tag{3.7}$$

For $A = \mathbf{E}$ we obtain

$$U_{i\mathbf{E}}(t, x) = \mathbb{P}\{L(t) \leqslant x \mid Z(0) = i\}, \quad i \in \mathbf{E}. \tag{3.8}$$

Proposition 3.1.1. *[20]*

The conditional cumulative distribution functions (CDF) of the process $\{L(t) : t \geqslant 0\}$ satisfy a system of the integral equations

$$U_{i\mathbf{E}}(t, x) = I_{[0,x]}(h(i)t)\,[1 - H_i(t)] + \sum_{j\in\mathbf{E}} \int_0^t U_{j\mathbf{E}}(t - v, x - h(i)v) dQ_{ij}(v). \tag{3.9}$$

From 3.4 we get

$$U_{iA}(t, \infty) = \lim_{x\to\infty} U_{iA}(t, x) = \mathbb{P}\{Z(t) \in A \mid Z(0) = i\}, \quad i \in \mathbf{E}.$$

Let

$$P_{iA}(t) = U_{iA}(t, \infty), \quad i \in \mathbf{E}.$$

The conditional probability $P_{iA}(t), i \in \mathbf{E}$ verifies a system of integral equations [20]

$$P_{iA}(t) = I_A(i)\,[1 - G_i(t)] + \sum_{j \in \mathbf{E}} \int_0^t P_{jA}(t - v) dQ_{ij}(v), \quad i \in \mathbf{E}. \tag{3.10}$$

If A denotes subset of "up" states of the object, then $P_{iA}(t)$ denotes it's availability under condition that an initial state is $i \in \mathbf{E}$.

A cumulative process $\{\mathrm{L}(t) : t \geqslant 0\}$ allows defining a random process $\{T(x) : x \geqslant 0\}$ by

$$T(x) = \inf\{t : L(t) > x\}. \tag{3.11}$$

A random variable $T(x)$ denotes an instant of a level x exceeding by the cumulative process. If x is the critical level of the process describing degradation of an object then

$$R(t) = \mathbb{P}(T(x) > t), \quad t \geqslant 0. \tag{3.12}$$

is a **reliability function**.

A stochastic process $\{K_j(t) : t \geqslant 0\}$ defined by

$$K_j(t) = \int_0^t \mathbf{1}_{(Z(u)=j)}\, du, \tag{3.13}$$

where

$$\mathbf{1}_{(Z(u)=j)} = \begin{cases} 1 & \text{for} \quad Z(u) = j \\ 0 & \text{for} \quad Z(u) \neq j \end{cases}, \tag{3.14}$$

is an example of an integral functional of a SM process. A value of the random variable $K_j(t)$ denotes a cumulated sojourn time of the SM process $\{Z(t) : t \geqslant 0\}$ in a state j, during the interval $[0, t]$. The process $\{K_j(t) : t \geqslant 0\}$ is connected with the process $\{T_j(x) : x \geqslant 0\}$ defined by

$$T_j(x) = \inf\{t : K_j(t) > x\}. \tag{3.15}$$

A random process $\{T_j(x) : x \geqslant 0\}$ denotes an instant of time of exceeding a level x by the cumulated sojourn time of the SM process. Those processes have asymptotically normal distribution with parameters depending on a kernel of the process $\{Z(t) : t \geqslant 0\}$. In renewal theory there is well known concept of an alternating process. We can treat it as the SM process $\{Z(t) : t \geqslant 0\}$ with a state space $\mathbf{E} = \{0, 1\}$, a kernel

$$\boldsymbol{Q}(t) = \begin{bmatrix} 0 & F_{\gamma}(t) \\ F_{\zeta}(t) & 0 \end{bmatrix}. \tag{3.16}$$

and an initial distribution

$$\alpha_0 = [0, 1]. \tag{3.17}$$

Where $F_{\gamma}(t), F_{\zeta}(t)$ are CDF of the nonnegative, independent random variables γ, ζ. From definition of the process $\{K_j(t) : t \geqslant 0\}$ it follows that the process is connected with the cumulated sojourn time in a state 1 of the alternating process. If $\zeta_n, n = 1, 2, \ldots$, denoting consecutive waiting times of the state j, are supposed to be the random variables with an identical probability

$$G_j(t) = \mathbb{P}(\zeta \leqslant t) = \mathbb{P}\left(T_j \leqslant t\right). \tag{3.18}$$

While $\gamma_n, n = 1, 2, \ldots$, denote the lengths of the time intervals, that pass from the instants of n-th going out from the state j to next going in this state, then the definition of the process $\{K_j(t) : t \geqslant 0\}$ which start with the state j is almost identical with definition of the simple alternating process. The only difference lies in that for general SM process, the random variables $\zeta_n, \gamma_n, n = 1, 2, \ldots$, can be dependent. But we can distinguish a class of the SM processes, that the mentioned above random variables are independent.

If $\{Z(t) : t \geqslant 0\}$ is SM process with a kernel $\boldsymbol{Q}(t) = [Q_{ij}(t) : i, j \in \mathbf{E}]$ such that $Q_{ij}(t) = p_{ij} G_j(t)$, where $p_{ii} = 0$ for $i \in \mathbf{E}$, then the random variables $\zeta_n, \gamma_n, n = 1, 2, \ldots$ are independent and

$$\Theta_{jj}^{(n)} = \zeta_n + \gamma_n. \tag{3.19}$$

Where $\Theta_{jj}^{(n)}$ is n-th return time to the j state .

The equations which allow to calculating the one dimensional distribution of the process $\{K_j(t) : t \geqslant 0\}$ are presented in books [19], [20]. But they are very complicated and difficult for applying. Then, in this case we can use an approximate formula which implies from the following theorem [19], [20]:

Theorem 3.1.2. *Let $\{Z(t) : t \geqslant 0\}$ be a SM process defined by a continuous type kernel $\boldsymbol{Q}(t) = [Q_{ij}(t) : i, j \in \mathbf{E}]$ such that*

$$Q_{ij}(t) = p_{ij} G_j(t), \quad \textit{Where} \quad p_{ii} = 0 \quad \textit{for} \quad i \in \mathbf{E}.$$

If moreover the random variables T_j, Θ_{jj} have positive finite second moments, then

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{K_j(t) - m_j(t)}{\sigma_j(t)} \leqslant x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \tag{3.20}$$

where

$$m_j(t) = \frac{E(T_j)}{E(\Theta_{jj})} t. \tag{3.21}$$

$$\sigma_j(t) = \sqrt{\frac{V(T_j)\left[E(\Theta_{jj}) - E(T_{jj})\right]^2 + \left[V(\Theta_{jj}) - E(T_j)\right]\left[E(T_j)\right]^2}{\left[E(\Theta_{jj})\right]^3} t}. \tag{3.22}$$

From the above theorem it follows that the process $\{K_j(t) : t \geqslant 0\}$ has an approximately normal distribution with an expectation given by an equality 3.14 and a standard deviation taking the form of 3.22.

Under the same assumptions the process $\{T_j(x) : x \geqslant 0\}$ denoting the moment of exceeding a level x by the sojourn time of the SM process has an approximately normal distribution

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{T_j(x) - m_j(x)}{\sigma_j(x)} \leqslant y\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{u^2}{2}} du, \tag{3.23}$$

where

$$m_j(x) = \frac{E(\Theta_{jj})}{E(T_j)} x. \tag{3.24}$$

$$\sigma_j(x) = \sqrt{\frac{V(T_j)\left[E(\Theta_{jj}) - E(T_j)\right]^2 + \left[V(\Theta_{jj}) - V(T_j)\right]\left[E(T_j)\right]^2}{\left[E(T_j)\right]^3} x}. \tag{3.25}$$

For the alternating process $\{Z(t) : t \geqslant 0\}$ with the kernel 3.16 the formulas 3.21, 3.22, 3.24, and 3.25 are of the form

$$\begin{aligned}
m(t) &= \frac{E(\zeta)}{E(\zeta) + E(\gamma)} t. \\
\sigma(t) &= \sqrt{\frac{[V(\zeta)][E(\gamma)]^2 + [V(\gamma)][E(\zeta)]^2}{[E(\zeta) + E(\gamma)]^3} t}. \\
m(x) &= \frac{E(\zeta) + E(\gamma)}{E(\zeta)} x. \\
\sigma(x) &= \sqrt{\frac{[V(\zeta)][E(\gamma)]^2 + [V(\gamma)][E(\zeta)]^2}{[E(\zeta)]^3} x}.
\end{aligned} \tag{3.26}$$

Example 3.1.2. *After* $c = 50000[\,\mathrm{h}\,]$ *of working a plane engine is treated as broken-down. We suppose that a sojourn time of one fly is random variable* ζ *with an expectation* $E(\zeta) = 3.48[\,\mathrm{h}\,]$ *and a variance* $V(\zeta) = 2.15\,\left[\,\mathrm{h}^2\right]$. *A time of the each plane stoppage is a positive random variable* γ *with the expectation* $E(\gamma) = 4.5[\,\mathrm{h}\,]$ *and the variance* $V(\gamma) = 4.21\,\left[\,\mathrm{h}^2\right]$. *Under those assumption the alternating process* $\{Z(t) : t \geqslant 0\}$ *defined by the kernel 3.16 and the initial distribution 3.17 is a reliability model of the plane engine operation process. A random variable* $T(c) = \inf\{t : K(t) > c\}$, *where* $K(t) = \int_0^t Z(u)du$ *denotes an instant of exceeding a level* c *by a summary sojourn time of the alternating process* $\{Z(t) : t \geqslant 0\}$. *From above presented theorem it follows that a random variable* $T(c)$ *has an approximately normal distribution* $N(m(c), \sigma(c))$, *where*

$$m(c) = \frac{E(\zeta) + E(\gamma)}{E(\zeta)} c.$$

and

$$\sigma(c) = \sqrt{\frac{V(\zeta)[E(\gamma)]^2 + [V(\gamma)][E(\zeta)]^2}{[E(\zeta)]^3} c}.$$

The estimated reliability function $R(t), t \geqslant 0$ *takes the form:*

$$R(t) = \mathbb{P}(T(c) > t) \approx 1 - \Phi\left(\frac{t - m(c)}{\sigma(c)}\right).$$

where $\Phi(\cdot)$ *is CDF of the standard normal distribution. For above assumed parameters we obtain the expectation and the standard deviation for a time to failure of the plane engine.*

$$m(50000) \approx 114655[\,\mathrm{h}] \approx 4777.29[\,\textit{days}\,], \quad \sigma(50000) \approx 334.0[\,\mathrm{h}] \approx 13.92[\,\textit{days}\,].$$

The estimated reliability function $R(t), t \geqslant 0$ *is*

$$R(t) \approx 1 - \Phi\left(\frac{t - 114655,0}{334.0}\right).$$

The function is shown in figure 3.3. The value of the function for $c = 50000$ *is 0.99865 .*

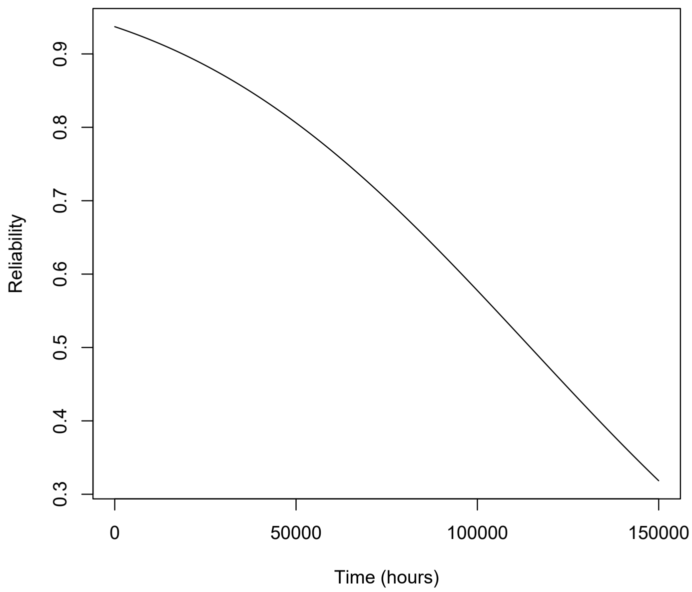

Figure 3.3: Approximate reliability function of the plain engine.

Remark 3.1.1. *Cumulative processes can be used as a probability models of the object degradation. Theoretical results give possibility to obtain approximate reliability parameters and characteristics.*

3.2 Discrete-time semi-Markov process reliability analysis

In reliability, the state space $\mathbf{E} = \{1, \ldots, s\}$, is naturally partitioned into two sets, U and D, where $U = \{1, \ldots, s_1\}$ is the set of working (or "up") states , and $D = \{s_1 + 1, \ldots, s\}$ is the set of repair (or "down") states , i.e. $\mathbf{E} = U \cup D$ and $U \cap D = \emptyset$. Finite semi-Markov reliability models whose state spaces are partitioned in the above manner will be considered here. The transition from one state to another means, physically speaking, the failure or the repair of one of the components of the system. In the set of up states, U, the system is operational. No service is delivered if the system is in the set of down states, D. However, a repair will return the system from D to U. To model this situation, it will be assumed that the MRP is irreducible.

Definition 3.2.1. *Consider a system $\mathcal{S}$ starting to function at time $k = 0$ and let T_D denote the first passage time in subset D, called the lifetime of the system, i.e.,*

$$T_D := \inf\{n \in \mathbb{N}; \quad Z_n \in D\} \textit{ and } \inf \emptyset := \infty.$$

The reliability of a discrete-time semi-Markov system $\mathcal{S}$ at time $k \in \mathbb{N}$, that is the probability that the system has functioned without failure in the period $[0, k]$ is

$$R(k) := \mathbb{P}\left(T_D > k\right) = \mathbb{P}\left(Z_n \in U, n = 0, \dots, k\right).$$

The following outcome describes the system's reliability in terms of the fundamental semi-Markov chain quantities, In the sequel, for a matrix X, we will denote X^U its restriction to U.

Proposition 3.2.1. *The reliability of a discrete-time semi-Markov system at time $k \in \mathbb{N}$ is given by:*

$$R(k) = \alpha^U \mathbf{P}^U(k)\mathbf{1}_{s_1} = \alpha^U \left(\delta I - \mathbf{q}^U\right)^{(-1)} * \left(I - \operatorname{diag}(\mathbf{Q}\cdot\mathbf{1})^U\right)(k)\mathbf{1}_{s_1}. \tag{3.27}$$

Where $\mathbf{1}_{s_1} = (1, 1, \dots, 1)^t$.

3.2.1 Asymptotic properties of the estimators

The expression of the reliability given in 3.27, together with the estimators of the semi-Markov transition function and of the cumulative semi-Markov kernel given above, allow us to obtain the estimator of the system's reliability at time k given by

$$\begin{aligned}\widehat{R}(k, M) :=& \widehat{\alpha}^U \cdot \widehat{\mathbf{P}}^U(k, M) \cdot \mathbf{1}_{s_1} \\ &= \widehat{\alpha}^U \left[\left(\delta I - \widehat{\mathbf{q}}^U\right)(\cdot, M) * \left(I - \operatorname{diag}(\widehat{\mathbf{Q}}(\cdot, M)\cdot\mathbf{1})^U\right)\right](k)\mathbf{1}_{s_1}.\end{aligned}$$

Let us give now the result concerning the consistency and the asymptotic normality of the reliability estimator.

Theorem 3.2.1. *For any fixed arbitrary positive integer $k \in \mathbb{N}$, the estimator of the reliability of a discrete-time semi-Markov system at instant k is* ***strongly consistent****, i.e.,*

$$|\widehat{R}(k, M) - R(k)| \xrightarrow[M\to\infty]{a.s} 0.$$

and ***asymptotically normal****, i.e., we have*

$$\sqrt{M}[\widehat{R}(k, M) - R(k)] \xrightarrow[M\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \sigma_r^2(k)\right),$$

with the asymptotic variance

$$\begin{aligned}\sigma_r^2(k) = \sum_{i=1}^{s} \mu_{ii} \Bigg\{ &\sum_{j=1}^{s} \left[D_{ij}^U - \mathbf{1}_{\{i\in U\}} \sum_{n\in U} \alpha_n \Psi_{ti}\right]^2 * q_{ij}(k) \\ &- \left[\sum_{j=1}^{s} \left(D_{ij}^U * q_{ij} - \mathbf{1}_{\{i\in U\}} \sum_{n\in U} \alpha_n \psi_{ti} * Q_{ij}\right)\right]^2 (k) \Bigg\},\end{aligned}$$

where

$$D_{ij}^U := \sum_{n\in U}\sum_{r\in U} \alpha_n \psi_{ni} * \psi_{jr} * (I - \mathrm{diag}(\mathbf{Q}\cdot\mathbf{1}))_{rr},$$

$$\psi_{ij}(k) := \sum_{n=0}^{k} q_{ij}^{(n)}(k), \quad \Psi_{ij}(k) := \sum_{n=0}^{k} Q_{ij}^{(n)}(k),$$

μ_{ii} : *the mean recurrence time of the state i for the chain* Z.

3.2.2 Asymptotic confidence intervals

The previously obtained asymptotic results allow one to construct the asymptotic confidence intervals for reliability. For this purpose, we need to construct a consistent estimator of the asymptotic variances.

Firstly, we can construct estimators of $\psi(k)$ and of $Q(k, M)$. One can check that these estimators are strongly consistent. Secondly, for $k \le M$, replacing $Q(k), \psi(k)$ respectively by $\widehat{Q}(k, M), \widehat{\psi}(k, M)$, we obtain an estimator $\widehat{\sigma}_r^2(k)$ of the variance $\sigma_r^2(k)$. From the strong consistency of the estimators $\widehat{Q}(k, M)$ and $\widehat{\psi}(k, M)$ (see [3], [2]), we obtain that $\widehat{\sigma}_r^2(k)$ converges almost surely to $\sigma_r^2(k)$, as M tends to infinity. Finally, the asymptotic confidence interval of $R(k)$ at level $100(1-\gamma)\%$, $\gamma \in (0, 1)$, is:

$$\widehat{R}(k, M) - u_{1-\gamma/2}\frac{\widehat{\sigma}_r(k)}{\sqrt{M}} \le R(k) \le \widehat{R}(k, M) + u_{1-\gamma/2}\frac{\widehat{\sigma}_r(k)}{\sqrt{M}}. \tag{3.28}$$

where u_γ is the γ - quantile of an $\mathcal{N}(0, 1)$ - distributed variable.

3.3 Continuous-time semi-Markov process reliability analysis

Definition 3.3.1. *The reliability* $R(t)$ *of the system is given by:*

$$R(t) = \sum_{i\in U} \alpha_i R_i(t) = \mathbb{P}\left[J_{N_s} \in U, \quad \text{for all} \ \ s \leqslant t\,\right].$$

Let $R_i(t)$ *to be the conditional probability that the first failure does not occur up to time* t*, given that the process started from state* $i \in U$*. So*

$$R(t) = \alpha^U \cdot P^U(t) \cdot \mathbf{1}_{s_1},$$

where

$$P^U(t) = \left(I - Q^U(t)\right)^{(-1)} * \left(I - \mathrm{diag}(Q(t)\mathbf{1}_{s_1})^U\right). \tag{3.29}$$

3.3.1 Asymptotic properties of the estimators

We shall give estimators of the reliability of semi-Markov systems and prove uniform strong consistency and weak convergence theorems, for these estimators when M, the censored time, tends to infinity.

Reliability estimator of a semi-Markov system can be expressed in closed forms as follows (Limnios, [31]).

$$\hat{R}(t, M) = \alpha^U \cdot \hat{P}^U(t, M).\mathbf{1}_{s_1}, \tag{3.30}$$

where

$$\hat{P}^U(t, M) = \Big(I - \hat{Q}^U(t, M)\Big)^{(-1)} * \Big(I - \mathrm{diag}(\hat{Q}(t, M)\mathbf{1}_{s_1})^U\Big).$$

In this part, we provide uniformly strong consistency and central limit theories for the reliability estimator.

Theorem 3.3.1. *[32] The estimator of the reliability of the semi-Markov system is uniformly strongly consistent in the sense that, for all $L \in R_+$, when $M \to \infty$*

$$\sup_{t\in[0,L]} |\hat{R}(t, M) - R(t)| \xrightarrow[M\to\infty]{a.s} 0.$$

Proof. Let us consider the matrices $B(t) = I - \mathrm{diag}(Q(t)\cdot\mathbf{1}_{s_1})$ and $\hat{B}(t, M) = I - \mathrm{diag}(\hat{Q}(t, M) \cdot \mathbf{1}_{s_1})$. Then $P(t) = [\Psi * B](t)$ and $\hat{P}(t, M) = [\hat{\Psi} * \hat{B}](t, M)$ is its estimator. Let $i, j \in \mathbf{E}$ be fixed. Then:

$$\sup_{t\in[0,L]} |\hat{R}(t, M) - R(t)| = \sup_{t\in[0,L]} \left| \sum_{i\in U} \sum_{j\in U} \Big\{ \hat{\alpha}_i (\hat{\Psi} * \hat{B})_{ij}(t, M) - \alpha_i (\Psi * B)_{ij}(t) \Big\} \right|$$

which has the same limit as

$$\begin{aligned}
&\sup_{t\in[0,L]} \left| \sum_{i\in U} \sum_{j\in U} \Big\{ \alpha_i (\hat{\Psi} * \hat{B})_{ij}(t, M) - \alpha_i (\Psi * B)_{ij}(t) \Big\} \right| \\
&\leqslant s_1^2 \left\{ \max_{i,j\in\mathbf{E}} \sup_{t\in[0,L]} \left| \hat{\Psi}_{ij}(t, M) - \Psi_{ij}(t) \right| \right. \\
&\left. + \max_{i,j\in\mathbf{E}} \sup_{t\in[0,L]} \left| \hat{\Psi}_{ij}(t, M) - \Psi_{ij}(t) \right| \cdot \mathrm{diag}(\hat{Q}(t, M) \cdot \mathbf{1}_{s_1}) \right\} \\
&+ s_1^2 \left\{ \max_{i,j\in\mathbf{E}} \sup_{t\in[0,L]} \left| \mathrm{diag}(\hat{Q} - Q)_{jj}(t, M) \right| \Psi_{ij}(L) \right\}.
\end{aligned}$$

Since s_1, the number of up states, is finite, the process is normal and therefore $\Psi_{ij}(t)$ is finite (see Pyke and Schaufele, [33]). From the uniform strong consistency of the estimators of the semi-Markov kernel and of the Markov renewal function in $[0, L]$ obtained in Theorems 2.2.6 and 2.2.7 , we get that $\operatorname{diag}(\hat{Q} - Q)_{jj}(t, M)$ and $\hat{\Psi}_{ij}(t, M) - \Psi_{ij}(t)$ on $[0, L]$ converges (a.s.) to zero as M tends to infinity. $\square$

Theorem 3.3.2. *For any Fixed* t, $t \in [0, M]$ *,*

$$M^{1/2}[\widehat{R}(t, M) - R(t)] \xrightarrow[M \to \infty]{\mathcal{D}} \mathcal{N}\left(0, \sigma_R^2(t)\right),$$

where

$$\sigma_R^2(t) = \sum_{i \in U} \sum_{j=1}^{s} \mu_{ii} \cdot \left\{ \left(B_{ij}^U \mathbf{1}_{\{j \in U\}} - \sum_{n \in U} \alpha_n \Psi_{li}^U \right)^2 * Q_{ij}(t) \right.$$
$$\left. - \left[\left(B_{ij}^U \mathbf{1}_{\{j \in U\}} - \sum_{n \in U} \alpha_n \Psi_{li}^U \right) * Q_{ij} \right]^2 \right\}.$$

and

$$B_{ij}^U = \sum_{n \in U} \sum_{k \in U} \alpha_n B_{nijk}^U * (I - \operatorname{diag}(Q \mathbf{1}_{s_1})_{kk}).$$

Proof. From 3.29 and 3.30, we get that

$$M^{1/2} \Delta R(t, M) = M^{1/2} \sum_{i \in U} \sum_{j \in U} \left\{ \hat{\alpha}_i \cdot \hat{\Psi}_{ij} * (I - \operatorname{diag}(\hat{Q} \cdot \mathbf{1}_{s_1}))_{jj} \right.$$
$$\left. - \alpha_i \cdot \Psi_{ij} * (I - \operatorname{diag}(Q \cdot \mathbf{1}_{s_1}))_{jj} \right\},$$

which has the same limit in law as

$$M^{1/2} \sum_{i \in U} \sum_{j \in U} \alpha_i \left\{ \Delta \Psi_{ij} * (I - \operatorname{diag}(Q \cdot \mathbf{1}_{s_1}))_{jj} \right.$$
$$\left. - \Psi_{ij} * \operatorname{diag}(\Delta Q \cdot \mathbf{1}_{s_1})_{jj} - \Delta \Psi_{ij} * \operatorname{diag}(\Delta Q \cdot \mathbf{1}_{s_1})_{jj} \right\}.$$

From Lemma 2.1, the last term, i.e., $M^{1/2} \Delta \Psi_{ij} * \operatorname{diag}(\Delta Q \cdot \mathbf{1}_{s_1})_{ij}$ converges in probability to zero, as M tends to infinity. On the other hand, from Theorem 2.2.7, $M^{1/2} \Delta R(t, M)$ has the same limit in law as

$$M^{1/2} \sum_{i \in U} \sum_{j \in U} \alpha_i \left\{ \sum_{n \in U} \sum_{k \in U} B_{inkj}^U * \Delta Q_{nk} * (I - \operatorname{diag}(Q \cdot \mathbf{1}_{s_1}))_{jj} - \Psi_{ij} * \operatorname{diag}(\Delta (Q \cdot \mathbf{1}_{s_1}))_{jj} \right\}.$$

$$= M^{1/2} \sum_{n \in U} \sum_{k \in U} \left\{ \sum_{i \in U} \sum_{j \in U} \alpha_i B_{inkj}^U * (I - \operatorname{diag}(Q \cdot \mathbf{1}_{s_1}))_{jj} * \Delta Q_{nk} \right\}$$
$$- M^{1/2} \sum_{n \in U} \sum_{k \in U} \alpha_n \Psi_{nk} * \operatorname{diag}(\Delta (Q \cdot \mathbf{1}_{s_1}))_{kk},$$

which can be written as

$$M^{1/2}\sum_{n\in U}\sum_{k\in U}B_{nk}^{U}*\Delta Q_{nk}-M^{1/2}\sum_{n\in U}\sum_{k\in U}\left(\sum_{i\in U}\alpha_i\Psi_{in}\right)*\Delta Q_{nk}. \tag{3.31}$$

Consider the real measurable function $f(\cdot,\cdot,\cdot)$ defined on $\mathbf{E}\times\mathbf{E}\times\mathbb{R}_+$ by

$$\begin{aligned} f(i,j,x) &= \mathbf{1}_{\{i=n,j=k\}}B_{nk}^{U}*\left(\mathbf{1}_{\{x\leqslant t\}}-Q_{nk}(t)\right)\\ &\quad -\mathbf{1}_{\{i=n,i\in U,j=k\}}\left(\sum_{r\in U}\alpha_r\Psi_{rn}\right)*\left(\mathbf{1}_{\{x\leqslant t\}}-Q_{nk}(t)\right). \end{aligned}$$

Then 3.31 can be written as $M^{1/2}W_f(t)$ where

$$W_f(t)=\sum_{n\in U}\sum_{k\in U}\frac{1}{N_n}\sum_{l=1}^{N_n}f\left(J_{l-1},J_l,X_l\right).$$

It is well known that for all $i\in\mathbf{E}$, M/N_i converges a.s. to μ_{ii}. Hence, by applying Pyke and Schaufele's ([33]) central limit theorem (see 2.2.3) to the function f, we get that $M^{1/2}\Delta R(t)$ converges weakly to a normal random variable with zero mean and variance

$$\begin{aligned} \sigma^2=\sum_{i\in U}\sum_{j\in U}\mu_{ii}&\left\{\left[B_{ij}^{U}-\sum_{k\in U}\alpha_k\Psi_{ki}\mathbf{1}_{\{i\in U\}}\right]^2*Q_{ij}(t)\right.\\ &\left.-\left[\left(B_{ij}^{U}-\sum_{k\in U}\alpha_k\Psi_{ki}\mathbf{1}_{\{i\in U\}}\right)*Q_{ij}(t)\right]^2\right\}. \end{aligned}$$

□

3.3.2 Asymptotic confidence intervals

For each time point $t\leq M$, we replace the true parameters $Q(t)$ and $\Psi(t)$ with their estimators $\widehat{Q}(t,M)$ and $\widehat{\Psi}(t,M)$. We use these estimators to construct $\widehat{\sigma}_R^2(t)$, an estimator of the variance $\sigma_R^2(t)$ of the reliability function $R(t)$.

the strong consistency of the estimators $\widehat{Q}(t,M)$ and $\widehat{\Psi}(t,M)$ implies that $\widehat{\sigma}_R^2(t)$ converges almost surely to $\sigma_R^2(t)$ as M tends to infinity.

the asymptotic confidence interval for $R(t)$ at a confidence level of $100(1-\gamma)\%$, where $\gamma\in(0,1)$ is given by:

$$\widehat{R}(t,M)-u_{1-\gamma/2}\frac{\widehat{\sigma}_R(t)}{\sqrt{M}}\leq R(t)\leq\widehat{R}(t,M)+u_{1-\gamma/2}\frac{\widehat{\sigma}_R(t)}{\sqrt{M}},$$

where u_γ is the γ-quantile of the standard normal distribution $\mathcal{N}(0,1)$.

Chapter 4

Simulation and estimation of semi-Markov models for reliability analysis using R

This chapter examines the application of R in simulating and estimating semi-Markov models, as well as for reliability and integral functional simulation.

4.1 R package for DTSMP analysis: smmR and SemiMarkov

This packages performs parametric and non-parametric estimation and simulation for multi-state discrete-time semi-Markov processes (Barbu [4]). For the parametric estimation, several discrete distributions are considered for the sojourn times: Uniform, Geometric, Poisson, Discrete Weibull of type 1 and Negative Binomial. The non-parametric estimation concerns the sojourn time distributions, where no assumptions are done on the shape of distributions. Moreover, the estimation can be done on the basis of one or several sample paths, with or without censoring at the beginning or/and at the end of the sample paths.

Semi-Markov models are specified by using the functions `smmparametric()` and `smmnonparametric()` for parametric and non-parametric specifications respectively. These functions return objects of S3 class (`smm`, `smmparametric`) and (`smm`, `smmnonparametric`) respectively (`smm` class inherits from S3 classes `smmparametric` or `smmnonparametric`). Thus, `smm` is like a wrapper class for semi-Markov model specifications.

Based on a model specification (an object of class `smm`), it is possible to:

- simulate one or several sequences with the method `simulate.smm()`;
- plot conditional sojourn time distributions (method `plot.smm()`);

- compute log-likelihood, AIC and BIC criteria (methods logLik(), AIC(), BIC());
- compute reliability, maintainability, availability, failure rates (methods `reliability()`, `maintainability()`, `availability()`, `failureRate()`).

Estimations of parametric and non-parametric semi-Markov models can be done by using the function `fitsmm()`. This function returns an object of S3 class `smmfit`. The class `smmfit` inherits from classes `(smm, smmparametric)` or `(smm, smmnonparametric)`.

Based on a fitted/estimated semi-Markov model (an object of class `smmfit`), it is possible to:

- simulate one or several sequences with the method `simulate.smmfit()`;
- plot estimated conditional sojourn time distributions (method `plot.smmfit()`);
- compute log-likelihood, AIC and BIC criteria (methods `logLik()`, `AIC()`, `BIC()`);
- compute estimated reliability, maintainability, availability, failure rates and their confidence intervals (methods `reliability()`, `maintainability()`, `availability()`, `failureRate()`).

In this work we consider four different semi-Markov models corresponding to the following four types of sojourn times:

- Sojourn times depending on the current state and on the next state:

$$f_{ij}(k) = \mathbb{P}\left(S_{n+1} - S_n = k \mid J_n = i, J_{n+1} = j\right).$$

- Sojourn times depending only on the current state:

$$f_{i\bullet}(k) = \mathbb{P}\left(S_{n+1} - S_n = k \mid J_n = i\right).$$

- Sojourn times depending only on the next state to be visited:

$$f_{\bullet j}(k) = \mathbb{P}\left(S_{n+1} - S_n = k \mid J_{n+1} = j\right).$$

- Sojourn times depending neither on the current state nor on the next state:

$$f(k) = \mathbb{P}\left(S_{n+1} - S_n = k\right).$$

Note that the sojourn times of the type $f_{i\bullet}(\cdot)$, $f_{\bullet j}(\cdot)$, or $f(\cdot)$ are particular cases of the general type $f_{ij}(\cdot)$. Nonetheless, in some specific applications, particular cases can be important because adapted to the phenomenon under study; that is the reason why we investigate these cases separately.

4.1.1 Simulation of semi-Markov model

Simulation according to classical distributions

In this part, we will consider the simulation according to classical distributions.

Parameters: This simulation is carried out by the function `simulSM()`. The different parameters of the function are:

- `E`: Vector of state space of length S.
- `NbSeq`: Number of simulated sequences.
- `lengthSeq`: Vector containing the lengths of each simulated sequence.
- `TypeSojournTime`: Type of sojourn time; it can be "fij", "fi", "fj" or "f" according to the four cases previously discussed.
- `init`: Vector of initial distribution of length S.
- `Ptrans`: Matrix of transition probabilities of the embedded Markov chain $J = (J_m)_m$ of size $S \times S$.
- `distr`: Sojourn time distributions.
  - is a matrix of distributions of size $S \times S$ if `TypeSojournTime` is equal to "fij",
  - is a vector of distributions of size S if `TypeSojournTime` is equal to "fi" or "fj",
  - is a distribution if `TypeSojournTime` is equal to "f", where the distributions to be used can be one of "uniform", "geom", "pois", "weibull" or "nbinom".
- `param`: Parameters of sojourn time distributions:
  - is an array of parameters of size $S \times S \times 2$ if `TypeSojournTime` is equal to "fij",
  - is a matrix of parameters of size $S \times 2$ if `TypeSojournTime` is equal to "fi" or "fj",

– is a vector of parameters if `TypeSojournTime` is equal to " f ".

The R commands below generate three sequences of size 1000, 10000, and 2000 respectively with the finite state space $\mathbf{E} = \{a, c, g, t\}$, where the sojourn times depend on the current state and on the next state.

```
install.packages("smmR")
install.packages("SemiMarkov")
library(smmR)
 library(SemiMarkov)
# state space
E <- c("a", "c", "g", "t")
S <- length(E)

# sequence sizes
lengthSeq3 <- c(1000, 10000, 2000)

# creation of the initial distribution
vect.init <- c(1/4, 1/4, 1/4, 1/4)

# creation of transition matrix
Pij <- matrix(c(0, 0.2, 0.3, 0.4, 0.2, 0, 0.5, 0.2, 0.5, 0.3, 0,
 0.4, 0.3, 0.5, 0.2, 0), ncol = 4)

# creation of the distribution matrix
distr.matrix <- matrix(c("dweibull", "pois", "geom", "nbinom",
                         "geom", "nbinom", "pois", "dweibull",
                         "pois", "pois", "dweibull", "geom",
                         "pois", "geom", "geom", "nbinom"),
                       nrow = S, ncol = S, byrow = TRUE)

# creation of an array containing the parameters
param1.matrix <- matrix(c(0.6, 2, 0.4, 4, 0.7, 2, 5, 0.6,
                          2, 3, 0.6, 0.6, 4, 0.3, 0.4, 4),
                        nrow = S, ncol = S, byrow = TRUE)
```

```
param2.matrix <- matrix(c(0.8, 0, 0, 2, 0, 5, 0, 0.8,
                           0, 0, 0.8, 0, 4, 0, 0, 4),
                         nrow = S, ncol = S, byrow = TRUE)

param.array <- array(c(param1.matrix, param2.matrix), c(S, S, 2))

# simulation of 3 sequences
seq3 <- simulSM(E = E, NbSeq = 3, lengthSeq = lengthSeq3,
                TypeSojournTime = "fij", init = vect.init,
                Ptrans = Pij, distr = distr.matrix,
                 param = param.array, File.out = "seq3.txt")
```

We note that the parameters of the distributions are given in the following way: for example, $f_{13}(\cdot)$ is a Geometric distribution with parameter 0.4, while $f_{14}(\cdot)$ is a Negative Binomial distribution with parameters 4 and 2. In other words, the parameters of $f_{13}(\cdot)$ are given in the vector `param.array` $[1,3]$, which is equal to $(0.4,0)$, and the parameters of $f_{14}(\cdot)$ are given in the vector `param.array` $[1,4]$, which is equal to $(4,2)$; that means that if a distribution has only 1 parameter, the corresponding vector of parameters will have 0 on the second position.

Values: The function `simulSM()` returns a list of simulated sequences. These sequences can be saved in a table file by using the parameter `File.out`.

```
seq3 [[1]][1:15]
 [1] "t" "t" "t" "t" "c" "c" "c" "c" "g" "g" "g" "g" "g" "g" "g"
```

Simulation according to distributions given by the user

Now we will consider the simulation according to distributions given by the user.

Parameters: This simulation is carried out by the function `simulSM()`. The various parameters of the function are the same as those of the previous function, with the addition of:

- `laws`: Sojourn time distributions introduced by the user:
  - is an array of size $S \times S \times K_{\max}$ if `TypeSojournTime` is equal to "fij",
  - is a matrix of size $S \times K_{\max}$ if `TypeSojournTime` is equal to "fi" or "fj",
  - is a vector of length $K_{\max}$ if `TypeSojournTime` is equal to "f",

where $K_{\max}$ is the maximum length for the sojourn times.

The R commands below generate three sequences of size 1000, 10000, and 2000 respectively with the finite state space $\mathbf{E} = \{a, c, g, t\}$, where the sojourn times depend only on the next state.

```
 # state space
E <- c("a", "c", "g", "t")
S <- length(E)

# sequence sizes
lengthSeq3 <- c(1000, 10000, 2000)

# creation of the initial distribution
vect.init <- c(1/4, 1/4, 1/4, 1/4)

# creation of transition matrix
Pij <- matrix(c(0, 0.2, 0.3, 0.4, 0.2, 0, 0.5, 0.2, 0.5,
0.3, 0, 0.4, 0.3, 0.5, 0.2, 0), ncol = 4)

# creation of a matrix corresponding to the conditional sojourn time
   distributions
Kmax <- 6
nparam.matrix <- matrix(c(0.2, 0.1, 0.3, 0.2, 0.2, 0, 0.4, 0.2,
0.1,0, 0.2, 0.1, 0.5, 0.3, 0.15, 0.05, 0, 0,0.3, 0.2, 0.1, 0.2,
 0.2, 0), nrow = S, ncol = Kmax, byrow = TRUE)

# simulation of 3 sequences with censoring at the beginning
seqNP3_begin <- simulSM(E = E, NbSeq = 3, lengthSeq = lengthSeq3,
      TypeSojournTime = "fj", init = vect.init, Ptrans = Pij,
      laws = nparam.matrix, File.out = "seqNP3_begin.txt",
      cens.beg = 1, cens.end = 0)

# simulation of 3 sequences with censoring at the end
seqNP3_end <- simulSM(E = E, NbSeq = 3, lengthSeq = lengthSeq3,
      TypeSojournTime = "fj", init = vect.init, Ptrans = Pij,
```

```
      laws = nparam.matrix, File.out = "seqNP3_end.txt",
      cens.beg = 0, cens.end = 1)

# simulation of 3 sequences censored at the beginning and at the end
seqNP3_begin_end <- simulSM(E = E, NbSeq 3, lengthSeq = lengthSeq3,
       TypeSojournTime = "fj", init = vect.init, Ptrans = Pij,
       laws = nparam.matrix, File.out = "seqNP3_begin_end.txt",
       cens.beg = 1, cens.end = 1)

# simulation of 3 sequences without censoring
seqNP3_no <- simulSM(E = E, NbSeq = 3, lengthSeq = lengthSeq3,
       TypeSojournTime = "fj", init = vect.init, Ptrans = Pij,
       laws = nparam.matrix, File.out = "seqNP3_no.txt")
```

Values: The function `simulSM()` returns a list of simulated sequences.

```
seqNP3_begin [[1]][1:15]
 [1] "g" "g" "g" "a" "g" "g" "g" "g" "t" "g" "g" "g" "g" "g" "g"
```

Estimation of semi-Markov model

In this section we explain and illustrate the estimation of a semi-Markov model in the non-parametric case.

4.1.2 Non-parametric estimation of DTSMP

Here we will consider two types of estimation for semi-Markov chains: a direct estimation, cf. Barbu and Limnios ([3], [1]) and an estimation based on a couple Markov chain associated to the semi-Markov chain (see [40]).

Parameters: The estimation is carried out by the function `estimSM()` and several parameters must be given.

- `file`: Path of the table file which contains the sequences from which to estimate.
- `seq`: List of the sequence(s) from which to estimate.
- `E`: Vector of state space of length S.

- `TypeSojournTime`: Type of sojourn time; always equal to "NP" for the non-parametric estimation.

Note that the sequences from which we estimate can be given either as an **R** list (`seq` argument) or as a file in table format (`file` argument). The parameter `distr` is always equal to "NP".

```
    ## data
seqNP3_no = read.table("seqNP3_no.txt")
E = c("a","c","g","t")
## estimation of simulated sequences
estSeqNP3= estimSM(seq = seqNP3_no, E = E, TypeSojournTime = "fj",
distr = "NP", cens.end = 0, cens.beg = 0)
```

Values: The function `estimSM()` returns a list containing:

- `init`: Vector of size S with estimated initial probabilities of the semi-Markov chain

```
    estSeqNP3\$init
    [1] 0.00000000 0.6666670 0.33333330 0.0000000
```

- `Ptrans`: Matrix of size $S \times S$ with estimated transition probabilities of the embedded Markov chain $J = (J_n)_n$

```
estSeqNP3\$Ptrans
           [ ,1]      [ ,2]      [ ,3]      [ ,4]
[1,] 0.0000000 0.2051948 0.5090909 0.2857143
[2,] 0.1938179 0.0000000 0.3107769 0.4954052
[3,] 0.3010169 0.4874576 0.0000000 0.2115254
[4,] 0.3881686 0.1936791 0.4181524 0.0000000
```

- `laws`: Array of size $S \times S \times K_{\max}$ with estimated values of the sojourn time distributions

```
estSeqNP3\$laws[,,1:2]
, , 1
           [ ,1]      [ ,2]      [ ,3]      [ ,4]
[1,] 0.0000000 0.3941423 0.4728997 0.2939271
[2,] 0.1896104 0.0000000 0.4728997 0.2939271
```

```
[3,] 0.1896104 0.3941423 0.0000000 0.2939271
[4,] 0.1896104 0.3941423 0.4728997 0.0000000
, , 2
        [ ,1]     [ ,2]     [ ,3]     [ ,4]
[1,] 0.0000000 0.1949791 0.3089431 0.1959514
[2,] 0.1073593 0.0000000 0.3089431 0.1959514
[3,] 0.1073593 0.1949791 0.0000000 0.1959514
[4,] 0.1073593 0.1949791 0.3089431 0.0000000
```

- q: Array of size $S \times S \times K\max$ with estimated semi-Markov kernel

```
    estSeqNP3\$q[,,1:3]
, , 1
        [ ,1]      [ ,2]      [ ,3]      [ ,4]
[1,] 0.00000000 0.07792208 0.2562771 0.06753247
[2,] 0.03508772 0.00000000 0.1378446 0.15956558
[3,] 0.05559322 0.18576271 0.0000000 0.06372881
[4,] 0.07698541 0.08670989 0.1920583 0.00000000
, , 2
        [ ,1]      [ ,2]      [ ,3]      [ ,4]
[1,] 0.00000000 0.04329004 0.1411255 0.05627706
[2,] 0.01670844 0.00000000 0.1019215 0.10025063
[3,] 0.03796610 0.09762712 0.0000000 0.03864407
[4,] 0.03889789 0.03160454 0.1385737 0.00000000
, , 3
        [ ,1]      [ ,2]      [ ,3]      [ ,4]
[1,] 0.00000000 0.02770563 0.07965368 0.04069264
[2,] 0.07101086 0.00000000 0.05179616 0.03926483
[3,] 0.09152542 0.05423729 0.00000000 0.02440678
[4,] 0.10940032 0.01782820 0.05591572 0.00000000
```

4.1.3 Simulation of the reliability function for DTSMP

In this section, we present a method to compute the reliability function and its confidence interval for a Discret-time SMP model. We detail the steps for the calculation, illustrate the process with a specific example, and provide the R code implementation.

parameters: To compute the reliability function for the semi-Markov model, we utilize the function `reliability()`.

- **k**: Number of periods to be examined for the reliability function.
- **upstates**: Vector specifying the states considered as upstates for the reliability calculation.
- **level**: Confidence level for the confidence interval calculation.
- **klim**: Optional parameter specifying the maximum number of iterations for convergence.

```
states <- c("a", "c", "g", "t")
s <- length(states)
# Creation of the initial distribution
vect.init <- c(1 / 4, 1 / 4, 1 / 4, 1 / 4)
# Creation of the transition matrix
pij <- matrix(c(0, 0.2, 0.5, 0.3,
                0.2, 0, 0.3, 0.5,
                0.3, 0.5, 0, 0.2,
                0.4, 0.2, 0.4, 0),
              ncol = s, byrow = TRUE)
# Creation of a matrix corresponding to the conditional sojourn time
   distributions
kmax <- 6
nparam.matrix <- matrix(c(0.2, 0.1, 0.3, 0.2,
                          0.2, 0, 0.4, 0.2,
                          0.1, 0, 0.2, 0.1,
                          0.5, 0.3, 0.15, 0.05,
                          0, 0, 0.3, 0.2,
                          0.1, 0.2, 0.2, 0),
                        nrow = s, ncol = kmax, byrow = TRUE)
# Initialize the semi-Markov model
library(smmR)
semimarkov <- smmnonparametric(states = states, init = vect.init,
   ptrans = pij, type.sojourn = "fj", distr = nparam.matrix)
```

```
# Calculate the reliability function
k <- 100
rr <- reliability(semimarkov, k, upstates = c("a", "g"), level =
   0.95, klim = 10000)
g <- rr[, 2]
t <- rr[, 1]
# Plot the reliability function
plot(t, type = "l",col="blue")
# Compute the confidence interval
ic1 <- t + 1.96 * (sqrt(g) / sqrt(k))
ic2 <- t - 1.96 * (sqrt(g) / sqrt(k))
# Plot the confidence intervals
lines(ic1, col = "red")
lines(ic2, col = "red")
legend("topright", legend = c("Reliability", "95% CI Upper", "95% CI
   Lower"), col = c("blue", "red", "red"), lty = c(1, 1, 1),
lwd = c(2, 1, 1))
```

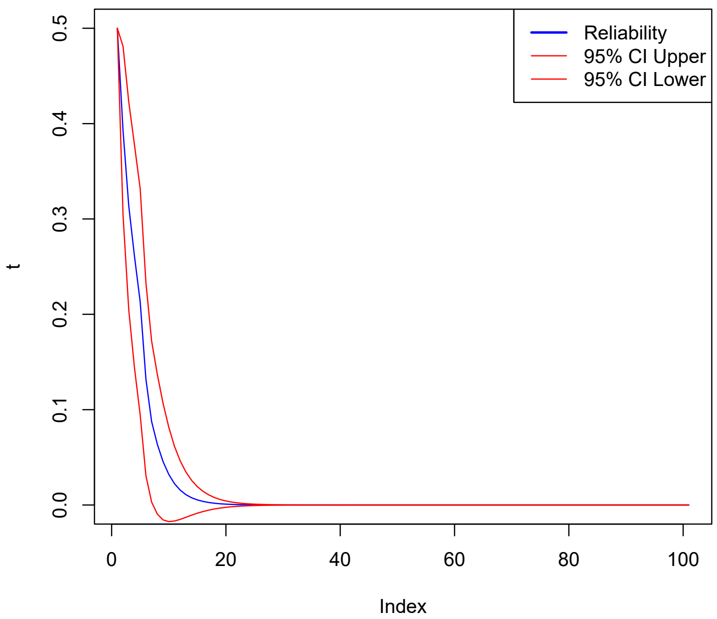

Figure 4.1: Reliability function with confidence intervals.

4.2 Continuous-time SMP algorithm

4.2.1 Monte carlo method

We shall give algorithm for realizing semi-Markov trajectories. This algorithm give realizations of a semi-Markov process into the time interval $[0, t]$. The output of the algorithms will be $(j_0, s_0, \ldots, j_k, s_k)$, the successive visited states and jump times, with $s_k \leq t < s_{k+1}$.

Consider a semi-Markov kernel $Q(t)$ and denote the transition probability matrix of the EMC $p = Q(\infty)$. Set also $F_{ij}(\cdot) = Q_{ij}(\cdot)/p_{ij}$, if $p_{ij} > 0$, $H_i(t) = \sum_{j \in \mathbf{E}} Q_{ij}(t)$, and $Q_{ij}(t) = \int_0^t q_{ij}(u) H_i(du)$. We consider here that the initial state of the system is fixed.

The algorithm is based on the EMC.

Algorithm

1. Put $k = 0$, $S_0 = 0$, and set j_0 as the initial state;
2. Sample random variable $J \sim p(j_k, \cdot)$ and set $j_{k+1} = J(\omega)$;
3. Sample random variable $X \sim F_{j_k j_{k+1}}(\cdot)$ and set $x = X(\omega)$;
4. Put $k := k + 1$ and $s_k = s_{k-1} + x$. If $s_k \geq t$ then end;
5. Set $j_k := j_{k+1}$ and continue to step 2.

Numerical example

In this section we carry out a simulation study to evaluate the finite sample performance of the estimation procedure described in the previous sections. We will apply our results to a three-state semi-Markov processes. The transitions between states are given in Fig.4.2.

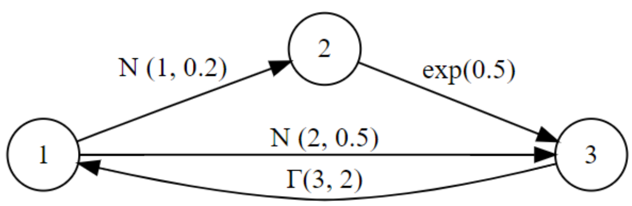

Figure 4.2: A three state semi-Markov system .

Taking the initial distribution $\alpha = (1/3, 1/3, 1/3)$, the transition matrix of the embedded Markov chain $(J_n)_{n\in\mathbb{R}^+}$ is given by:

$$(p_{ij})_{ij} = \begin{pmatrix} 0 & 0.9 & 0.1 \\ 0.8 & 0 & 0.2 \\ 1 & 0 & 0 \end{pmatrix}$$

and the conditional sojourn time distributions are defined by:

- f_{12} is normally distributed, $\mathcal{N}(\mu, \sigma^2)$, with parameters $\mu = 1.0$ and $\sigma^2 = 0.2$.
- f_{13} is gamma distributed, $\Gamma(\alpha, \beta)$, with parameters $\alpha = 3$ and $\beta = 2$.
- f_{31} is normally distributed, $\mathcal{N}(\mu, \sigma^2)$, with parameters $\mu = 2.0$ and $\sigma^2 = 0.5$.
- f_{23} is exponentially distributed, with parameter $\lambda = 0.5$.

```
$realisation
$realisation[[1]]
   [1] 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 3 1 2 1
       2 1 2 1 2 1 2
  [39] 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1
       2 1 2 1 2 3 1
  [77] 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 3 1 2 1 2 1
       2 1 2 1 2 1 2
 [115] 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1
       2 1 2 1 2 1 2
$saut
$saut[[1]]
  [1]    8.007594    9.203026    9.770048   19.143830   28.217834
         30.768822
   [7]   33.335021   42.466593   42.879345   49.355742   51.626033
         58.951192
  [13]   59.597311   64.305667   77.287072   80.907378   85.729960
         88.478846
  [19]   89.108953   92.867506  106.755540  111.191515  115.952540
        126.819882
  [25]  129.388958  136.824663  144.531240  147.938703  156.030915
```

```
        158.115849
  [31]  158.158408  159.573728  163.305283  166.754757  171.593676
        173.281841
$sejourn
$sejourn[[1]]
   [1]  8.00759382  1.19543239  0.56702195  9.37378145  9.07400434
        2.55098803
   [7]  2.56619889  9.13157176  0.41275244  6.47639699  2.27029129
        7.32515853
  [13]  0.64611947  4.70835602 12.98140506  3.62030550  4.82258204
        2.74888570
  [19]  0.63010761  3.75855250 13.88803434  4.43597442  4.76102527
        10.86734270
  [25]  2.56907526  7.43570526  7.70657709  3.40746317  8.09221189
        2.08493358
```

4.2.2 Simulation of the integral functional

In this section, we describe a procedure that uses a piecewise-defined function to calculate the integral functional of a random process. We give an example of how to calculate the functional in detail, walk through the processes in detail, and provide the R code implementation.

The integral functional $\mathrm{L}(t)$ can be defined as:

$$\mathrm{L}(t)=\int_0^t h\left(Z_s\right) ds=\sum_{k=1}^{N(t)} h\left(J_{k-1}\right) S_k+\left(t-X_{N(t)}\right) h\left(J_{N(t)}\right), \quad t \geq 0, \tag{4.1}$$

where

- The function $h(x)$ modifies the state values.
- The sequences J and S represent the states and holding times, respectively.
- Sequence of time points t with a specified increment.

The steps to compute $\mathrm{L}(t)$ for each time point:

- Calculate the cumulative sum of S to determine $N(t)$, the number of jumps up to time t.

- Sum the terms $h(J_{k-1})S_k$ for k from 1 to $N(t)$.
- Add the remaining term $(t - X_{N(t)})h(J_{N(t)})$.

```
# Define the sequences J and S
J <- c(3, 1, 4, 1, 5)
S <- c(1.5, 3.3, 2.7, 0.8, 2.2)
# Define the function h(x)
h <- function(x) {
  return(0.2 * x^2 + 0.3 * x + 0.1)
}
# Define the time points and initialize L values
t <- seq(0, 12, by = 0.01)
L_values <- numeric(length(t))
# Calculate L(t) for each time point t
for (i in seq_along(t)) {
  current_t <- t[i]
  cumulative_S <- cumsum(S)
  # Find N(t), the number of jumps up to time t
  N_t <- sum(cumulative_S <= current_t)
  if (N_t == 0) {
    L_values[i] <- current_t
  } else {
    sum_term <- sum(h(J[1:N_t]) * S[1:N_t])
    if (N_t < length(S)) {
      remaining_term <- (current_t - cumulative_S[N_t]) * h(J[N_t +
         1])
    } else {
      remaining_term <- (current_t - cumulative_S[N_t]) * h(J[N_t])
    }
    L_values[i] <- sum_term + remaining_term
  }
}
# Plot the trajectory
plot(t, L_values, type = "l", col = "blue", lwd = 2, xlab = "Time",
```

```
  ylab = "L(t)")
```

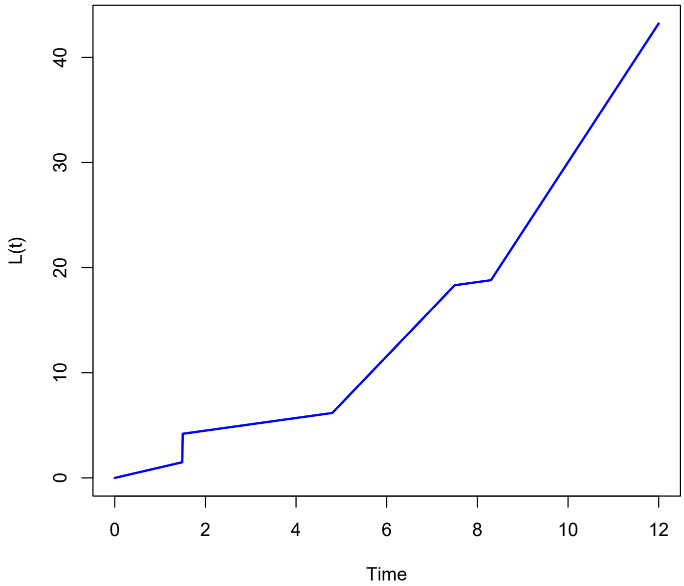

Figure 4.3: Trajectory of L(t).

4.2.3 Reliability algorithm

Generally speaking, it is clear that, for the purpose of application, it is more worth while solving the Equation 3.29

In order to numericaly solve this equation and give an approximation of the reliability estimator, we use the same algorithm of Corradi [12] (we use the restriction on U).

The variables involved are the following:

- $s =$ number of states of the SMP.
- $M =$ number of periods to be examined for the transient analysis of the SMP.
- $\mathbf{P} =$ matrix of order s of the embedded MC in the SMP.
- ${}^{M}\mathbf{F} =$ square lower-triangular block matrix of order $M + 1$ whose blocks are of order m.

The algorithm for the solution of 3.29 works with the following steps:

Reads the inputs: $s, M, \mathbf{P}, {}^{M}\mathbf{F}$

Constructs ${}^{M}\mathbf{Q}, {}^{M}\mathbf{U}, {}^{M}\mathbf{D}$

$U_{(0)} = \mathrm{I}$

$Q_{(0)} = 0$

$S_{(0)} = 0$

$D_{(0)} = \mathrm{I}$

for $t = 1$ to M

  $Q_{(t)} = \mathbf{P} * F_{(t)}$

  for $i = 1$ to m

    $s_{ii}(t) = Q_{ii}(t) \cdot \mathbf{1}$

  endfor

  $U_{(t)} = Q_{(t)} = Q_{(t-1)}$

  $D_{(t)} = D_{(0)} = S_{(t)}$

endfor

Solves the system: $\Phi_{(0)} = D_{(0)}$

  for $t = 1$ to T

    $\Phi_{(t)} = D_{(t)}$

    for $s = 1$ to t

      $\Phi_{(t)} = \Phi_{(t)} + U_{(s)} \cdot \Phi_{(t-s)}$

    endfor

  endfor

Prints the results:

${}^{T}\mathbf{Q}, {}^{T}\Phi$

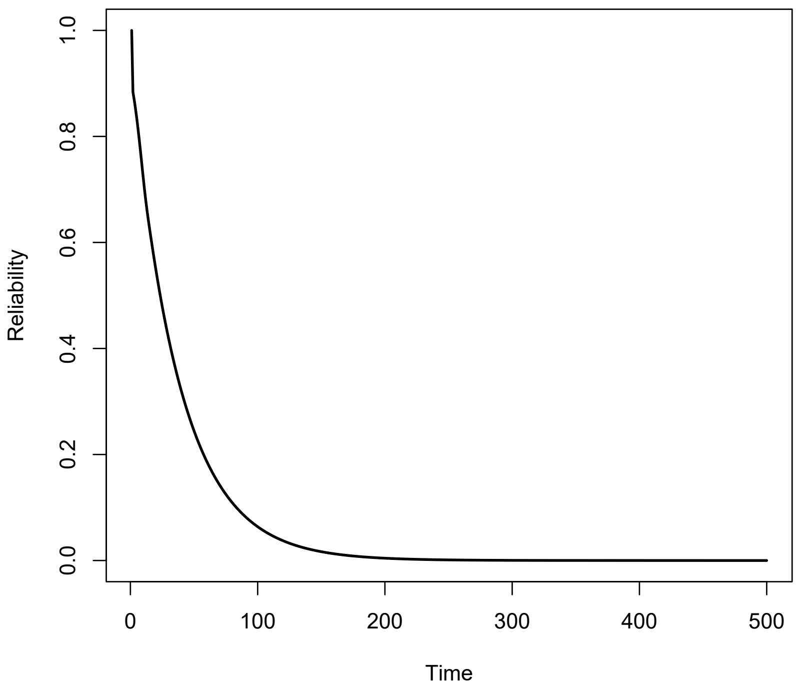

Figure 4.4: Reliability of a three-state system by the Monte Carlo method.

Conclusion

Integral functionals of semi-Markov processes are highly valuable in various reliability studies. They play a crucial role in analyzing complex engineering systems, which include a vast range of applications such as infrastructure networks, manufacturing processes, and telecommunications systems. By incorporating these mathematical tools into reliability models, researchers can significantly improve their predictive capabilities regarding system reliability. This integration allows for better optimization of maintenance schedules, ensuring that systems operate efficiently and with minimal downtime. Additionally, it aids in the assessment of overall system performance, providing a clear picture of how systems function under different conditions.

Consequently, this detailed analysis enables researchers and engineers to make well-informed decisions that enhance the reliability and robustness of the systems they work with. By leveraging integral functionals of semi-Markov processes, it is possible to achieve a more thorough and nuanced understanding of system behavior, leading to improvements in design, operation, and overall system reliability.

Bibliography

[1] V.S. Barbu and N. Limnios, Empirical estimation for discrete time semi-Markov processes with applications in reliability. J. Nonparametr.Statist. 2006a.

[2] V.S. Barbu, and N. Limnios, Empirical estimation for discrete time semi-Markov processes with applications in reliability. to appear in Journal of Nonparametric Statistics, 2005.

[3] V.S. Barbu and N. Limnios. Semi-Markov chains and Hidden Semi Markov Models toward Applications vol.191, Spring, New York.(2008a).

[4] V.S. Barbu, Vlad Stefan, et al. "SMM: An R Package for Estimation and Simulation of Discrete-time semi-Markov Models." R Journal 10.2 (2018).

[5] V.S. Barbu, C. Berard, D. Cellier, F. Lecocq, C. Lothode, and M. Sautreuil. smmR: Semi-Markov models, Markov models and reliability. `https://cran.r-project.org/web/packages/smmR/index.html`, 2021.

[6] V.S. Barbu and N. Limnios, Discrete time semi-Markov processes for reliability and survival analysis, Communications in statistic theory and methods. Taylor and Francis publisher 2004.

[7] P. Billingsley. Probability and Measure. Wiley, New York, 3rd edition, 1995.

[8] P. Billingsley. Convergence of Probability Measures. Wiley, New York, 2nd edition, 1999.

[9] E. Cinlar, Markov renewal theory, Adv. Appl. Prob. 1 (1969).

[10] O. Chryssaphinou, M. Karaliopoulou, and N. Limnios. On discrete time semi-Markov chains and applications in words occurrences. Comm. Statist. Theory Methods, 2008.

[11] H. Cramer, Mathematical Methods of Statistics, Princeton University Press, 1946.

[12] G. Corradi, J. Jacques , and M. Raimondo , "Numerical treatment of homogeneous semi-Markov processes in transient caseâa straightforward approach." Methodology and Computing in Applied Probability 6 (2004): 233-246.

[13] D.A. Darling, "The kolmogorov-smirnov, cramer-von mises tests." The Annals of Mathematical Statistics 28.4 (1957): 823-838.57.

[14] E.H. Moore, R. Pyke, Estimation of the transition distributions of a Markov renewal process, Ann. Inst. Statist. Math.,1968.

[15] R.P. Feynman, "Space-time approach to non-relativistic quantum mechanics." Reviews of modern physics 20.2 (1948): 367.

[16] F. Grabski, Semi-Markov Processes: Applications in System Reliability and Maintenance,Polish Naval University Gdynia, Poland, 2015.

[17] F. Grabski, Franciszek. Semi-Markov processes: applications in system reliability and maintenance. Elsevier, 2014.

[18] F. Grabski, Theory of semi-Markov processes of the operation. Zeszyty naukowe Marynarki Wojennej 75A, s. 253, (1982).

[19] F. Grabski, Semi-Markov Models of Reliability and Operation, IBS PAN, Warsaw, 2002.

[20] F. Grabski, J. Jaźwiński, Functions of random arguments: applications in reliability, safety, and logistics, WKL, Warszawa, (2009).

[21] A. Gut. Stopped Random Walks. Limit Theorems and Applications, volume 5 of Applied Probability. A Series of the Applied Probability Trust. Springer, New York, 1988.

[22] I. B. Gertsbakh, "Models of preventive service." Sovetskoe radio. Moscow (in Russian) (1969).

[23] N. Heutte and C. Huber-Carol. Semi-Markov models for quality of life data with censoring. In M. Mesbah, M.-L. T. Lee, and B. F. Cole, editors, Statistical Methods for Quality of Life Studies, Kluwer, Dordrecht, 2002.

[24] J. Janssen and R. Manca. Applied Semi-Markov Processes. Springer-Verlag, 2006.

[25] V.S. Korolyuk, A.F. Turbin, Decompositions of Large Scale Systems, Kluwer Academic, Singapore, 1993.

[26] V.S. Korolyuk, A.F. Turbin, Semi-Markov Processes and Their Appli- cations, Naukova Dumka, Kiev, 1976.

[27] P. Levy, Proceesus semi-Markoviens, in: Proc. Int. Cong. Math., Amsterdam, 1954.

[28] N. Limnios and G. Oprisan, Semi-Markov Processes and Reliability (Boston) 2001.

[29] E.H. Moore, R. Pyke, Estimation of the transition distributions of a Markov renewal process, Ann. Inst. Statist. Math.,1968.

[30] B. Ouhbi and N. Limnios. Nonparametric reliability estimation for semi-Markov processes. J. Statist. Plann. Inference, 2003.

[31] B. Ouhbi, N. Limnios, Nonparametric estimation for semi- Markov kernels with application to reliability analysis, Appl. Stochas- tic Models Data Anal., 1996.

[32] B. Ouhbi, N. Limnios, Nonparametric esimation for semi- Markov processes based on its hazard rate functions, Statist. Inference Stochastic Proc., 2, No.2, 151-173.(1999).

[33] R. Pyke, Ronald, and Ronald Schaufele. "Limit theorems for Markov renewal processes." The Annals of Mathematical Statistics (1964): 1746-1764.

[34] R. Pyke, Markov renewal processes: definitions and preliminary properties, Ann. Math. Stat. 32 (1961).

[35] R. Pyke, Markov renewal processes with finitely many states, Ann. Math. Stat. 32 (1961).

[36] D.S. Silvestrov, Semi-Markov processes with discrete state space. Sovietskoe Radio, Moskva (1980) (in Russian).

[37] W.L. Smith, Regenerative stochastic processes, Proc. R. Soc. Lond. A 232 (1955).

[38] L. Takacs, Some investigations concerning recurrent stochastic processes of a certain type, Magyar Tud. Akad. Mat. Kutato Int. Kzl. 3 (1954).

[39] L. Takacs, On a sojourn time problem in the theory of stochastic pro- cesses, Trans. Am. Math. Soc. 1959.

[40] Trevezas, Samis, and Nikolaos Limnios. "Exact MLE and asymptotic properties for nonparametric semi-Markov models." Journal of Nonparametric Statistics 23.3 (2011): 719-739.

[41] I. Votsi, et al. "Hidden semi-Markov modeling for the estimation of earthquake occurrence rates." Communications in Statistics-Theory and Methods 43.7 (2014): 1484-1502.