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Theme:

Stochastic Koopman Operator for Random Dynamical Systems

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Dedication

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Abstract

The stochastic Koopman operator characterizes the average evolution of observables in dynamical systems subject to uncertainty or noise. This study focuses on random dynamical systems (RDS) in both discrete and continuous time. We analyze the spectrum and eigenfunctions of several linear RDS. In particular, we examine cases where the family of operators forms a semigroup, which enables the definition of an associated generator. A stochastic Hankel-DMD algorithm is introduced to numerically approximate the Koopman eigenvalues and eigenfunctions, along with a proof of convergence. The method is applied to various examples, demonstrating spectral decomposition and model reduction. This approach extends deterministic Koopman theory to systems influenced by randomness. Moreover, we highlight the advantages of refined DMD algorithms in improving the accuracy of the approximated spectra.

Keywords: stochastic Koopman operator, random dynamical systems, spectral analysis, semigroup theory, Hankel-DMD, dynamic mode decomposition, numerical approximation.

Introduction

The Koopman operator theory represents an alternative formulation of dynamical systems theory, offering a versatile framework for data-driven study of high-dimensional nonlinear systems. This theory originated in the 1930s through the work of Bernard Koopman and John Von Neumann [24, 25]. In particular, Koopman realized that the evolution of observables on the state space of a Hamiltonian system could be described by a linear transformation, which was later named the Koopman operator [2]. Decades later, research such as that in [32] and [30] revived interest in this formalism by proving the Koopman spectral decomposition and introducing the concept of Koopman modes. These theoretical advancements were complemented by data-driven algorithms for approximating the Koopman operator's spectrum and modes, opening a new path for data-driven study of high-dimensional systems (see, e.g.,[7, 18, 34, 35, 38]).

The two main candidates for studying systems via operators are the Koopman operator and the Perron-Frobenius operator. In appropriate function spaces, these operators are duals to each other, theoretically implying that there is no fundamental distinction between working with one or the other. However, as previously mentioned, practical considerations come into play. For example, how do we construct or represent the chosen operator from the problem description and available data? How well does a finite approximation of this operator represent the ideal theoretical framework? And to what extent is the intuition gained from numerical artifacts rather than reality? The Perron-Frobenius operator represents a "dynamics of densities" view, focusing on the evolution of groups of trajectories. The idea is to observe the evolution of a mass distribution under the action of a flow. From a numerical perspective, constructing this operator relies on selecting a set of initial conditions and simulating over a short period of time to avoid accumulating numerical errors. Through these short simulations, transient dynamics can be captured well. However, much attention has been focused on computing invariant densities, theoretical objects at infinity, by approximating the Perron-Frobenius operator with a Markov chain.

On the other hand, the Koopman operator presents a "dynamics of observables" view, akin to the Lagrangian perspective in fluid mechanics, where measurements are made along trajectories. For this operator, the numerical construction relies on potentially fewer initial conditions but requires longer run-times, making it more suitable for physical experiments. For example, when testing a jet engine, one starts with a relatively small number of initial conditions and runs it for a long time, rather than preparing thousands of initial conditions and running the engine for only a few seconds for each one[8].

Dynamic Mode Decomposition (DMD) is an essential tool in computational data-driven analysis of fluid flows. More generally, it is a computational device for Koopman spectral analysis of nonlinear dynamical systems, with numerous applications in applied sciences and engineering. Its effectiveness has led to the development of several modifications that make DMD more reliable and functional. This approach allows for more precise selection of Ritz pairs, providing more accurate spectral information about the Koopman operator. These improvements are illustrated through numerical experiments, enhancing the effectiveness of the method known as DMD RRR (Refined Rayleigh-Ritz Dynamic Mode Decomposition).

The objective of this master thesis is to provide a comprehensive study of the Koopman operator theory in the context of Random Dynamical Systems (RDS). The primary focus is to elucidate the fundamental concepts of the Koopman operator and its spectral properties, and to explore the application of data-driven algorithms for approximating the Koopman operator's spectrum and modes. A further objective is to demonstrate the effectiveness of these methodologies through various numerical examples, with a particular emphasis on the stochastic Koopman operator.

This work is presented in three chapters:

Chapter 1: Theoretical Principles of Random Dynamical Systems
This chapter lays the theoretical groundwork by introducing the fundamental concepts of both deterministic and random dynamical systems. It delves into the definition of a Wiener process and establishes the basic definitions of discrete and continuous Random Dynamical Systems (RDS). Furthermore, the chapter provides an extensive overview of the Koopman operator, covering its fundamental concepts, spectral properties, and the analysis of eigenfunctions and the spectrum of eigenvalues.

Chapter 2: Koopman Operator Spectrum for Random Dynamical Systems
Chapter 2 focuses on the stochastic Koopman operator within the framework of Random Dynamical Systems. It explores the operator's characteristics in various contexts, including discrete-time RDS, RDS generated by random differential equations, and RDS generated by stochastic differential equations.Additionally, the concept of semigroups of Koopman operators and their generators is also examined.

Chapter 3: Numerical Examples
Chapter 3 delves into numerical approximations of the stochastic Koopman operator, with a detailed discussion of the DMD algorithms for RDS.The theoretical concepts and methodologies developed in the previous chapters are illustrated through a variety of numerical examples. These examples include discrete RDS scenarios such as noisy rotation on the circle and discrete linear RDS, as well as continuous-time linear RDS. Furthermore, the chapter presents examples involving stochastic differential equations, such as linear scalar SDE, nonlinear scalar SDE, and the noisy van der Pol oscillator, to demonstrate the practical application and effectiveness of the studied approaches.

Contents

List Of Notations and Symbols

$\mathcal{F}$ Sigma-algebra

P Probability measure

$\mathbb{T}$ Semigroup (representing time)

$(\theta(t))_{t\in\mathbb{T}}$ Group or semigroup of measurable transformations

$(M, \mathcal{B})$ Measurable space

$\mathcal{B}$ Sigma-algebra on M

φ Random Dynamical System (RDS) map

θ Measure-preserving transformation

ψ_n A map defined as $\psi \circ \theta^{n-1}$

$T^n(\omega, x)$ Action of the discrete RDS on x at the n-th step

RDE Random Differential Equation

SDE Stochastic Differential Equation

$X_t(\omega)$ Stochastic process

W_t Wiener process

F^t Flow map operator, or time-t map

$\mathcal{K}^t$ Koopman operator

$L^2(X, d\mu)$ Hilbert space or reproducing kernel Hilbert spaces

∇ Gradient operator

Chapter 1

Theoretical Principles of Random Dynamical Systems

Random Dynamical Systems (RDS) are formally defined as cocycles over an ergodic base flow, acting on a state space.

1.1 Introduction to deterministic and random dynamical systems

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathbb{T}$ a semigroup (we can think of it as time). Suppose that $\theta := (\theta(t))_{t\in\mathbb{T}}$ is a group or semigroup of measurable transformations of $(\Omega, \mathcal{F}, P)$ preserving a measure P (i.e. $\theta(t)P = P$), such that the map $(t, \omega) \mapsto \theta(t)\omega$ is measurable. The quadruple $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$ is called a metric dynamical system.

Definition 1.1.1. A Random Dynamical System (RDS) on a measurable space $(M, \mathcal{B})$ over $(\theta_t)_{t\in\mathbb{T}}$ the dynamical system $(\Omega, \mathcal{F}, P)$ is a measurable map:

$$\varphi : \mathbb{T} \times \Omega \times M \to M,$$

such that $\varphi(0, \omega) = \mathrm{id}_M$ and

$$\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) \tag{1.1.1}$$

for all $t, s \in \mathbb{T}$ and for all ω outside a P-null set, where $\varphi(t, \omega) : M \to M$ is the map which arises when $t \in \mathbb{T}$ and $\omega \in \Omega$ are fixed, and $\circ$ means composition.
A family of maps $\varphi(t, \omega)$ satisfying (1.1.1) is called a cocycle, and (1.1.1) is the cocycle property.

The simplest case of a random dynamical system is a non-random, deterministic dynamical system.

Definition 1.1.2. A random dynamical system is deterministic if φ does not depend on ω, i.e., $\varphi(t, x, \omega) = \varphi(t, x)$.

Then the cocycle property (1.1.1) reads $\varphi(t + s) = \varphi(t) \circ \varphi(s)$, hence $(\varphi(t))_{t\in T}$ consists of the iterates of a measurable map on M if $\mathbb{T} = \mathbb{Z}^{(+)}$, and $(\varphi(t))_{t\in T}$ is a measurable (semi-)flow if $\mathbb{T} = \mathbb{R}^{(+)}$, respectively.

Remark 1. By the cocycle property, $\varphi(t, \omega)$ is automatically invertible (for all $t \in \mathbb{T}$ and for P-almost all ω) if $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, and $\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t\omega)$.

Definition 1.1.3. Let $\theta : \Omega \to \Omega$ be a measure preserving transformation, and let $\psi : M \times \Omega \to M$ be a measurable map. Put $\psi_n = \psi \circ \theta^{n-1}$. Then

$$\varphi(n, \omega) = \begin{cases} \psi_n(\omega) \circ \psi_{n-1}(\omega) \ldots \circ \psi_1(\omega) & \text{for } n > 0 \\ \mathrm{id} & \text{for } n = 0 \\ \psi_{n+1}^{-1}(\omega) \circ \psi_{n+2}^{-1}(\omega) \ldots \circ \psi_0^{-1}(\omega) & \text{for } n < 0 \end{cases}$$

defines an RDS (of course, defining $\varphi(n, \omega)$ for $n < 0$ requires θ and $\psi(\cdot, \omega)$ to be invertible P-a.s.). In particular, if $M = \mathbb{R}^d$ and $x \mapsto \psi(x, \omega)$ is linear, then φ becomes a product of random matrices.

1.2 The Wiener process

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t\}_{t\in[0,T]}$ a system of σ-subalgebras of $\mathcal{F}$ which are increasing in t, and $\{X_t\}_{t\in[0,T]}$ a stochastic process on $(\Omega, \mathcal{F}, P)$. In the following discussion, the time interval $[0, T]$ will be fixed.

Definition 1.2.1. When $(X_t, \mathcal{F}_t, P)$ satisfies the following conditions 1), 2) and 3), it is called a Wiener process:
1) The sample paths of X_t are continuous in t, and $X_0 = 0$.
2) For $t \geq s$, $t, s \in [0, T]$, $E(X_t \mid \mathcal{F}_s) = X_s$ with P-measure 1, where $E(\cdot \mid \cdot)$ denotes the conditional exectation with respect to the measure P.
3) $E((X_t - X_s)^2 \mid \mathcal{F}_s) = t - s$ with P-measure 1, for $t \geq s$, $t, s \in [0, T]$

Definition 1.2.2. A stochastic process Y_t defined on $(\Omega, \mathcal{F}, P)$ (or simply, (Y_t, P)) is called a Gaussian process, when the distribution of $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$ with respect to P is subject to an n-dimensional Gaussian distribution.

1.3 Discrete and Continuous Random Dynamical Systems

1.3.1 Discrete RDS

For a discrete time RDS we have $\mathbb{T} = \mathbb{Z}^+ \cup \{0\}$. Let $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$ be a given metric dynamical system and let $\psi = \theta(1)$.

Definition 1.3.1. The discrete RDS $\varphi(n,\omega)$ over θ can be defined by the one step maps $T(\omega,\cdot) : M \to M$ as

$$T(\omega,\cdot) := \varphi(1,\omega),$$

since by applying the cocycle property one gets

$$\varphi(n,\omega) = T(\psi^{n-1}(\omega),\cdot) \circ \cdots \circ T(\psi(\omega),\cdot) \circ T(\omega,\cdot), \quad n \geq 1. \tag{1.3.1}$$

From the fact that maps $\psi^i = \theta(i)$ preserve the measure P, it follows that the maps $T(\psi^i(\omega),\cdot)$ are identically distributed, thus $(T(\psi^i(\omega),\cdot))_{i\in\mathbb{T}}$ is a stationary sequence of random maps on M [4, Section 2.1]. According to (1.1.1), $\varphi(0,\omega) = \mathrm{id}_M$. The action of the discrete RDS $\varphi(n,\omega)$ on $\mathrm{x} \in M$ gives its value at the n-th step and we denote it as

$$T^n(\omega,\mathrm{x}) = \varphi(n,\omega)\mathrm{x}. \tag{1.3.2}$$

Remark 2. We have a continuous-time RDS,if $\mathbb{T}$ is continuous (for example, $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}^+$).

1.3.2 RDS generated by the random differential equations

Suppose that $T = \mathbb{R}$ or $T = \mathbb{R}^+ \cup \{0\}$. Let $(\Omega,\mathcal{F},P,(\theta(t))_{t\in\mathbb{T}})$ be a metric dynamical system. We consider a continuous-time RDS generated by the random differential equation (RDE) of the following form

$$\dot{\mathrm{x}} = F(\theta(t)\omega,\mathrm{x}), \tag{1.3.3}$$

defined on the manifold M, where $\theta(t)\omega$ is associated with the random dynamics. In this type of equations, the randomness refers just to the random parameters, which do not depend on the state of the system. RDE (1.3.3) generates an RDS φ over θ, whose action is defined by

$$\varphi(t,\omega)\mathrm{x} = \mathrm{x} + \int_0^t F(\theta(s)\omega,\varphi(s,\omega)\mathrm{x})\,ds. \tag{1.3.4}$$

The properties of this RDS under different regularity properties of the function F can be found in [4, Section 2.2]. A set of trajectories starting at x that are generated by (1.3.3) is given by $\varphi(t,\omega)\mathrm{x}$ and it defines the family of random variables. We say that this is a solution of the RDE with the initial condition $\varphi(0,\omega)\mathrm{x} = \mathrm{x}$. Since the solutions of the RDE are defined pathwise, for each fixed ω, the trajectory can be determined as a solution of deterministic ordinary differential equations, so that the RDE (1.3.3) can be seen as a family of ordinary differential equations.

1.3.3 RDS generated by the stochastic differential equations

Let $\mathbb{T} = \mathbb{R}^+$, $M = \mathbb{R}^d$ and $T \in \mathbb{T}$.Suppose that the stochastic process $X_t(\omega), t \in [0,T], \omega \in \Omega$ is obtained as a solution of the nonautonomous stochastic differential equation (SDE)

$$dX_t = G(t,X_t)\,dt + \sigma(t,X_t)\,dW_t, \tag{1.3.5}$$

where $G : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ are L^2 measurable. Here, $W_t = (W_t^1, \ldots, W_t^r)$ denotes the r-dimensional Wiener process with independent components and standard properties, i.e., $\mathbb{E}(W_t^i) = 0$, $i = 1, \ldots, r$, $\mathbb{E}(W_t^i W_s^j) = \min(t,s)\delta_{ij}$, $i, j = 1, \ldots, r$ (δ_{ij} is the Kronecker delta symbol). The solution $X_t(\omega)$ of

$$X_t(\mathrm{x}) = \mathrm{x} + \int_0^t G(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s \tag{1.3.6}$$

with the initial condition $X_{t_0}(\omega)$ is formally defined in terms of Itô integral as (see [1, Chapter 6])

$$X_t(\omega) = X_{t_0}(\omega) + \int_{t_0}^t G(s, X_s(\omega))\, ds + \int_{t_0}^t \sigma(s, X_s(\omega))\, dW_s. \tag{1.3.7}$$

The probability space on which the process is considered can be identified with $\Omega = C_0(\mathbb{R}^+, \mathbb{R}^r)$ (space of continuous functions satisfying $\omega(t_0) = 0$). Then, $\omega \in \Omega$ is identified with the canonical realization of the Wiener process such that $\omega(t) = W_t(\omega)$. If $\mathcal{F}$ is the Borel σ-algebra, P the measure generated by the Wiener process, and $\theta(t)$ defined by the "Wiener shifts"

$$\theta(t)\omega(\cdot) = \omega(t + \cdot) - \omega(t), \tag{1.3.8}$$

$(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$ becomes a metric dynamical system (see [4, Appendix A]). It is a driving dynamical system for the two-parameter family of RDS $\varphi(t, t_0, \omega)$ that is given by

$$\varphi(t, t_0, \omega)\mathrm{x} = X_t(\omega),$$

where $X_t(\omega)$ is given by (1.3.7) for the initial condition $X_{t_0}(\omega) = \mathrm{x}$. Under certain regularity and boundedness properties of the functions G and σ, the basic existence and uniqueness results for the SDE of the form (1.3.5) can also be found in [1, Section 6.3], [33, Section 3.3].

1.4 Random Dynamical Systems and Markov Processes

1.4.1 RDS with independent increments, Brownian RDS

A random dynamical system (RDS) $\varphi(t, \omega)$ over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{T}})$ is said to have independent increments if for all
$t_0 \le t_1 \le \ldots \le t_n$ the random variables

$$\varphi(t_1 - t_0, \theta_{t_0}\omega), \varphi(t_2 - t_1, \theta_{t_1}\omega), \ldots, \varphi(t_n - t_{n-1}, \theta_{t_{n-1}}\omega) \tag{1.4.1}$$

are independent. If, in addition, for $\mathbb{T} = \mathbb{R}^+$ or $\mathbb{R}$ the map $t \mapsto \varphi(t, \omega)\mathrm{x}$ is continuous for all $\mathrm{x} \in M$ P-a.s., then the RDS or cocycle is said to be a Brownian RDS or cocycle.

Remark 3. (i) An RDS with independent increments automatically has stationary (time homogeneous) increments, as, by the θ_t invariance of P, $\varphi(h, \theta_t\omega) \stackrel{\mathcal{D}}{=} \varphi(h, \omega)$ for all $t \in \mathbb{T}$.
(ii) If $\varphi(t, \omega)$ consists of invertible mappings then, by the cocycle property,

$$\varphi_{s,t}(\omega) := \varphi(t, \omega) \circ \varphi(s, \omega)^{-1} = \varphi(t - s, \theta_s\omega)$$

for $s \leq t$, so (1.4.1) meansthat for $t_0 \leq t_1 \leq \ldots \leq t_n$

$$\varphi_{t_0,t_1}, \varphi_{t_1,t_2}, \ldots, \varphi_{t_{n-1},t_n} \tag{1.4.2}$$

are independent.

1.4.2 RDS and Markov chains, discrete time $\mathbb{T} = \mathbb{Z}^+$ or $\mathbb{Z}$

Case $\mathbb{T} = \mathbb{Z}^+$: Here $\varphi(n, \omega) = \varphi(1, \theta^{n-1}\omega) \circ \ldots \circ \varphi(1, \omega)$, so the cocycle has independent increments if and only if $\varphi(1, \omega), \varphi(1, \theta\omega), \ldots$ are iid. We thus have a product of iid random mappings, i.e., a classical 'iterated function system'. The mapping

$$x \mapsto \varphi(n, \omega)x$$

defines a homogeneous Markov chain with transition kernel,for all $B \in \mathcal{B}$

$$P(x, B) = P\{\omega \mid \varphi(1, \omega)x \in B\}. \tag{1.4.3}$$

Putting $x_n = \varphi(n, \omega)x_0$ we have

$$x_{n+1} = \varphi(1, \theta^n\omega)x_n, \tag{1.4.4}$$

i.e., a stochastic difference equation generating the Markov chain.

Conversely, given a transition kernel $P(x, B)$ on M, we want to construct an RDS with independent increments over a dynamical system $(\Omega, \mathcal{F}, P, \theta)$, i.e., a cocycle $\varphi(n, \omega)$ with $(\varphi(1, \theta^n\omega))_{n\in\mathbb{Z}^+}$ iid, such that (1.4.3) holds. This question has been dealt with by Kifer [23, Section 1.1]. It always has a positive answer as soon as M is a Borel subset of a Polish space and if we are content with a measurable mapping $(x, \omega) \mapsto \varphi(1, \omega)x$. If we want $x \mapsto \varphi(1, \omega)x$ to be continuous or homeomorphisms or smooth etc., a general answer to this representation problem is not known up to now (compare Kifer [23, p. 12]).

Case $\mathbb{T} = \mathbb{Z}$: Now $\varphi(n, \omega)$ is invertible, and the cocycle has independent increments if and only if $(\varphi(1, \theta^n\omega))_{n\in\mathbb{Z}}$ is iid. The mapping $x \mapsto \varphi(n, \omega)x$ defines a homogeneous Markov chain on all of $\mathbb{Z}$ starting at $x_0 = x$, and (1.4.4) can be inverted to give

$$x_n = \varphi(1, \theta^n\omega)^{-1}x_{n+1} = \varphi(-1, \theta^{-n}\omega)x_{n+1}.$$

We can now look at the forward transition kernel

$$P^+(x, B) = P\{\omega \mid \varphi(1, \omega)x \in B\}$$

and the backward transition kernel

$$P^-(x,B) = P\{\omega \mid \varphi(-1,\omega)x \in B\} = P\{\omega \mid \varphi(1,\omega)^{-1}x \in B\}.$$

Note that in general P^+ and P^- do not have the same invariant measures: a forward invariant measure ν^+ has to satisfy

$$\nu^+ = \int P^+(x,\cdot)d\nu^+(x) = E\varphi(1,\omega)\nu^+,$$

whereas a backward invariant measure ν^- is characterized by

$$\nu^- = \int P^-(x,\cdot)d\nu^-(x) = E\varphi(-1,\omega)\nu^- = E\varphi(1,\omega)^{-1}\nu^-.$$

1.4.3 RDS and Markov processes, continuous time $\mathbb{T} = \mathbb{R}^+$ or $\mathbb{R}$

Case $\mathbb{T} = \mathbb{R}^+$: Assume $M = \mathbb{R}^d$ (similar things hold on manifolds). Let $\varphi(t,\omega)$ be a Brownian RDS of homeomorphisms (or diffeomorphisms of some smoothness class). Then $(\varphi(t,\omega))_{t\in\mathbb{R}^+}$ is a Brownian motion with values in the group $\mathrm{Hom}(\mathbb{R}^d)$ (or $\mathrm{Diff}^*(\mathbb{R}^d)$ with a suitable $*$) in the sense of Baxendale [5], or

$$\varphi_{s,t}(\omega) = \varphi(t,\omega)\circ\varphi(s,\omega)^{-1},$$

$s,t \in \mathbb{R}^+$, is a temporally homogeneous Brownian flow. By studying the infinitesimal mean

$$\lim_{h\to 0}\frac{1}{h}\mathbb{E}(\varphi_{t,t+h}(\omega)x - x) = b(x)$$

and the infinitesimal covariance

$$\lim_{h\to 0}\frac{1}{h}\mathbb{E}((\varphi_{t,t+h}(\omega)x - x)(\varphi_{t,t+h}(\omega)y - y)') = a(x,y),$$

Kunita [27] constructs a vector field valued Brownian motion $(F(x,t,\omega))_{x\in\mathbb{R}^d,t\in\mathbb{R}^+}$, i.e., a continuous (in t) Gaussian process $(F(\cdot,t,\omega))_{t\in\mathbb{R}^+}$ with values in the space of vector fields on $\mathbb{R}^d$
(so $x \mapsto F(x,t,\omega)$ is a vector field), which has additively stationary independent increments and satisfies $F(\cdot,0,\omega) = 0$ (P-a.s.).
The Brownian motion F is related to the Brownian flow φ by $\mathbb{E}(F(x,t,\omega)) = tb(x)$ and

$$\mathrm{cov}(F(x,t,\omega),F(y,s,\omega)) = \min\{t,s\}a(x,y).$$

This implies that for all $s \in \mathbb{R}^+$

$$\varphi_{s,t}(\omega)x = x + \int_s^t F(\varphi_{s,u}(\omega)x, du, \omega), \quad t \in [s,\infty), \tag{1.4.5}$$

which has to be understood in the sense that (1.4.5) has a solution which coincides in distribution with the original Brownian flow φ. In short: The (forward) flow satisfies an Itô SDE driven by

vector field valued Gaussian white noise. F is called the random infinitesimal generator of φ. All n-point motions $(\varphi(t,\omega)x_1,\dots,\varphi(t,\omega)x_n)$ are homogeneous Feller-Markov processes. In particular, $(\varphi(t,\omega)x)_{t\in\mathbb{R}^+}$ is a Markov process whose transition semigroup has generator

$$L=\sum_{i=1}^{d} b^i(x)\frac{\partial}{\partial x^i}+\frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x,x)\frac{\partial^2}{\partial x^i\partial x^j}. \tag{1.4.6}$$

The backward flow $\varphi_{s,t}(\omega)^{-1}=\varphi(s,\omega)\circ\varphi(t,\omega)^{-1},\ 0\le s\le t$, satisfies for each $t\in\mathbb{R}^+$ a backward Itô equation in $s\in[0,t]$,

$$\varphi_{s,t}(\omega)^{-1}x=x-\int_s^t \hat{F}(\varphi_{u,t}(\omega)^{-1}x,du,\omega),$$

where

$$\hat{F}(x,t,\omega)=F(x,t,\omega)-tc(x),\quad c_i(x)=\sum_{j=1}^{d}\frac{\partial a^{ij}}{\partial x^j}(x,y)\Big|_{y=x}, \tag{1.4.7}$$

and the backward integral $\int_s^t \hat{F}(\varphi_{u,t}(\omega)^{-1}x,d\hat{u},\omega)$ is formally defined by the same definition as the forward integral — the difference being the inverted measurability counting from t backward to s. As usual, things get more symmetric if we use Stratonovich forward and backward integrals.Put

$$F^0(x,t,\omega)=F(x,t,\omega)-\frac{t}{2}c(x),$$

then F^0 is the forward as well as the backward Stratonovich infinitesimal generator of φ.

Conversely, given a temporally homogeneous $V(\mathbb{R}^d)$ (=vector fields on $\mathbb{R}^d$) -valued Brownian motion F, we can write down the SDE (1.4.5) to generate a Brownian flow with generator F. We can easily construct a Brownian RDS describing the same object. Indeed, put

$$\begin{aligned}\Omega&=\{\omega\mid\omega(0)=0,\omega(\cdot)\in C(\mathbb{R}^+,V(\mathbb{R}^d))\}\\ \mathcal{F}&=\text{Borel field}\\ P&=\text{distribution of } F=\text{`Wiener measure'}\\ \theta_t\omega(\cdot)&=\omega(\cdot+t)-\omega(t),\ t\in\mathbb{R}^+,\end{aligned}$$

so θ_t leaves P invariant and is ergodic. Moreover, representing $F(\cdot,t,\omega)\equiv\omega(t)$, the uniqueness of the solution flow of (1.4.5) yields

$$\varphi_{s,t}(\omega)=\varphi(t-s,\theta_s\omega),\quad 0\le s\le t.$$

The flow property $\varphi_{0,t+s}=\varphi_{s,t+s}\circ\varphi_{0,s}$ reads

$$\varphi(t+s,\omega)=\varphi(t,\theta_s\omega)\circ\varphi(s,\omega),$$

which is nothing but the cocycle property. As (1.4.2) is satisfied, and (1.4.2) is equivalent to (1.4.1), we have constructed a Brownian RDS from the given $V(\mathbb{R}^d)$-valued temporally homogeneous Brownian motion F.
The result is (modulo technical assumptions): There is a one-to-one relation between Brownian RDS and temporally homogeneous vector field valued Brownian motions (defined via the SDE (1.4.5)).
Since the law of $F(\cdot, t)$ (Gaussian!) is uniquely determined by the infinitesimal characteristics $b(x)$ and $a(x, y)$, it follows that (a, b) uniquely determine the law of the corresponding Brownian RDS. Consequently, as a and b appear in the generator of the two-point motion, the law of the two-point motion uniquely determines the law of the Brownian RDS.
(The last statement is in general not true in the case of discrete time.)
Note that, however, the law of the Brownian RDS is in general not determined by the law of the one-point motion. The point is that the generator (1.4.6) only contains $a(x, x)$ instead of $a(x, y)$. In general there are thus many distinct Brownian RDS whose one-point motions have the same law. For instance, if $a(x, x) \in C_b^2(\mathbb{R}^d; \mathbb{R}^{d\times d})$ is given, it can be factorized as $a(x, x) = \sigma(x)\sigma(x)'$ with $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ Lipschitz. Now put $a(x, y) = \sigma(x)\sigma(y)'$, and generate a Brownian RDS by solving

$$d\varphi_{s,t}(\omega)x = b(\varphi_{s,t}(\omega)x)\, dt + \sigma(\varphi_{s,t}(\omega)x)\, dB(t, \omega),$$

where B is a standard Brownian motion in $\mathbb{R}^d$. In this case

$$F(x, t, \omega) = tb(x) + \sigma(x)B(t, \omega).$$

In general countably many B_j's are necessary to represent F.

1.5 The Koopman operator

1.5.1 Background

The Koopman operator advances measurement functions of the state of a dynamical system with the flow of the dynamics. To explain the basic properties of the Koopman operator, we begin with an autonomous ordinary differential equation

$$\frac{d}{dt}\mathrm{x}(t) = f(\mathrm{x}(t)) \tag{1.5.1}$$

on a finite-dimensional state space $M \subseteq \mathbb{R}^n$. The flow map operator, or time-t map, $F^t : M \to M$ advances initial conditions x(0) forward along the trajectory by a time t, so that trajectories evolve according to

$$\mathrm{x}(t) = F^t(\mathrm{x}(0)). \tag{1.5.2}$$

The family of Koopman operators $\mathcal{K}^t : \mathcal{G}(M) \to \mathcal{G}(M)$, parameterized by t, are given by

$$\mathcal{K}^t g(\mathrm{x}) = g(F^t(\mathrm{x})), \tag{1.5.3}$$

where $\mathcal{G}(M)$ is a set of measurement functions $g : M \to \mathbb{C}$. Another name for g, derived from this framework's origin in quantum mechanics, is an observable function, although this should not be confused with the unrelated observability from control theory. We can interpret (1.5.3) as defining a family of functions

$$g_t := \mathcal{K}^t g, \quad g_0 := g. \tag{1.5.4}$$

that corresponds to the trajectory $t \mapsto g_t$ in the set $\mathcal{G}(M)$ of measurement functions. In most applications, the set of functions $\mathcal{G}(M)$ is not defined a priori, but is loosely specified by a set of properties it should satisfy, e.g., that it is a vector space, that it possesses an inner product, that it is complete, or that it contains certain functions of interest, such as continuous functions on M. Hilbert spaces, such as $L^2(M, d\mu)$ or reproducing kernel Hilbert spaces (RKHS), are a common choice in modern applied mathematics, although historically other Banach spaces, such as integrable functions L^1 or continuous functions $C(M)$ have also been used. The choice of the space, whether explicit or implicit, can have consequences on the properties of the operator and its approximations. In all cases, however, $\mathcal{G}(M)$ is of significantly higher dimension than M, i.e., countably or uncountably infinite.

The most significant property of the Koopman operator is that it is linear when $\mathcal{G}(M)$ is a linear (vector) space of functions:

$$\begin{aligned} \mathcal{K}^t(\alpha_1 g_1(\mathrm{x}) + \alpha_2 g_2(\mathrm{x})) &= \alpha_1 g_1(F^t(\mathrm{x})) + \alpha_2 g_2(F^t(\mathrm{x})) \\ &= \alpha_1 \mathcal{K}^t g_1(\mathrm{x}) + \alpha_2 \mathcal{K}^t g_2(\mathrm{x}). \end{aligned} \tag{1.5.5}$$

This property holds regardless of whether $F^t : M \to M$ is linear itself, as it is simply a consequence of definition (1.5.3), since the argument function g is on the "outside" of the composition, allowing linearity to carry over from the vector space of observables. In this sense, the Koopman framework obtains linearity of $\mathcal{K}^t$ despite the nonlinearity of F^t by trading the finite-dimensional state space M for an infinite-dimensional function space $G(M)$.

When time t is discrete, $t \in \mathbb{N}$, and the dynamics are autonomous, then F^t is a repeated t-fold composition of $F \equiv F^1$ given by $F^t(\mathrm{x}) = F(F(\ldots(F(\mathrm{x}))))$, so that $\mathcal{K}^t g$ is likewise generated by repeated application of $\mathcal{K} \equiv \mathcal{K}^1$. The generator $\mathcal{K}$ of the (countable) composition semigroup is then called the Koopman operator, which results in a dynamical system

$$g_{k+1} = \mathcal{K} g_k, \tag{1.5.6}$$

analogous to $\mathrm{x}_{k+1} = F(\mathrm{x}_k)$, except that (1.5.6) is linear and infinite-dimensional.

When time t is continuous, the flow map family satisfies the semigroup property

$$F^{t+s}(\mathrm{x}) = F^t(F^s(\mathrm{x})), \quad \forall \mathrm{x}, t, s \geq 0, \tag{1.5.7}$$

which can be strengthened to a group property $t, s \in \mathbb{R}$ if the flow map is invertible. The Koopman operator family $\mathcal{K}^t$ inherits these properties as well. Given a continuous and sufficiently smooth dynamics, it is also possible to define the continuous-time infinitesimal generator $\mathcal{L}$ of the Koopman operator family as

$$\mathcal{L} g := \lim_{t \to 0} \frac{\mathcal{K}^t g - g}{t} = \lim_{t \to 0} \frac{g \circ F^t - g}{t}. \tag{1.5.8}$$

The generator $\mathcal{L}$ has been called the Lie operator [24], as it is the Lie derivative of g along the vector field f(x) when the dynamics is given by (1.5.1). This follows from applying the chain rule to the time derivative of $g(\mathrm{x})$:

$$\frac{d}{dt}g(\mathrm{x}(t)) = \nabla g \cdot \dot{\mathrm{x}}(t) = \nabla g \cdot f(\mathrm{x}(t)) \tag{1.5.9}$$

and

$$\frac{d}{dt}g(\mathrm{x}(t)) = \lim_{\tau \to 0} \frac{g(\mathrm{x}(t+\tau)) - g(\mathrm{x}(t))}{\tau} = \mathcal{L}(g(\mathrm{x}(t))). \tag{1.5.10}$$

resulting in

$$\mathcal{L}g = \nabla g \cdot \mathrm{f}. \tag{1.5.11}$$

The adjoint of the Koopman operator is the Perron-Frobenius operator [15, 16]. In many ways, the operator-theoretic framework for applied dynamical systems has two dual perspectives, corresponding either to the Koopman operator or the Perron-Frobenius operator.

Similar to (1.5.6), $\mathcal{L}$ induces a linear dynamical system in continuous-time:

$$\frac{d}{dt}g = \mathcal{L}g. \tag{1.5.12}$$

The linear dynamical systems in (1.5.6) and (1.5.12) are analogous to the dynamical systems in (1.5.1).

1.5.2 Eigenfunctions and the spectrum of eigenvalues

A Koopman eigenfunction $\varphi(\mathrm{x})$ corresponding to an eigenvalue λ satisfies

$$\phi(\mathrm{x}_{k+1}) = \mathcal{K}\phi(\mathrm{x}_k) = \lambda\phi(\mathrm{x}_k). \tag{1.5.13}$$

In continuous-time, a Lie operator eigenfunction $\phi(\mathrm{x})$ satisfies

$$\frac{d}{dt}\phi(\mathrm{x}) = \mathcal{L}\phi(\mathrm{x}) = \mu\phi(\mathrm{x}), \tag{1.5.14}$$

where μ is a continuous-time eigenvalue. In general, eigenvalues and eigenvectors are complex-valued scalars and functions, respectively, even when the state space M and dynamics $F(\mathrm{x})$ are real-valued.

It is simple to show that Koopman eigenfunctions $\phi(\mathrm{x})$ that satisfy (1.5.13) for $\lambda \neq 0$ are also eigenfunctions of the Lie operator, although with a different eigenvalue. Applying the Lie operator (1.5.8) to such a ϕ leads to

$$\mathcal{K}^t\phi = \lambda^t\phi \implies \mathcal{L}\phi = \lim_{t\to 0}\frac{\mathcal{K}^t\phi - \phi}{t} = \lim_{t\to 0}\frac{\lambda^t - 1}{t}\phi = \log(\lambda)\phi. \tag{1.5.15}$$

Conversely, the induced dynamics (1.5.12) applied to an eigenfunction of $\mathcal{L}$ leads to

$$L\phi = \mu\phi \implies \frac{d}{dt}\phi = L\phi = \mu\phi. \tag{1.5.16}$$

An eigenfunction ϕ of $\mathcal{L}$ with eigenvalue μ is then an eigenfunction of $\mathcal{K}^t$ with eigenvalue

$$\lambda^t = \exp(\mu t).$$

Thus, we will typically not make a distinction between Lie and Koopman eigenfunctions in the context of autonomous dynamical systems.

Eigenfunctions of $\mathcal{K}$ and $\mathcal{L}$ that are induced by already-linear dynamics further illustrate the connection between linear discrete dynamics $\mathrm{x}_{n+1} = A\mathrm{x}_n$ with analogous concepts for

$$g_{n+1} = \mathcal{K}g_n.$$

Given a left-eigenvector $w^T A = \lambda w^T$ of the matrix A, we form a corresponding Koopman eigenfunction as

$$\phi(\mathrm{x}) := w^T \mathrm{x} \tag{1.5.17}$$

since

$$\mathcal{K}\phi(\mathrm{x}) = \phi(A\mathrm{x}) = w^T A\mathrm{x} = \lambda w^T \mathrm{x} = \lambda\phi(\mathrm{x}). \tag{1.5.18}$$

In other words, while right-eigenvectors of A giver rise to time-invariant directions in the state space M, which will be known as Koopman modes or dynamic modes, the left-eigenvectors give rise to Koopman eigenfunctions, which are similarly time-invariant directions in the space of observables $\mathcal{G}(M)$.

In general systems, a set of Koopman eigenfunctions may be used to generate more eigenfunctions. In discrete time, we find that the product of two eigenfunctions $\phi_1(\mathrm{x})$ and $\phi_2(\mathrm{x})$ is also an eigenfunction:

$$\begin{aligned} \mathcal{K}(\phi_1(\mathrm{x})\phi_2(\mathrm{x})) &= \phi_1(F(\mathrm{x}))\phi_2(F(\mathrm{x})) \\ &= \lambda_1\lambda_2\phi_1(\mathrm{x})\phi_2(\mathrm{x}) \end{aligned} \tag{1.5.19}$$

with a new eigenvalue $\lambda_1\lambda_2$ given by the product of the two eigenvalues of $\phi_1(\mathrm{x})$ and $\phi_2(\mathrm{x})$. This argument assumes implicitly that the product of the two eigenfunctions is again an eigenfunction or, more strongly, that the space of observables is closed under multiplication. The corresponding relationship for $\mathcal{L}$ can be found by applying (1.5.15),

$$\lambda_1\lambda_2 = e^{\mu_1}e^{\mu_2} = e^{\mu_1+\mu_2}, \tag{1.5.20}$$

resulting in

$$\mathcal{L}(\phi_1(\mathrm{x})\phi_2(\mathrm{x})) = (\mu_1 + \mu_2)\phi_1(\mathrm{x})\phi_2(\mathrm{x}). \tag{1.5.21}$$

A simple consequence is that a complex conjugate pair of eigenfunctions of $\mathcal{L}$, (μ, ϕ), $(\bar{\mu}, \bar{\phi})$, additionally implies the existence of a real-valued eigenfunction $|\varphi| = \sqrt{\varphi\bar{\varphi}}$ with an associated eigenvalue $(\mu + \bar{\mu})/2 = \mathrm{Re}(\mu)$, thus leading to a non-oscillatory growth/decay of the eigenvalue.

Algebraically, when the space of observables is closed under multiplication, the set of Koopman eigenfunctions establishes a commutative monoid under point-wise multiplication. Thus, depending on the dynamical system, there may be a finite set of generator eigenfunction elements that may be used to construct all other eigenfunctions. Cardinality of the set of eigenfunctions and the relationships between them are further explored in [6]. The corresponding Koopman eigenvalues form a multiplicative lattice, or an additive lattice for Lie eigenvalues due to (1.5.15).

Observables that can be formed as linear combinations of eigenfunctions, i.e., $g \in \text{span}\{\phi_k\}_{k=1}^K$, have a particularly simple evolution under the Koopman operator

$$g(\mathrm{x}) = \sum_k v_k \phi_k \implies \mathcal{K}^t g(\mathrm{x}) = \sum_k v_k \lambda_k^t \phi_k. \tag{1.5.22}$$

This implies that the subspace $\text{span}\{\phi_k\}_{k=1}^K$ is invariant under the action of $\mathcal{K}$.

1.5.3 Analytic Series Expansions for Eigenfunctions

Given the dynamics in (1.5.1), it is possible to solve the PDE

$$\nabla\phi(x) \cdot f(x) = \lambda\phi(x) \tag{1.5.23}$$

using standard techniques, such as recursively solving for the terms in a Taylor or Laurent series. A number of simple examples are explored below.

Linear Dynamics. Consider the simple linear dynamics:

$$\frac{d}{dt}x = x \tag{1.5.24}$$

Assuming a Taylor series expansion for $\phi(x)$:

$$\phi(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Then the gradient and directional derivatives are given by:

$$\nabla\phi = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$\nabla\phi \cdot f = c_1 x + 2c_2 x^2 + 3c_3 x^3 + 4c_4 x^4 + \cdots$$

Solving the Koopman eigenfunction PDE (1.5.23), we find that $c_0 = 0$ must hold. For any positive integer λ in (1.5.23), only one of the coefficients may be nonzero. Specifically, for $\lambda = k \in \mathbb{Z}^+$, then $\phi(x) = cx^k$ is an eigenfunction for any constant c. For instance, if $\lambda = 1$, then $\phi(x) = x$.

Quadratic Nonlinear Dynamics. Consider a nonlinear dynamical system:

$$\frac{d}{dt}x = x^2 \tag{1.5.25}$$

There is no Taylor series that satisfies (1.5.23), except the trivial solution $\phi = 0$ for $\lambda = 0$. Instead, we assume a Laurent series:

$$\phi(x) = \cdots + c_{-3}x^{-3} + c_{-2}x^{-2} + c_{-1}x^{-1} + c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$

The gradient and directional derivatives are:

$$\nabla\phi = \cdots - 3c_{-3}x^{-4} - 2c_{-2}x^{-3} - c_{-1}x^{-2} + c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots$$

$$\nabla\phi \cdot f = \cdots - 3c_{-3}x^{-2} - 2c_{-2}x^{-1} - c_{-1} + c_1x^2 + 2c_2x^3 + 3c_3x^4 + 4c_4x^5 + \cdots$$

Solving for the coefficients of the Laurent series that satisfy (1.5.23), we find that all coefficients with positive index are zero, i.e., $c_k = 0$ for all $k \geq 1$. However, the nonpositive index coefficients are given by the recursion:

$$\lambda c_{k+1} = kc_k, \quad \text{for } k \leq -1$$

Thus, the Laurent series becomes:

$$\phi(x) = c_0\left(1 - \frac{\lambda}{x} + \frac{\lambda^2}{2x^2} - \frac{\lambda^3}{3!x^3} + \cdots\right) = c_0e^{-\lambda/x}$$

This holds for all values of $\lambda \in \mathbb{C}$. There are also other Koopman eigenfunctions that can be identified from the Laurent series.

Polynomial Nonlinear Dynamics. For a more general nonlinear dynamical system:

$$\frac{d}{dt}x = ax^n, \tag{1.5.26}$$

$\phi(x) = e^{\frac{\lambda}{(1-n)a}x^{1-n}}$ is an eigenfunction for all $\lambda \in \mathbb{C}$.

Chapter 2

The Koopman Operator Spectrum for Random Dynamical Systems

In the context of random dynamical systems, the stochastic Koopman operator acts on the space of observable functions and allows the system's evolution to be represented in terms of dynamic modes. This spectral approach offers a unique perspective for analyzing systems where traditional mathematical models are too complex or non-existent, but where data can be obtained by observing the system. The growing interest in this approach lies in its ability to provide data-driven model reduction and prediction methods. Using algorithms such as stochastic Hankel-Dynamic Mode Decomposition (sHankel-DMD), it is possible to numerically approximate the spectral objects (eigenvalues, eigenfunctions) of the stochastic Koopman operator and apply this methodology to various practical examples[10].

2.1 Stochastic Koopman operator

Definition 2.1.1. The stochastic Koopman operator $\mathcal{K}^t$ associated with the RDS φ is defined on a space of functions (observables) $f : M \to \mathbb{C}$ for which the functional

$$\mathcal{K}^t f(\mathrm{x}) = \mathbb{E}[f(\varphi(t,\omega)\mathrm{x})], \quad \mathrm{x} \in M \tag{2.1.1}$$

exists. We refer to the family of operators $(\mathcal{K}^t)_{t\in\mathbb{T}}$ as the stochastic Koopman operator family.

Given a continuous function on a compact metric space, the functional exists. If spectral expansions of the Koopman operator are required, the space might require further specification (e.g., in many cases, a Hilbert space will suffice, see Example 2). However, to maintain simplicity, we generally do not specify the domain of the operator (except in specific examples), as our analysis is largely unaffected by it.[10]

Definition 2.1.2. The observables $\phi^t : M \to \mathbb{C}$ that satisfy equation

$$\mathcal{K}^t\phi^t(\mathrm{x}) = \lambda^S(t)\phi^t(\mathrm{x}) \tag{2.1.2}$$

we call the eigenfunctions of the stochastic Koopman operator, while the associated values $\lambda^S(t)$ we call the stochastic Koopman eigenvalues.

The above definition can be viewed geometrically: the level sets of an eigenfunction ϕ of a stochastic Koopman operator are unique in the sense that the expectation of the value of the level set at time t is $\lambda^S(t)\phi^t(\mathrm{x})$, where λ^S is the stochastic eigenvalue, and $\phi^t(\mathrm{x})$ represents the eigenfunction evaluated at time t. This expectation depends only on the initial level set at $t = 0$, from which the dynamics started.

An even stronger statement is available for eigenvalues on the unit circle: in that case, the precise location of the state of the system after time t is within a level set of the associated eigenfunction. For example, if $\phi(\mathrm{x})$ is an eigenfunction associated with eigenvalue 1, then its level sets are invariant sets for the underlying RDS.(see the proof in [32] for the discrete-time RDS).

2.2 Stationarity and Ergodicity

Definition 2.2.1. A measure μ is called invariant, or stationary, if

$$\mu(A) = \int_M \int_\Omega \chi_A(\varphi(1, \omega, x))\, dP\, d\mu,$$

where χ_A is the indicator function for $A \subseteq M$. If μ is a stationary measure, we have the equality

$$\int_M \int_\Omega f(\varphi(t_1 + s, \omega, x), \ldots, \varphi(t_n + s, \omega, x))\, dP\, d\mu = \int_M \int_\Omega f(\varphi(t_1, \omega, x), \ldots, \varphi(t_n, \omega, x))\, dP\, d\mu \tag{2.2.1}$$

for any $s, t_1, \ldots, t_n$.

Definition 2.2.2. A set $A \subseteq M$ is called invariant if

$$\int_\Omega \chi_A(\varphi(1, \omega, x))\, dP = \chi_A(x)$$

for almost every x.

Definition 2.2.3. A stationary measure μ is called ergodic if every invariant set has measure 0 or 1. The ergodicity assumption ensures that almost every trajectory samples the entire space, not just some invariant subset. With this assumption, we can use time averages to evaluate integrals over the space.

Lemma 1. Suppose μ is an ergodic measure. Let

$$h(x, \omega) = \hat{h}(\varphi(t_1, \omega, x), \varphi(t_2, \omega, x), \ldots, \varphi(t_n, \omega, x))$$

for some $t_1, t_2, \ldots, t_n$, with $h \in L^1(\mu \times P)$. Then we have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} h(x_j, \omega_j) = \int_M \int_\Omega h(x, \omega)\, dP\, d\mu \tag{2.2.2}$$

for almost every (x_0, ω_0) with respect to $\mu \times P$.

2.3 Stochastic Koopman operator for discrete time RDS

Proposition 1. Suppose that $A : \Omega \to \mathbb{R}^{d\times d}$ is measurable and that the one step map $T : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ of a discrete RDS is defined by

$$T(\omega, \mathrm{x}) = A(\omega)\mathrm{x}. \tag{2.3.1}$$

Denote by $\Phi(n, \omega)$ the linear RDS satisfying $T^n(\omega, \mathrm{x}) = \Phi(n, \omega)\mathrm{x}$, i.e.,

$$\Phi(n, \omega) = A(\psi^{n-1}(\omega)) \cdots A(\psi(\omega))A(\omega).$$

Assume that $\hat{\Phi}(n) = \mathbb{E}[\Phi(n, \omega)]$ are diagonalizable, with simple eigenvalues $\hat{\lambda}_j(n)$ and left and right eigenvectors $\hat{w}_j^n, \hat{v}_j^n$, $j = 1, \ldots, d$. Then the eigenfunctions of the stochastic Koopman operator $\mathcal{K}^n$ are

$$\phi_j^n(\mathrm{x}) = \langle \mathrm{x}, \hat{w}_j^n \rangle, \quad j = 1, \ldots, d, \tag{2.3.2}$$

with the corresponding eigenvalues $\lambda_j^S(n) = \hat{\lambda}_j(n)$. Moreover, if matrices $A(\omega), \omega \in \Omega$ commute and are diagonalizable with the simple eigenvalues $\lambda_j(\omega)$ and corresponding left eigenvectors w_j, $j = 1, \ldots, d$, then

$$\hat{w}_j^n = w_j \quad \text{and} \quad \lambda_j^S(n) = \mathbb{E}\left[\prod_{i=1}^{n} \lambda_j(\psi^{i-1}(\omega))\right].$$

Furthermore, $\hat{v}_j^n$, $j = 1, \ldots, d$ are the Koopman modes of the full-state observable and the following expansion is valid

$$\mathcal{K}^n\mathrm{x} = \sum_{j=1}^{d} \lambda_j^S(n)\langle \mathrm{x}, \hat{w}_j^n \rangle \hat{v}_j^n. \tag{2.3.3}$$

Proof. The action of the stochastic Koopman operator on functions defined by (2.3.2) is equal to

$$\begin{aligned}
\mathcal{K}^n\phi_j^n(\mathrm{x}) &= \mathbb{E}[\langle \Phi(n, \omega)\mathrm{x}, \hat{w}_j^n \rangle] \\
&= \mathbb{E}[\langle \mathrm{x}, \Phi(n, \omega)^* \hat{w}_j^n \rangle] \\
&= \langle \mathrm{x}, \mathbb{E}[\Phi(n, \omega)^*]\hat{w}_j^n \rangle \\
&= \langle \mathrm{x}, \hat{\Phi}(n)^* \hat{w}_j^n \rangle \\
&= \hat{\lambda}_j(n)\langle \mathrm{x}, \hat{w}_j(n) \rangle \\
&= \hat{\lambda}_j(n)\phi_j^n(\mathrm{x})
\end{aligned}$$

where we have used that

$$\mathbb{E}[\Phi(n, \omega)^*] = (\mathbb{E}[\Phi(n, \omega)])^* = \hat{\Phi}(n)^*.$$

In the case when matrices $A(\omega), \omega \in \Omega$ commute and are diagonalizable, they are simultaneously diagonalizable (see [20, Theorem 1.3.12]), i.e., there exists a single invertible matrix V, so that $A(\omega) = V\Lambda(\omega)V^{-1}$, where $\Lambda(\omega) = \mathrm{diag}(\lambda_1(\omega), \ldots, \lambda_d(\omega))$. It is clear that the columns of the matrix V are the common right eigenvectors of the matrices $A(\omega), \omega \in \Omega$, and that $W^* = V^{-1}$, where W is the matrix of the left eigenvectors. It is straightforward that

$$\Phi(n, \omega) = V \prod_{i=1}^{n} \Lambda(\psi^{i-1}(\omega))W^*.$$

Therefore,

$$\hat{\Phi}(n) = V\mathbb{E}\left[\prod_{i=1}^{n} \Lambda(\psi^{i-1}(\omega))\right] W^*,$$

and we easily conclude that $\hat{w}_j = w_j$ and

$$\lambda_j^S(n) = \mathbb{E}\left[\prod_{i=1}^{n} \lambda_j(\psi^{i-1}(\omega))\right].$$

Equation (2.3.3) can be easily proved by using the decomposition of the state x in the base $\hat{w}_j^n, \hat{v}_j^n$, $j = 1, \ldots, d$, the linearity of the Koopman operator, and its action on the eigenfunctions given with (2.3.2), i.e.,

$$\begin{aligned}
\mathcal{K}^n \mathrm{x} &= \mathcal{K}^n \sum_{j=1}^{d} \langle \mathrm{x}, \hat{w}_j^n \rangle \hat{v}_j^n \\
&= \sum_{j=1}^{d} \mathcal{K}^n \langle \mathrm{x}, \hat{w}_j^n \rangle \hat{v}_j^n \\
&= \sum_{j=1}^{d} \lambda_j^S(n) \langle \mathrm{x}, \hat{w}_j^n \rangle \hat{v}_j^n.
\end{aligned}$$

Remark 4. We will use the term *principal eigenfunctions* for the eigenfunctions of the form $\phi_j(\mathrm{x}) = \langle \mathrm{x}, w_j \rangle$, $j = 1, \ldots, d$;In the linear case, the action of the Koopman operators on the full state observable can be derived by using just the principal eigenfunctions, eigenvalues, and modes.

2.4 Stochastic Koopman operator for RDS generated by the random differential equations

Proposition 2. If $A : \Omega \to \mathbb{R}^{d \times d}$ and $A \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then RDE

$$\dot{\mathrm{x}} = A(\theta(t)\omega)\mathrm{x}, \tag{2.4.1}$$

generates a linear RDS Φ satisfying

$$\Phi(t,\omega) = I + \int_0^t A(\theta(s)\omega)\Phi(s,\omega)\,ds. \tag{2.4.2}$$

Assume that $\hat{\Phi}(t) = \mathbb{E}[\Phi(t,\omega)]$ is diagonalizable, with simple eigenvalues $\hat{\mu}_j^t$ and left eigenvectors $\hat{w}_j^t$, $j = 1,\dots,d$. Then

$$\phi_j^t(\mathrm{x}) = \langle \mathrm{x}, \hat{w}_j^t \rangle, \quad j = 1,\dots,d, \tag{2.4.3}$$

are the principal eigenfunctions of the stochastic Koopman operator $\mathcal{K}^t$ with corresponding principal eigenvalues $\lambda_j^S(t) = \hat{\mu}_j(t)$, $j = 1,\dots,d$. Moreover, if matrices $A(\omega)$ commute and are diagonalizable with the simple eigenvalues $\lambda_j(\omega)$ and corresponding left eigenvectors w_j, $j = 1,\dots,d$, then

$$\hat{w}_j^t = w_j \quad \text{and} \quad \lambda_j^S(t) = \mathbb{E}\left[e^{\int_0^t \lambda_j(\theta(s)\omega)\,ds}\right].$$

Furthermore, $\hat{v}_j^t$, $j = 1,\dots,d$ are the Koopman modes of the full-state observable and the following expansion is valid

$$\mathcal{K}^t\mathrm{x} = \sum_{j=1}^{d} \lambda_j^S(t)\langle \mathrm{x}, \hat{w}_j^t\rangle \hat{v}_j^t. \tag{2.4.4}$$

Proof. The first part of the proposition follows from [4, Example 2.2.8]. Furthermore, the action of the stochastic Koopman operator on functions defined by (2.4.3) is equal to

$$\begin{aligned}
\mathcal{K}^t\phi_j^t(\mathrm{x}) &= \mathbb{E}[\langle \Phi(t,\omega)\mathrm{x}, \hat{w}_j^t\rangle] \\
&= \mathbb{E}[\langle \mathrm{x}, \Phi(t,\omega)^*\hat{w}_j^t\rangle] \\
&= \langle \mathrm{x}, \mathbb{E}[\Phi(t,\omega)^*]\hat{w}_j^t\rangle \\
&= \langle \mathrm{x}, \hat{\Phi}(t)^*\hat{w}_j^t\rangle \\
&= \hat{\mu}_j^t\langle \mathrm{x}, \hat{w}_j^t\rangle \\
&= \hat{\mu}_j^t\phi_j^t(\mathrm{x})
\end{aligned}$$

With the same argument as in the proof of Proposition 1, we have $A(\omega) = V\Lambda(\omega)W^*$, where W and V are matrices of common left and right eigenvectors,
and $\Lambda(\omega) = \mathrm{diag}(\lambda_1(\omega),\dots,\lambda_d(\omega))$. It is straightforward that

$$\Phi(t,\omega) = e^{\int_0^t A(\theta(s)\omega)\,ds} = Ve^{\int_0^t \Lambda(\theta(s)\omega)\,ds}W^*,$$

and

$$\hat{\Phi}(t) = V\mathbb{E}\left[e^{\int_0^t \Lambda(\theta(s)\omega)\,ds}\right]W^*.$$

We easily conclude that $\hat{w}_j = w_j$ and $\lambda_j^S(t) = \mathbb{E}\left[e^{\int_0^t \lambda_j(\theta(s)\omega)\,ds}\right]$.
The proof of (2.4.4) is the same as the proof of (2.3.3) .

Example 1. (Linear scalar RDS). Suppose that linear scalar RDE is given by

$$\dot{\mathrm{x}} = a(\omega)\mathrm{x}, \tag{2.4.5}$$

where $a : \Omega \to \mathbb{R}$ is a random variable with finite moments. Observe that $\theta(t) = \mathrm{id}$, which means that the probability space does not change with time. If the moment generating function defined by $\mathcal{M}_a(t) = \mathbb{E}[e^{ta(\omega)}]$ is analytic for $|t| < R$, then for the initial condition $\varphi(0, \omega)x = x$ and $t < R$ there exists a unique solution of (2.4.5), which can be expressed as[36]

$$\varphi(t, \omega)x = xe^{\int_0^t a(\omega)\, ds}. \tag{2.4.6}$$

The action of the stochastic Koopman operator on the full state observable function $\phi(x) = x$ is then

$$\begin{aligned} \mathcal{K}^t x &= \mathbb{E}[\varphi(t, \omega)x] \\ &= \mathbb{E}\left[xe^{\int_0^t a(\omega)\, ds}\right] \\ &= \mathbb{E}\left[e^{a(\omega)t}\right] x. \end{aligned}$$

Thus, $\phi(x) = x$ is the eigenfunction of the stochastic Koopman operator and the corresponding eigenvalue satisfies

$$\lambda^S(t) = \mathbb{E}\left[e^{a(\omega)t}\right] = \mathcal{M}_a(t). \tag{2.4.7}$$

2.5 Stochastic Koopman operator for RDS generated by the stochastic differential equations

The Koopman operator family $\mathcal{K}^{t,t_0}$ related to this RDS is defined by

$$\mathcal{K}^{t,t_0} f(\mathrm{x}) = \mathbb{E}[f(\varphi(t, t_0, \omega)\mathrm{x})]. \tag{2.5.1}$$

In this more general setting with the two-parameter family of Koopman operators (2.5.1), the eigenfunctions $\phi^{t,t_0} : M \to \mathbb{C}$ and eigenvalues $\lambda^S(t, t_0)$ of the Koopman operator $\mathcal{K}^{t,t_0}$ defined on a finite-time interval satisfy

$$\mathcal{K}^{t,t_0}\phi^{t,t_0}(\mathrm{x}) = \lambda^S(t, t_0)\phi^{t,t_0}(\mathrm{x}). \tag{2.5.2}$$

The following two propositions treat two classes of linear SDE. In the first one the random part of equations models the additive noise and in the second one it models the multiplicative noise.

Proposition 3. Let the linear SDE with additive noise be defined by

$$dX_t = A(t)X_t\, dt + \sum_{i=1}^{r} b^i(t)\, dW_t^i. \tag{2.5.3}$$

where $A(t)$ is a $d \times d$ matrix of functions and $b^i(t)$, $i = 1, \dots, m$ are d-dimensional vector functions. Assume that the fundamental matrix $\Phi(t, t_0)$ satisfying the matrix differential equation

$$\dot{\Phi} = A(t)\phi, \quad \Phi(t_0) = I \tag{2.5.4}$$

is diagonalizable, with simple eigenvalues $\hat{\mu}_j^{t,t_0}$ and left eigenvectors $\hat{w}_j^{t,t_0}$, $j = 1, \dots, d$. Then

$$\phi_j^{t,t_0}(\mathrm{x}) = \langle \mathrm{x}, \hat{w}_j^{t,t_0} \rangle, \quad j = 1, \dots, d, \tag{2.5.5}$$

are the eigenfunctions of the stochastic Koopman operator $\mathcal{K}^{t,t_0}$, with corresponding eigenvalues

$$\lambda_j^S(t, t_0) = \hat{\mu}_j(t, t_0).$$

If matrices $A(t)$ commute and are diagonalizable with the simple eigenvalues $\lambda_j(t)$ and corresponding left eigenvectors w_j, $j = 1, \dots, d$, then

$$\hat{w}_j^{t,t_0} = w_j \quad \text{and} \quad \lambda_j^S(t, t_0) = e^{\int_{t_0}^t \lambda_j(s)\, ds}. \tag{2.5.6}$$

Proof. Since the solution of (2.5.3) with the initial condition $X_{t_0}(\omega) = \mathrm{x}$ is given by, (see [1, Section 8.2])

$$X_t(\omega) = \Phi(t, t_0)\left(\mathrm{x} + \sum_{i=1}^r \int_{t_0}^t \Phi^{-1}(s, t_0) b^i(s)\, dW_s^i\right),$$

we have

$$\begin{aligned}
\mathcal{K}_{t,t_0}\phi_j^{t,t_0}(\mathrm{X}) &= \mathbb{E}\left[\phi_j^{t,t_0}(X_t(\omega))\right] \\
&= \mathbb{E}\left[\left\langle \Phi(t, t_0)\mathrm{x}, \hat{w}_j^{t,t_0} \right\rangle + \left\langle \sum_{i=1}^r \int_{t_0}^t \Phi(t, t_0)\Phi^{-1}(s, t_0) b^i(s)\, dW_s^i, \hat{w}_j^{t,t_0} \right\rangle\right] \\
&= \left\langle \Phi(t, t_0)\mathrm{x}, \hat{w}_j^{t,t_0} \right\rangle + \sum_{i=1}^r \left\langle \mathbb{E}\left[\int_{t_0}^t \Phi(t, t_0)\Phi^{-1}(s, t_0) b^i(s) dW_s^i\right], \hat{w}_j^{t,t_0} \right\rangle \\
&= \hat{\mu}_j^{t,t_0} \langle \mathrm{x}, \hat{w}_j^{t,t_0} \rangle \\
&= \hat{\mu}_j^{t,t_0} \phi_j^{S,t,t_0}(\mathrm{x}),
\end{aligned}$$

where we used that $\mathbb{E}\left[\int_{t_0}^t F(s)\, dW_s\right] = 0$ (see [1, Theorem 4.4.14]) applied to

$$F(s) = \Phi(t, t_0)\Phi^{-1}(s, t_0) b^i(s).$$

With this we proved the first statement. Since in the commutative case the fundamental matrix can be expressed in the form

$$\Phi(t, t_0) = e^{\int_{t_0}^t A(s)\, ds},$$

its eigenvectors coincide with the eigenvectors of the matrix $A(t)$ and eigenvalues are given by (2.5.6).

Proposition 4. Let the linear SDE with multiplicative noise be defined by

$$dX_t = A(t)X_t\,dt + \sum_{i=1}^{r} B^i(t)X_t\,dW_t^i, \tag{2.5.7}$$

where $A(t)$, $B^i(t)$, $i = 1,\dots,r$ are $d\times d$ matrices of functions. Denote with $\Phi(t,t_0)$ the fundamental matrix satisfying the matrix SDE

$$d\Phi = A\Phi\,dt + \sum_{i=1}^{r} B^i(t)\Phi\,dW_t^i, \quad \Phi(t_0) = I. \tag{2.5.8}$$

and assume that $\hat{\Phi}(t,t_0) = \mathbb{E}[\Phi(t,t_0)]$ is diagonalizable, with simple eigenvalues $\hat{\mu}_j^{t,t_0}$ and left eigenvectors $\hat{w}_j^{t,t_0}$, $j = 1,\dots,d$. Then

$$\phi_j^{t,t_0}(\mathrm{x}) = \langle \mathrm{x}, \hat{w}_j^{t,t_0}\rangle, \quad j = 1,\dots,d, \tag{2.5.9}$$

are the eigenfunctions of the stochastic Koopman operator $\mathcal{K}^{t,t_0}$, with corresponding eigenvalues

$$\lambda_j^S(t,t_0) = \hat{\mu}_j(t,t_0).$$

If the matrices $A(t)$ and $B^i(t)$, $i = 1,\dots,r$ commute, i.e., if $A(t)A(s) = A(s)A(t)$,

$$\begin{aligned} A(t)B^i(s) &= B^i(s)A(t) \\ &= B^i(t)B^j(s) \\ &= B^j(s)B^i(t) \end{aligned}$$

for $i,j = 1,\dots,r$ and all s,t, and if the matrices $A(t)$ are diagonalizable with the simple eigenvalues $\lambda_j(t)$ and corresponding left eigenvectors w_j, $j = 1,\dots,d$, then

$$\hat{w}_j^{t,t_0} = w_j \quad \text{and} \quad \lambda_j^S(t,t_0) = e^{\int_{t_0}^{t}\lambda_j(s)\,ds}. \tag{2.5.10}$$

Proof. For the fundamental matrix $\Phi(t,t_0)$ and the initial condition $X_{t_0}(\omega) = \mathrm{x}$, the solution of (2.5.7) is equal to, (see [1, Section 8.5])

$$X_t(\omega) = \Phi(t,t_0)\mathrm{x}, \tag{2.5.11}$$

thus

$$\begin{aligned} \mathcal{K}^{t,t_0}\phi_j^{t,t_0}(\mathrm{x}) &= \mathbb{E}\left[\phi_j^{t,t_0}(X_t(\omega))\right] \\ &= \mathbb{E}\left[\left\langle \Phi(t,t_0)\mathrm{x}, \hat{w}_j^{t,t_0}\right\rangle\right] \\ &= \left\langle \hat{\Phi}(t,t_0)\mathrm{x}, \hat{w}_j^{t,t_0}\right\rangle \\ &= \hat{\mu}_j^{t,t_0}\langle \mathrm{x}, \hat{w}_j^{t,t_0}\rangle \\ &= e^{\hat{\lambda}_j(t,t_0)}\phi_j^{t,t_0}(\mathrm{x}), \end{aligned}$$

For the case with commutative matrices $A(t)$ and $B^i(t)$, $i = 1, \dots, r$, the fundamental matrix $\Phi(t, t_0)$ can be expressed in an explicit form as

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} \left(A(s) - \frac{1}{2}\sum_{i=1}^{r} B^i(s)B^i(s)^T\right) ds + \int_{t_0}^{t} \sum_{i=1}^{r} B^i(s)\, dW_s^i}. \tag{2.5.12}$$

Since

$$\begin{aligned}
\hat{\Phi}(t, t_0) &= \mathbb{E}[\Phi(t, t_0)] \\
&= e^{\int_{t_0}^{t} \left(A(s) - \frac{1}{2}\sum_{i=1}^{r} B^i(s)B^i(s)^T\right) ds}\, \mathbb{E}\left[e^{\int_{t_0}^{t} \sum_{i=1}^{r} B^i(s)\, dW_s^i}\right] \\
&= e^{\int_{t_0}^{t} \left(A(s) - \frac{1}{2}\sum_{i=1}^{r} B^i(s)B^i(s)^T\right) ds} e^{\frac{1}{2}\int_{t_0}^{t} \sum_{i=1}^{r} B^i(s)B^i(s)^T\, ds} \\
&= e^{\int_{t_0}^{t} A(s)\, ds},
\end{aligned}$$

the eigenvectors of $\hat{\Phi}(t, t_0)$ coincide with the eigenvectors of the matrix $A(t)$ and eigenvalues are given by (2.5.10), which proves the statement. Here we used the fact that

$$\mathbb{E}\left[e^{\int_{t_0}^{t} B(s)\, dW_s}\right] = e^{\frac{1}{2}\int_{t_0}^{t} B(s)B(s)^T\, ds}.$$

(see [1, Theorem 8.4.5])

2.6 Semigroups of Koopman operators and their generators

In this section, we examine Random Dynamical Systems (RDS) whose associated stochastic Koopman operators satisfy the semigroup property. We focus on situations where M is a Polish space and B is the Borel sigma-algebra. An RDS defines a family of stochastic processes taking values in M. Each process is associated with a probability measure P_x on the canonical space M^T, induced by a probability measure P related to the driving dynamical system. Significant results regarding the properties of stochastic Koopman operators can be obtained in specific contexts, such as linear or Markovian settings. Therefore, we limit our considerations to particular types of RDS and only consider those that are Markovian[9], or more precisely, to those for which the family of processes $\{\varphi(t,\omega)\mathrm{x}\}_{\mathrm{x}\in M}$ is a time-homogeneous Markov family. Let $\mathcal{F}_t^x$ denote the sigma-algebra generated by the "past" of the stochastic process, i.e.,

$$\mathcal{F}_t^x = \sigma(\varphi(s,\omega)\mathrm{x},\, s \le t)$$

The Markov property implies that for $s \le t$ and any random variable Y, measurable with respect to $\mathcal{F}_t^x$, the following relation holds:

$$\mathbb{E}[Y \mid \mathcal{F}_s^x] = \mathbb{E}[Y \mid \varphi(s,\omega)x] \tag{2.6.1}$$

Moreover, for such Y, the following equality, known as the Chapman-Kolmogorov equation, is valid:

$$\mathbb{E}[Y \mid \varphi(s,\omega)x] = \mathbb{E}[\mathbb{E}[Y \mid \varphi(t,\omega)x] \mid \varphi(s,\omega)x] \tag{2.6.2}$$

The Chapman-Kolmogorov equation implies the semigroup property of stochastic Koopman operators, i.e.,

$$\mathcal{K}^{t+s} = \mathcal{K}^s \circ \mathcal{K}^t \tag{2.6.3}$$

In the context of Markov processes, the Koopman operator is commonly referred to as the Markov propagator or transition operator, and its properties have been extensively studied [12], [41]. There are two significant and well-known classes of Random Dynamical Systems (RDS) that can be identified with the Markov family: the discrete-time RDS generated by an independent identically distributed process, and the continuous-time RDS generated by stochastic differential equations where noise is modeled using a Wiener process. In both cases, the probability space associated with the stochastic process modeling the noise can be identified with the canonical measure-preserving dynamical system ([4]). In the following, we briefly describe the canonical dynamical system $(\theta(t))_{t\in T}$ induced by a given stochastic process $\tilde{\xi}$. Suppose $\tilde{\xi} = (\tilde{\xi}_t)_{t\in T}$, where $\tilde{\xi}_t : \tilde{\Omega} \to B$ is a B-valued stochastic process on a probability space $\tilde{\Omega}$, where $(B, \mathcal{B})$ is a measurable state space. The given process and the probability measure on $\tilde{\Omega}$ induce a probability measure P on B^T, allowing us to define a new probability space $(\Omega, \mathcal{F}, P) = (B^{\mathbb{T}}, \mathcal{B}^{\mathbb{T}}, P)$, where $\mathcal{B}^{\mathbb{T}}$ is the sigma-algebra generated by the collection of cylinder sets. The canonical realization ξ of the stochastic process $\tilde{\xi}$ is defined on $(\Omega, \mathcal{F}, P)$ by the coordinate functions $\xi_t(\omega) = \omega(t), \omega \in \Omega$. The shift transformations
$\theta(t) : B^{\mathbb{T}} \to B^{\mathbb{T}}$ given by

$$\theta(t)\omega(\tau) := \omega(t+\tau), \quad t \in \mathbb{T} \tag{2.6.4}$$

constitute the semigroup or group of measurable transformations. Notice that the canonical realization $\xi_t(\omega)$ can be viewed as the composition of the shift transformation (2.6.4) and the canonical projection $\pi : B^{\mathbb{T}} \to B$ defined by

$$\pi(\omega(\tau)) = \omega(0) \tag{2.6.5}$$

i.e., $\xi_t(\omega) = \omega(t) = \pi(\theta(t)\omega)$. If time $\mathbb{T}$ is discrete ($\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{Z}^+$),
a map $(t, \omega) \mapsto \theta(t)\omega$ is measurable and $(\theta(t))_{t\in T}$ is a measurable dynamical system. In the continuous time case ($\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}^+$), when $B = \mathbb{R}^m$ or when B is a Polish space, $(\theta(t))_{t\in\mathbb{T}}$ could become a measurable dynamical system after some redefinition of the probability measure P and of the σ-algebra set (see [4]).

2.6.1 Discrete-time RDS

Let now assume that for $\mathbb{T} = \mathbb{Z}^+ \cup \{0\}$, $\omega = (\omega_i)_{i\in\mathbb{T}}$ is a canonical realization of the stochastic process with the associated driving system composed by the shift transformation maps (2.6.4) as described in the previous paragraph. If we assume that the discrete RDS $\varphi(n, \omega)$ is defined by the one step map $T(\omega, \cdot) : M \to M$ of the form

$$T(\omega, \cdot) = T_0(\pi(\omega), \cdot), \tag{2.6.6}$$

where π denotes the canonical projection (2.6.5), by taking into account (2.6.6) in (1.3.1), we get

$$\varphi(n, \omega) = T_0(\pi(\psi^{n-1}(\omega)), \cdot) \circ \cdots \circ T_0(\pi(\psi(\omega)), \cdot) \circ T_0(\pi(\omega), \cdot), \quad n \geq 1. \tag{2.6.7}$$

If ω is an i.i.d. stochastic process, the sequence

$$\{T(\psi^i(\omega), \cdot) = T_0(\pi(\psi^{i-1}(\omega)), \cdot); i \geq 1\}$$

is an i.i.d. sequence of random maps, so that RDS (2.6.7) generates the time-homogeneous Markov family $\{\varphi(n, \omega)\mathrm{x}\}_{\mathrm{x}\in M}$ [4].
In this case, the stochastic Koopman operator family is a semigroup, so that

$$\mathcal{K}^n = \mathcal{K}^1 \circ \cdots \circ \mathcal{K}^1 = (\mathcal{K}^1)^n, \quad n > 0.$$

Therefore one can think about $\mathcal{K}^1$ as the generator of the stochastic Koopman semigroup, and we denote it by $\mathcal{K}^S$. According to (2.1.1), $\mathcal{K}^1$ is determined by using the one step map $T(\omega, \cdot)$ as

$$\mathcal{K}^S f(\mathrm{x}) = \mathcal{K}^1 f(\mathrm{x}) = \mathbb{E}[f(T(\omega, \mathrm{x}))]. \tag{2.6.8}$$

It follows from the semigroup property that if λ^S is the eigenvalue of the stochastic Koopman generator with the associated eigenfunction $\phi(\mathrm{x})$, then $(\lambda^S)^n$ and $\phi(\mathrm{x})$ are the eigenvalue and the eigenfunction of the operator $\mathcal{K}^n$.

Remark 5. The random maps were identically distributed but not necessarily independent, so that the future state of the process obtained by the action of their composition (1.3.1) could depend on the past behavior of the system and not only on the present state.

Example 2. (Noisy rotation on the circle). We describe here the example considered in [10, 22]. A deterministic dynamical system representing the rotation on the unit circle S^1 is defined by

$$T(\mathrm{x}) = \mathrm{x} + \vartheta, \tag{2.6.9}$$

where $\vartheta \in S^1$ is a constant number. We consider here its stochastic perturbation, i.e., a discrete RDS over the dynamical system $\theta = (\theta(t))_{t\in\mathbb{T}}$, where $\theta(t)$ are shift transformations (2.6.4), defined by the one step map $T : \Omega \times S^1 \to S^1$ of the form (2.6.6):

$$T(\omega, x) = x + \vartheta + \pi(\omega). \tag{2.6.10}$$

Here $\omega \in [-\delta/2, \delta/2]^{\mathbb{Z}}$ is a canonical realization of a stochastic process and $\pi(\cdot)$ is the canonical projection defined by (2.6.5). We suppose that the coordinates ω_i are i.i.d. with uniform distribution on the interval $[-\delta/2, \delta/2]$ for some $\delta > 0$. According to (2.6.8), the action of the associated stochastic Koopman generator on an observable function $f : S^1 \to \mathbb{C}$ is given by

$$\mathcal{K}^S f(x) = \mathbb{E}[f(T(\omega, x))] = \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} f(x + \vartheta + \omega_0)\, d\omega_0. \tag{2.6.11}$$

For the functions

$$\phi_j(x) = \exp(i2\pi jx), \quad j \in \mathbb{Z}, \tag{2.6.12}$$

the following equality holds

$$\begin{aligned}\mathcal{K}^S \phi_j(x) &= \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} \exp(i2\pi j(x + \vartheta + \omega_0))\, d\omega_0 \\ &= \frac{\sin(j\pi\delta)}{j\pi\delta}\exp(i2\pi j\vartheta)\exp(i2\pi jx) \\ &= \frac{\sin(j\pi\delta)}{j\pi\delta}\exp(i2\pi j\theta)\phi_j(x).\end{aligned}$$

We easily conclude that (2.6.12) are the eigenfunctions of the stochastic Koopman generator with corresponding eigenvalues

$$\lambda_j^S = \frac{\sin(j\pi\delta)}{j\pi\delta}\exp(i2\pi j\vartheta), \quad j \in \mathbb{Z}. \tag{2.6.13}$$

For any function $f : L^2(S^1) \to \mathbb{C}$ we have the spectral expansion

$$\mathcal{K}^n f(x) = \sum_{j\in\mathbb{Z}} c_j \left(\frac{\sin(j\pi\delta)}{j\pi\delta}\right)^n \exp(i2\pi jn\vartheta)\exp(i2\pi jx), \tag{2.6.14}$$

where c_j are the Fourier coefficients of f. Clearly, $\mathcal{K}^n f(x) \to c_0$ as $t \to \infty$. It is known that the eigenvalues of the Koopman generator related to the deterministic dynamical system (2.6.9) lie

on the unit circle and are equal to $\lambda_j = \exp(i2\pi j\vartheta)$; while the eigenfunctions are the same as in the random case. Moreover, it is interesting to observe that for rational θ the eigenspaces in the deterministic case, i.e., for $\delta = 0$, are infinite dimensional, while they are finite dimensional in the stochastic case ($\delta > 0$) due to the compactness of the operator. To be more precise, in this example, the eigenspaces become one-dimensional.

Remark 6. Consider the case when randomness is additive, i.e., when the one-step map $T : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$T(\omega, \mathrm{x}) = A\mathrm{x} + \pi(\theta(t)\omega),$$

where $\theta(t)$ is a shift transformation and $\omega : \Omega \to \mathbb{R}^d$ is the canonical realization of a process with i.i.d. components. Suppose that $\mathbb{E}[\omega] = 0$ and that the matrix A is diagonalizable with simple eigenvalues λ_j, $j = 1, \ldots, d$. Then the eigenfunctions of the stochastic Koopman generator $\mathcal{K}^S$ are principal eigenfunctions of the form
$\phi_j(\mathrm{x}) = \langle \mathrm{x}, w_j \rangle$, where w_j are left eigenvectors of A, while its eigenvalues coincide with the eigenvalues of the matrix A.

2.6.2 Continuous-time RDS

Let suppose that the stochastic Koopman operators family satisfies the semigroup property. Define the generator of the stochastic Koopman family $(\mathcal{K}^t)_{t\in T}$ acting on the observable functions $f \in C_b^1(\mathbb{R}^d)$ by the limit

$$\mathcal{K}^S f(\mathrm{x}) = \lim_{t\to 0^+} \frac{\mathcal{K}^t f(\mathrm{x}) - f(\mathrm{x})}{t}, \tag{2.6.15}$$

if it exists. For the Koopman operators associated with the RDS generated by RDE we have the following proposition.

Proposition 5. If the solution of RDE (1.3.3) is differentiable with respect to t and the stochastic Koopman family $(\mathcal{K}^t)_{t\in T}$ is a semigroup, then the action of the generator $\mathcal{K}^S$ on $f \in C_b^1(\mathbb{R}^d)$ is equal to

$$\mathcal{K}^S f(\mathrm{x}) = \mathbb{E}[F(\omega, \mathrm{x})] \cdot \nabla f(\mathrm{x}). \tag{2.6.16}$$

Proof.

$$\begin{aligned}
\mathcal{K}^S f(\mathrm{x}) &= \lim_{t\to 0^+} \frac{\mathcal{K}^t f(\mathrm{x}) - f(\mathrm{x})}{t} \\
&= \lim_{t\to 0^+} \frac{\mathbb{E}[f(X(t,\omega,\mathrm{x}))] - f(\mathrm{x})}{t} \\
&= \lim_{t\to 0^+} \mathbb{E}\left[\frac{f(X(t,\omega,\mathrm{x})) - f(\mathrm{x})}{t}\right] \\
&= \mathbb{E}\left[\lim_{t\to 0^+} \frac{f(X(t,\omega,\mathrm{x})) - f(X(0,\omega,\mathrm{x}))}{t}\right] \\
&= \mathbb{E}\left[\frac{d}{dt} f(X(t,\omega,\mathrm{x}))\Big|_{t=0}\right] \\
&= \mathbb{E}\left[\nabla f(\mathrm{x}) \cdot \frac{d}{dt} X(t,\omega,\mathrm{x})\Big|_{t=0}\right] \\
&= \mathbb{E}[F(\omega,\mathrm{x})] \cdot \nabla f(\mathrm{x}).
\end{aligned}$$

The swapping of the order of limit and expectation in the second line is justified by the dominated convergence theorem and the fact that the convergence $\frac{f(X(t,\omega,\mathrm{x}))-f(\mathrm{x})}{t}$ is uniform for all ω and x, since the derivative of f is bounded and the solution is differentiable (see [28, Section 7.6] for the proof in the deterministic case).

Corollary 1. Suppose that a stochastic Koopman generator $\mathcal{K}^S$ associated with RDE (1.3.3) exists. If ϕ_1 and ϕ_2 are the eigenfunctions of $\mathcal{K}^S$ with the associated eigenvalues λ_1 and λ_2, then $\phi_1\phi_2$ is also an eigenfunction with the associated eigenvalue $\lambda_1 + \lambda_2$.

Proof. Since $\mathcal{K}^S \phi_i(\mathrm{x}) = \lambda_i \phi_i(\mathrm{x}) = \mathbb{E}[F(\omega,\mathrm{x})] \cdot \nabla\phi_i(\mathrm{x})$, for $i = 1, 2$, we have

$$\begin{aligned}
\mathcal{K}^S(\phi_1\phi_2)(\mathrm{x}) &= \mathbb{E}[F(\omega,\mathrm{x})] \cdot \nabla(\phi_1\phi_2)(\mathrm{x}) \\
&= \mathbb{E}[F(\omega,\mathrm{x})] \cdot \nabla\phi_1(\mathrm{x})\phi_2(\mathrm{x}) + \mathbb{E}[F(\omega,\mathrm{x})] \cdot \nabla\phi_2(\mathrm{x})\phi_1(\mathrm{x}) \\
&= \lambda_1\phi_1(\mathrm{x})\phi_2(\mathrm{x}) + \lambda_2\phi_2(\mathrm{x})\phi_1(\mathrm{x}) \\
&= (\lambda_1 + \lambda_2)(\phi_1\phi_2)(\mathrm{x}). \qquad (*)
\end{aligned}$$

Corollary 2. Let $\phi \in C_b^1(\mathbb{R}^d)$ be an eigenfunction associated with eigenvalue λ of the stochastic Koopman generator $\mathcal{K}^S$ associated with an RDE (1.3.3). Then

$$\frac{d}{dt}\phi(\varphi(t,\omega)\mathrm{x}) = \lambda\phi(\varphi(t,\omega)\mathrm{x}) + \tilde{F}(\omega,\mathrm{x}) \cdot \nabla\phi(\varphi(t,\omega)\mathrm{x}), \qquad (2.6.17)$$

where

$$\tilde{F}(\omega,\mathrm{x}) = F(\omega,\mathrm{x}) - \mathbb{E}[F(\omega,\mathrm{x})].$$

Corollary 3. Suppose that a stochastic Koopman generator $\mathcal{K}^S$ associated with linear RDE (2.4.1) exists. Also, assume that $\hat{A} = \mathbb{E}[A(\omega)]$ is diagonalizable, with simple eigenvalues $\hat{\lambda}_j$ and left eigenvectors $\hat{w}_j$, $j = 1, \ldots, d$. Then

$$\phi_j(\mathrm{x}) = \langle \mathrm{x}, \hat{w}_j \rangle, \quad j = 1, \ldots, d, \tag{2.6.18}$$

are the principal eigenfunctions of the generator $\mathcal{K}^S$, while $\lambda_j^S = \hat{\lambda}_j$ are the associated principal eigenvalues.

Proof. According to (2.6.16), the action of the generator $\mathcal{K}^S$ of the Koopman operator family associated with the linear RDS generated by (2.4.1) on $f \in C_b^1(\mathbb{R}^d)$ is equal to

$$\begin{aligned} \mathcal{K}^S f(\mathrm{x}) &= \mathbb{E}[A(\omega)\mathrm{x}] \cdot \nabla f(\mathrm{x}) \\ &= \hat{A}\mathrm{x} \cdot \nabla f(\mathrm{x}). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{K}^S \phi_j(\mathrm{x}) &= \langle \hat{A}\mathrm{x}, w_j \rangle \\ &= \langle \mathrm{x}, \hat{A}^* w_j \rangle \\ &= \hat{\lambda}_j \phi_j(\mathrm{x}), \end{aligned}$$

which proves the statement.

Remark 7. Provided that the assumptions of Corollary 3 are valid, the principal eigenfunctions $\phi_j(\mathrm{x})$ given by (2.6.18) are the eigenfunctions of each Koopman operator $\mathcal{K}^t$ also, with the corresponding principal eigenvalues $\lambda_j^S(t) = e^{\hat{\lambda}_j t}$. The set of principal eigenfunctions does not cover all the eigenfunctions of the Koopman operator as we discuss next. According to Corollary 1, over the space of real analytic functions,

$$\phi(\mathrm{x}) = \phi_1^{n_1}(\mathrm{x}) \cdots \phi_d^{n_d}(\mathrm{x}), \quad \lambda = \sum_{j=1}^{d} n_j \hat{\lambda}_j,$$

with $n_j \in \mathbb{N}^+ \cup \{0\}$, $j = 1, \ldots, d$, are the eigenvalues and eigenfunctions of the Koopman generator. Thus, like in the deterministic case, any analytic observable function f can be represented as a linear combination of powers of the principal eigenfunctions [31] and its evolution under the RDS can be obtained using the spectral expansion formula. Another type of RDS which could be identified with the time-homogeneous Markov family are the RDS generated by the SDE of the form (1.3.5) with autonomous functions G and σ, i.e.,

$$dX_t = G(X_t)\, dt + \sigma(X_t)\, dW_t. \tag{2.6.19}$$

In this case the stochastic differential equation generates the one-parameter family of RDS

$$\varphi(t, \omega) := \varphi(t, 0, \omega) = \varphi(t + t_0, t_0, \omega),$$

so that the corresponding stochastic Koopman operator family and the associated stochastic eigenvalues and eigenfunctions depend on parameter t only. In this autonomous setting, we denote by $X_t(\mathrm{x})$ the solution of equation (2.6.19) for the initial condition $X_0(\omega) = \mathrm{x}$. In accordance with (1.3.7) it is equal to

$$X_t(\mathrm{x}) = \mathrm{x} + \int_0^t G(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s. \tag{2.6.20}$$

Proposition 6. The action of the generator of the stochastic Koopman family $\mathcal{K}^S$ on $f \in C_b^2(\mathbb{R}^d)$ is given by

$$\mathcal{K}^S f(\mathrm{x}) = G(\mathrm{x})\nabla f(\mathrm{x}) + \frac{1}{2}\mathrm{Tr}\left(\sigma(\mathrm{x})(\nabla^2 f(\mathrm{x}))\sigma(\mathrm{x})^T\right), \tag{2.6.21}$$

where Tr denotes the trace of the matrix.

Proposition 7. Let $\phi \in C_b^2(\mathbb{R}^d)$ be an eigenfunction of the stochastic Koopman generator $\mathcal{K}^S$ associated with RDS generated by SDE (2.6.19) with the corresponding eigenvalue λ. Then

$$d\phi(X_t) = \lambda\phi(X_t)dt + \nabla\phi(X_t)\sigma(X_t)dW_t. \tag{2.6.22}$$

ϕ is the eigenfunction of the stochastic Koopman operator $\mathcal{K}^t$ also, i.e.

$$\mathcal{K}^t\phi(\mathrm{x}) = e^{\lambda t}\phi(\mathrm{x}). \tag{2.6.23}$$

Proof. Suppose $d = 1$. It follows from Itô's formula

$$dY_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)\sigma(X_t)^2dt \tag{2.6.24}$$

that the eigenfunction $\phi(X_t)$ evolves according to

$$\begin{aligned} d\phi(X_t) &= \phi'(X_t)G(X_t)dt + \frac{1}{2}\phi''(X_t)\sigma(X_t)^2dt + \phi'(X_t)\sigma(X_t)dW_t \\ &= \mathcal{K}^S\phi(X_t)dt + \phi'(X_t)\sigma(X_t)dW_t \\ &= \lambda\phi(X_t)dt + \phi'(X_t)\sigma(X_t)dW_t, \end{aligned}$$

where in the last equality we used that $\mathcal{K}^S\phi(x) = \lambda\phi(x)$. By using a similar procedure, the equations (2.6.22) valid in multidimensional case can be easily derived. The fact that ϕ is an eigenfunction of each Koopman semigroup member $\mathcal{K}^t$ follows from the spectral mapping theorem [13, Chapter IV.3].

Remark 8. Unlike in the case of RDS generated by the RDE, the product of eigenfunctions of the stochastic Koopman generator associated with RDS generated by SDE is not necessarily an eigenfunction. This easily follows from Proposition 6. However, the eigenfunctions in many cases satisfy a recurrence relationship (e.g., Hermite polynomials, which are Koopman eigenfunctions of Ornstein-Uhlenbeck processes obtained as a solution of the Ornstein-Uhlenbeck SDE [33, Section 4.4]) and thus can be deduced from the principal eigenfunctions. This reduces the problem to analysis of principal eigenfunctions, and thus d objects in an d dimensional space, remarkable for a nominally infinite-dimensional representation.

Chapter 3

Numerical approximations of the stochastic Koopman operator

One of the goals of this work is to compute the numerical approximations of the spectral objects of the stochastic Koopman operators, i.e., their eigenvalues and eigenfunctions. This is performed by extending the DMD algorithms, that are originally developed for approximating the spectral object of the Koopman operators in deterministic settings, to the stochastic framework. DMD algorithms and the spectral analysis of the Koopman operator are connected in the following way. Suppose that the restriction of the infinite dimensional stochastic Koopman operator $\mathcal{K}^t$ to an appropriate n dimensional subspace of functions is closed under the action of the Koopman operator (see [11]). Let $\mathbb{K} \in \mathbb{C}^{n\times n}$ denote its representation by an $n \times n$ dimensional matrix. The goal of the DMD numerical algorithms is to determine the spectrum and associated eigenvectors of the finite dimensional linear operator $\mathbb{K}$, and those are in turn the eigenvalues and coordinates of eigenfunctions of the associated Koopman operator in the chosen finite-dimensional basis. If the considered subspace is not closed under the action of the Koopman operator, one can expect that under certain assumptions, the operator $\mathbb{K}$ at least approximates the underlying dynamics [2, 26].

3.1 The DMD algorithm for RDS

The DMD algorithm provides us with a decomposition of the pair $(\mathrm{X}_m, \mathrm{Y}_m)$ given by the eigenvalues and the eigenvectors of the operator $\mathbb{K}$, which is in the data-driven settings not known explicitly, while it is known that its action on the range X_m should be equal to Y_m [40]. For $n \gg m$ and full column range of X_m there exists an exact solution of the equation $\mathrm{Y}_m = \mathbb{K}\mathrm{X}_m$, while for $m \leq n$ the equation is satisfied in a least squares sense [40].
Let $\mathrm{f} = (f_1, \ldots, f_n)^T : M \to \mathbb{C}^n$ be a vector-valued observable on the state space. For an $\mathrm{x} \in M$, let $\mathrm{f}^k(\omega, \mathrm{x}) = f(T^k(\omega, \mathrm{x})); k = 0, 1, \ldots$ be a vector-valued observable series on the trajectory of the considered discrete-time RDS. Denote its expectation by

$$\mathrm{f}^k(\mathrm{x}) = \mathbb{E}[\mathrm{f}^k(\omega, \mathrm{x})] = \mathcal{K}^k\mathrm{f}(\mathrm{x}). \tag{3.1.1}$$

For the continuous-time RDS φ, we choose the time step Δt and define the values of the observables in the same way as in the discrete-time case by taking

$$T^k(\omega, \mathrm{x}) = \varphi(k\Delta t, \omega)\mathrm{x}. \tag{3.1.2}$$

The value of the observable at the time moment $t_k = k\Delta t$ is evaluated as

$$\mathrm{f}^k(\mathrm{x}) = \mathcal{K}^k_{\Delta t}\mathrm{f}(\mathrm{x}) = \mathbb{E}[\mathrm{f}(\varphi(k\Delta t, \omega)\mathrm{x})]. \tag{3.1.3}$$

In this stochastic framework, the DMD algorithm is applied to the matrices $\mathrm{X}_m, \mathrm{Y}_m \in \mathbb{C}^{n\times m}$, which are for the chosen initial states $\mathrm{x}_1, \ldots, \mathrm{x}_m \in M$ defined by

$$\mathrm{X}_m = \begin{pmatrix} \mathrm{f}^0(\mathrm{x}_1) & \mathrm{f}^0(\mathrm{x}_2) & \cdots & \mathrm{f}^0(\mathrm{x}_m) \end{pmatrix} \qquad \mathrm{Y}_m = \begin{pmatrix} \mathrm{f}^k(\mathrm{x}_1) & \mathrm{f}^k(\mathrm{x}_2) & \cdots & \mathrm{f}^k(\mathrm{x}_m) \end{pmatrix} \tag{3.1.4}$$

In this case, we expect that the DMD algorithm provides us with the eigenvalues and eigenfunctions of the finite dimensional approximation operator associated with the Koopman operator $\mathcal{K}^{t_k}$. In the case when the Koopman operator family is a semigroup, we have $\mathcal{K}^k_{\Delta t} = (\mathcal{K}^1_{\Delta t})^k$, thus it makes sense to apply the DMD algorithm to the time-delayed snapshots, which means that for the chosen initial condition x_0, we define the matrices X_m and Y_m as

$$\mathrm{X}_m = \begin{pmatrix} \mathrm{f}^0(\mathrm{x}_0) & \mathrm{f}^1(\mathrm{x}_0) & \cdots & \mathrm{f}^{m-1}(\mathrm{x}_0) \end{pmatrix} \qquad \mathrm{Y}_m = \begin{pmatrix} \mathrm{f}^1(\mathrm{x}_0) & \mathrm{f}^2(\mathrm{x}_0) & \cdots & \mathrm{f}^m(\mathrm{x}_0) \end{pmatrix} \tag{3.1.5}$$

In this case, we expect that the algorithm provides us with the eigenvalues and eigenfunctions of the approximation operator associated with $\mathcal{K}^1_{\Delta t}$. The proof of the convergence of the DMD type algorithm with input matrices (3.1.5) where $f_1, \ldots, f_n$ span the finite dimensional invariant subspace of the stochastic Koopman operator, to its eigenvalues and eigenfunctions is given in [39] for the RDS in which the noise is modeled by i.i.d. random variables and under the assumption of ergodicity. We describe now the crucial steps of the DMD RRR algorithm.

3.2 DMD-RRR:Refined Rayleigh Ritz Dynamic Mode Decomposition

The **DMD-RRR** is an improved version of the Dynamic Mode Decomposition (DMD) that aims to extract "dynamic modes" (spatial patterns evolving with specific frequencies and growth/decay rates) from dynamic system data. Unlike classical DMD, DMD-RRR incorporates **Refined Rayleigh-Ritz** techniques and numerical rank management, making it more robust to noise and high-dimensional data.

In [11] a data-driven algorithm for computing DMD, called DMD refined Rayleigh Ritz (DMD-RRR) algorithm, which enables selection of Ritz pairs based on the data-driven computation of the residual, and substantially improves the quality of the retained spectral objects. The DMD RRR algorithm starts in the same way as other SVD-based DMD algorithms, i.e., with the SVD decomposition for low dimensional approximation of data:

$$\mathrm{X}_m = U\Sigma V^* \approx U_r\Sigma_r V_r^*, \tag{3.2.1}$$

where $\Sigma = \mathrm{diag}(\sigma_i)_{i=1}^{\min(m,n)}$, σ_i are singular values arranged in descending order, i.e., $\sigma_1 \geq \ldots \geq \sigma_{\min(m,n)} \geq 0$, and r is the dimension of the approximation space. Then

$$\mathrm{Y}_m = \mathbb{K}\mathrm{X}_m \approx \mathbb{K}U_r\Sigma_r V_r^*. \tag{3.2.2}$$

Since $\mathbb{K}U_r = \mathrm{Y}_m V_r \Sigma_r^{-1}$, the Rayleigh quotient matrix $\mathrm{S}_r = U_r^* \mathbb{K} U_r$ with respect to the range U_r can be computed in this data-driven setting as

$$\mathrm{S}_r = U_r^* \mathrm{Y}_m V_r \Sigma_r^{-1}. \tag{3.2.3}$$

Each eigenpair (λ, w) of S_r generates the corresponding Ritz pair $(\lambda, U_r w)$, that is a candidate for the approximation of the eigenvalue and eigenvector of the Koopman operator. Here we emphasize a few crucial points at which the DMD RRR algorithm is improved in comparison with the standard DMD algorithms [35, 40]. The first point of difference refers to the dimension r of the reduction space. Instead of defining the dimension of the Instead of defining the dimension of the space *a priori* or taking into account the spectral gap in singular values, it is proposed in [11] to take into account user-supplied tolerance ϵ, which is then used for defining r as the largest index satisfying $\sigma_r \geq \sigma_1 \epsilon$. The algorithm is, according to [11], further enhanced with the residual computation for each Ritz vector pair $(\lambda, U_r w)$

$$\eta = \|\mathbb{K}(U_r w) - \lambda(U_r w)\|_2 = \|(Y_m V_r \Sigma_r^{-1})w - \lambda(U_r w)\|_2, \tag{3.2.4}$$

where $\|\cdot\|_2$ stands for the L^2 norm. Then the vectors at which the required accuracy is not attained are not taken into account. The final improvement compared to the standard algorithms refers to scaling of the initial data, i.e., if the matrix D_x is defined with

$$D_x = \mathrm{diag}(\|\mathrm{X}_m(:, i)\|_2)_{i=0}^{m-1},$$

we set $\mathrm{X}_m = \mathrm{X}_m^{(1)} D_x$ and $\mathrm{Y}_m = \mathrm{Y}_m^{(1)} D_x$ and proceed with $\mathrm{X}_m^{(1)}$ and $\mathrm{Y}_m^{(1)}$ as data matrices.

Algorithm 1 DMD-RRR for Random Dynamical Systems

Require:
- Data matrices $\mathbf{X}_m = (\mathrm{f}^0(\mathrm{x}_0), \mathrm{f}^1(\mathrm{x}_0), \cdots, \mathrm{f}^{m-1}(\mathrm{x}_0))$ *and* $\mathbf{Y}_m = (\mathrm{f}^1(\mathrm{x}_0), \mathrm{f}^2(\mathrm{x}_0), \cdots, \mathrm{f}^m(\mathrm{x}_0))$ from a single trajectory starting at x_0
- Tolerance ε for numerical rank determination

Normalize the data: $D_x = \mathrm{diag}(\|\mathbf{X}_m(:,i)\|_2)_{i=1}^m$
$\mathbf{X}_m^{(1)} = \mathbf{X}_m D_x^+$ and $\mathbf{Y}_m^{(1)} = \mathbf{Y}_m D_x^+$
Compute the thin SVD: $\mathbf{X}_m^{(1)} = U\Sigma V^*$, where $\Sigma = \mathrm{diag}(\sigma_i)_{i=1}^m$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{min} \geq 0$
Determine the rank r as the largest index i satisfying $\sigma_i \geq \sigma_1 \varepsilon$
Extract the first r columns: $U_r = U(:, 1:r)$, $V_r = V(:, 1:r)$, $\Sigma_r = \Sigma(1:r, 1:r)$
Compute the Rayleigh quotient: $S_r = U_r^* \mathbf{Y}_m V_r \Sigma_r^{-1}$
Compute eigenvalues and eigenvectors of S_r: $S_r W = W\Lambda$, where $\Lambda = \mathrm{diag}(\lambda_i)_{i=1}^r$
Compute the dynamic modes: $\Phi = \mathbf{Y}_m V_r \Sigma_r^{-1} W$
For $i = 1, \ldots, r$ **do**
    Compute the residual for each Ritz pair $(\lambda_i, U_r w_i)$:
    $\eta_i = \|\mathbb{K}(U_r w_i) - \lambda_i (U_r w_i)\|_2 = \|(Y_m V_r \Sigma_r^{-1}) w_i - \lambda_i (U_r w_i)\|_2$
End For

Ensure: Φ (dynamic modes), Λ (eigenvalues), residuals $\{\eta_i\}_{i=1}^r$

3.3 Stochastic Hankel DMD (sHankel-DMD) algorithm

This section defines the stochastic Hankel matrix and describes the application of a DMD-type algorithm to this structure. The approach is based on the Hankel DMD method introduced in [2] for deterministic dynamical systems, with the underlying principles extended to the stochastic setting. The analysis is restricted to discrete-time random dynamical systems (RDS), under the assumption that the associated family of stochastic Koopman operators satisfies the semigroup property. This assumption is crucial, as it ensures the validity of the results produced by applying the DMD algorithm to the Hankel matrix. In the absence of this property, a situation similar to that of non-autonomous deterministic systems arises: applying the DMD algorithm to time-delayed snapshots leads to the approximation of different operators, which results in significant errors [29]. The Hankel matrix in the stochastic framework is defined as follows. For a scalar observable $f : M \to \mathbb{C}$, we define the vector of n observations along the trajectory that starts at $\mathrm{x} \in M$ and is generated by the one-step discrete random map T:

$$\mathrm{f}^n(\omega, \mathrm{x}) = (f(\mathrm{x}), f(T(\omega, \mathrm{x})) \ldots f(T^{n-1}(\omega, \mathrm{x})))^T. \tag{3.3.1}$$

Let $\mathrm{f}_n^k(\mathrm{x})$, $k = 0, 1, \ldots$ denote the expectation of $\mathrm{f}^n(\theta(k)\omega, T^k(\omega, \mathrm{x}))$ over the trajectories of length k, i.e.,

$$\begin{aligned} \mathrm{f}_n^k(\mathrm{x}) &= \mathbb{E}[\mathrm{f}^n(\theta(k)\omega, T^k(\omega, \mathrm{x}))] \\ &= (\mathcal{K}^k f(\mathrm{x}), \mathcal{K}^k f(T(\omega, \mathrm{x})), \ldots, \mathcal{K}^k f(T^{n-1}(\omega, \mathrm{x})))^T. \end{aligned}$$

Observe that the components of $\mathrm{f}_n^k(\mathrm{x})$ are the values of the function $\mathcal{K}^k f$ along the trajectory of length n starting at $\mathrm{x} \in M$. The stochastic Hankel matrix of dimension $n \times m$, associated with the trajectories starting at $\mathrm{x} \in M$ and generated by the map T is defined by

$$\begin{aligned}\mathrm{H}_{n\times m}(\omega, \mathrm{x}) &= (\mathrm{f}_n^0(\mathrm{x}) \quad \mathrm{f}_n^1(\mathrm{x}) \dots \mathrm{f}_n^{m-1}(\mathrm{x})) \\ &= \begin{pmatrix} f(\mathrm{x}) & \mathcal{K}^1 f(\mathrm{x}) & \cdots & \mathcal{K}^{m-1} f(\mathrm{x}) \\ f(T(\omega, \mathrm{x})) & \mathcal{K}^1 f(T(\omega, \mathrm{x})) & \cdots & \mathcal{K}^{m-1} f(T(\omega, \mathrm{x})) \\ \vdots & \vdots & \ddots & \vdots \\ f(T^{n-1}(\omega, \mathrm{x})) & \mathcal{K}^1 f(T^{n-1}(\omega, \mathrm{x})) & \cdots & \mathcal{K}^{m-1} f(T^{n-1}(\omega, \mathrm{x})) \end{pmatrix}.\end{aligned}$$

The columns of $\mathrm{H}_{n\times m}$ are approximations of functions in the Krylov subspace

$$\mathbb{K}_m(\mathcal{K}, f) = \mathrm{span}(f, \mathcal{K}^1 f, \dots, \mathcal{K}^{m-1} f), \tag{3.3.2}$$

obtained by sampling the values of functions $\mathcal{K}^k f$, $k = 0, \dots, m-1$ along the trajectory of length n starting at $\mathrm{x} \in M$. When the DMD algorithm is applied to the stochastic Hankel matrix, the input data matrix X_m is defined by taking the first m columns of the Hankel matrix $\mathrm{H}_{n\times(m+1)}(\omega, \mathrm{x})$, while the data matrix Y_m is formed from the last m columns of the same matrix. We refer to this methodology as the stochastic Hankel DMD (sHankel-DMD) algorithm. As already mentioned, when we consider the continuous-time RDS, we could for the chosen Δt associate to it a discrete RDS by defining the one-step map as $T(\omega, \mathrm{x}) = \varphi(\Delta t, \omega)\mathrm{x}$. Then the Koopman operator $\mathcal{K}^k$ should be replaced with the operator $\mathcal{K}_{\Delta t}^k$.

Algorithm 2 Stochastic Hankel DMD

Require:
- Scalar observable $f : M \to \mathbb{C}$
- One-step discrete random map $T : M \to M$
- Initial state $x \in M$
- Number of observations n along trajectory
- Number of columns m for the Hankel matrix
- Tolerance ε for numerical rank determination

Form the vector of n observations along the trajectory starting at x:
$\mathbf{f}_n(\omega, x) = (f(x), f(T(\omega, x)), \ldots, f(T^{n-1}(\omega, x)))^T$
Compute the components of the expected trajectory:
$\mathbf{f}_n^k(x) = \mathbb{E}[\mathbf{f}_n(\theta(k)\omega, T^k(\omega, x))]$
Construct the stochastic Hankel matrix $\mathbf{H}_{n\times m}(\omega, x)$ of dimension $n \times m$:
$\mathbf{H}_{n\times m}(\omega, x) = (\mathrm{f}_n^1(\mathrm{x}) \ldots \mathrm{f}_n^{m-1}(\mathrm{x}))$
Define the data matrices $\mathbf{X}_m$ and $\mathbf{Y}_m$:
$\mathbf{X}_m$ = first m columns of $\mathbf{H}_{n\times(m+1)}(\omega, x)$
$\mathbf{Y}_m$ = last m columns of $\mathbf{H}_{n\times(m+1)}(\omega, x)$
Normalize the data: $D_x = \mathrm{diag}(\|\mathbf{X}_m(:, i)\|_2)_{i=1}^m$
$\mathbf{X}_m^{(1)} = \mathbf{X}_m D_x^+$ and $\mathbf{Y}_m^{(1)} = \mathbf{Y}_m D_x^+$
Compute the thin SVD: $\mathbf{X}_m^{(1)} = U\Sigma V^*$, where $\Sigma = \mathrm{diag}(\sigma_i)_{i=1}^m$
Determine the rank r as the largest index i satisfying $\sigma_i \geq \sigma_1\varepsilon$
Extract the first r columns: $U_r = U(:, 1:r)$, $V_r = V(:, 1:r)$, $\Sigma_r = \Sigma(1:r, 1:r)$
Compute the Rayleigh quotient: $S_r = U_r^* \mathbf{Y}_m V_r \Sigma_r^{-1}$
Compute eigenvalues and eigenvectors of S_r: $S_r W = W\Lambda$, where $\Lambda = \mathrm{diag}(\lambda_i)_{i=1}^r$
Compute the dynamic modes: $\Phi = \mathbf{Y}_m V_r \Sigma_r^{-1} W$
For $i = 1, \ldots, r$ **do**
  Compute the residual for each Ritz pair $(\lambda_i, U_r w_i)$:
  $\eta_i = \|\mathbb{K}(U_r w_i) - \lambda_i(U_r w_i)\|_2 = \|\mathbf{Y}_m V_r \Sigma_r^{-1} w_i - \lambda_i(U_r w_i)\|_2$
End For

Ensure: Φ (dynamic modes), Λ (eigenvalues), residuals $\{\eta_i\}_{i=1}^r$

3.3.1 Convergence of the sHankel-DMD algorithm

It was proved in [2] that for ergodic systems, under the assumption that observables are in an invariant subspace of the Koopman operator, the eigenvalues and eigenfunctions obtained by the extended DMD algorithm applied to Hankel matrix, converge to the true Koopman eigenvalues and eigenfunctions of the considered system. The convergence of the DMD algorithm with input matrices X_m and Y_m defined by (3.1.5) to the eigenvalues and the eigenfunctions of the Koopman operator is proved in [39] for the class of RDS in which the noise is modeled by i.i.d. random variables, under the assumption of ergodicity, and of the existence of the finite dimensional invariant subspace. Here we prove that under the same assumptions, the convergence is accomplished for the sHankel-DMD algorithm. Our proof is based on the fact that the eigenvalues and the eigenfunctions obtained by DMD algorithm correspond to the matrix that is similar to the companion matrix, which represents the finite dimensional approximation of the Koopman operator in the Krylov basis. We limit the considerations to the discrete-time RDS. Suppose that the dynamics on the compact invariant set $A \subseteq M$ is generated by the measure-preserving map $T(\omega, \cdot) : A \to A$ for each $\omega \in \Omega$. We recall from [4] that a probability measure ν is invariant for RDS $\phi(n, \omega) = T^n(\omega, \cdot)$ if it is invariant for the skew product flow

$$\Theta(n)(\omega, x) = (\theta(n)\omega, T^n(\omega, x))$$

generated by T and $\theta(t)$, i.e., if $\Theta(n)\nu = \nu$ and if $\pi_\Omega \nu = \mathbb{P}$ where π_Ω denotes the canonical projection $\Omega \times A \to \Omega$. Invariant measures always exist for a continuous RDS on compact space A (see [4, Theorem 1.5.10]). If A is a Polish space, the measure ν on $\Omega \times A$ could be written as a product of measures, i.e., $d\nu(\omega, x) = d\mu_\omega(x)d\mathbb{P}(\omega)$, i.e., for $f \in L^1(\nu)$

$$\int_{\Omega \times A} f d\nu = \int_\Omega \int_A f(\omega, x) d\mu_\omega(x) d\mathbb{P}(\omega). \tag{3.3.3}$$

If the skew product dynamical system Θ is ergodic on $\Omega \times A$ with respect to some invariant measure ν and if $\theta(n)$ is ergodic with respect to $\mathbb{P}$, we say that φ is ergodic with respect to the invariant measure ν. Under the assumption of ergodicity of Θ with respect to the measure ν, the Birkhoff's ergodic theorem states that the time average of observable $f \in L^2(\Omega \times A, \nu)$ under Θ is given by

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\theta(k)\omega, T^k(\omega, \mathrm{x})) = \int_{\Omega \times A} f(\omega, x) d\nu, \quad \text{a.e. on } \Omega \times A. \tag{3.3.4}$$

The following proposition shows that under the assumption of ergodicity and Markovian property of RDS, the sHankel-DMD algorithm provide us with approximations of eigenvalues and eigenfunctions that converge to the true Koopman eigenvalues and eigenfunctions.

Proposition 8. Suppose that the dynamics on the compact invariant set $A \subseteq M$ is given by the one step map $T(\omega, \cdot) : A \to A$ for each $\omega \in \Omega$ and that the associated discrete time RDS φ is ergodic with respect to some invariant measure ν. Assume additionally that the processes $\{\varphi(n, \omega)\mathrm{x}; \mathrm{x} \in A\}$

form a Markov family. Denote by μ the marginal measure $\mu = \mathbb{E}_{\mathbb{P}}(\nu)$ on A. Let the Krylov subspace $\mathbb{K}_m(\mathcal{K}, f)$ span an r-dimensional subspace of the Hilbert space $\mathcal{H} = L^2(A, \mu)$, with $r < m$, invariant under the action of the stochastic Koopman operator. Then for almost every $\mathrm{x} \in A$, as $n \to \infty$, the eigenvalues and eigenfunctions obtained by applying the DMD algorithm to the first $r + 1$ columns of the $n \times (m + 1)$-dimensional stochastic Hankel matrix converge to the true eigenvalues and eigenfunctions of the stochastic Koopman operator.

Proof. Consider the observables $f : A \to \mathbb{R}$ belonging to the Hilbert space $\mathcal{H} = L^2(A, \mu)$. Due to the ergodicity of Θ, in accordance with Birkhoff's ergodic theorem, (3.3.4) is valid, we get:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(\omega, \mathrm{x})) = \int_{\Omega\times A} f(x) d\nu = \int_{\Omega} \int_{A} f(x) d\mu_\omega(x) d\mathbb{P}(\omega) = \int_A f d\mu. \tag{3.3.5}$$

where the last equality follows from the fact that $\mu = \mathbb{E}_{\mathbb{P}}(\nu) = \mathbb{E}_{\mathbb{P}}(\mu_\omega)$ For observables $f, g \in \mathcal{H}$, let the vectors of n observations along the trajectory starting at $\mathrm{x} \in A$ of the RDS generated by the map T be denoted by $\mathrm{f}_n(\omega, \mathrm{x})$ and $g_n(\omega, \mathrm{x})$ and defined by (3.3.1). If we denote the data-driven inner product by $\langle \mathrm{f}_n(\omega, \mathrm{x}), g_n(\omega, \mathrm{x}) \rangle$, we have:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} [\langle \mathrm{f}_n(\omega, \mathrm{x}), g_n(\omega, \mathrm{x}) \rangle] &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(\omega, \mathrm{x})) g^*(T^k(\omega, \mathrm{x})) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f g^*(T^k(\omega, \mathrm{x})) \\
&= \int_A f g^* d\mu \\
&= \langle f, g \rangle_{\mathcal{H}} \quad \text{for a.e. } \mathrm{x} \qquad (**)
\end{aligned}$$

with respect to the measure μ. Using the assumption that $\mathbb{K}_m(\mathcal{K}, f)$ spans an r-dimensional subspace of $\mathcal{H}$, which is invariant for the stochastic Koopman operator, the restriction of the Koopman operator to this subspace is finite-dimensional and can be realized by an $r \times r$ matrix. The representation of this matrix in the basis formed by the functions $(f, \mathcal{K}f, \ldots, \mathcal{K}^{m-1}f)$ is given with the companion matrix:

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 1 & 0 & \cdots & 0 & c_1 \\ 0 & 1 & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{r-1} \end{pmatrix} \tag{3.3.6}$$

where the vector $c = (c_0, c_1, \ldots, c_{r-1})^T$, obtained by using least squares approximation, is equal to

$$c = G^{-1}\left(\langle f, \mathcal{K}^r f\rangle_{\mathcal{H}}, \langle \mathcal{K}^1 f, \mathcal{K}^r f\rangle_{\mathcal{H}}, \ldots, \langle \mathcal{K}^{r-1} f, \mathcal{K}^r f\rangle_{\mathcal{H}}\right)^T. \tag{3.3.7}$$

Here G denotes the Gramian matrix with elements

$$G_{ij} = \langle \mathcal{K}^{i-1} f, \mathcal{K}^{j-1} f\rangle_{\mathcal{H}}, \quad i, j = 1, \ldots, r \tag{3.3.8}$$

Consider now the stochastic Hankel matrix $H_{n\times(r+1)}(\omega, \mathrm{x})$ of dimension $n\times(r+1)$ along a trajectory starting at x and the companion matrix algorithm [2, 11] applied to the matrices

$$\mathrm{X}_r = (\mathrm{f}_n^0(\mathrm{x})\ \ \mathrm{f}_n^1(\mathrm{x})\ \ \ldots\ \ \mathrm{f}_n^{r-1}(\mathrm{x})) \quad \text{and} \quad \mathrm{Y}_r = (\mathrm{f}_n^1(\mathrm{x})\ \ \mathrm{f}_n^2(\mathrm{x})\ \ \ldots\ \ \mathrm{f}_n^r(\mathrm{x})).$$

We denote by $\tilde{C}$ the numerical companion matrix computed as a best approximation of the least squares problem

$$\tilde{C} = \arg\min_{B\in\mathbb{C}^{r\times r}} \|\mathrm{Y}_r - \mathrm{X}_r B\|.$$

Using the assumption that the matrix X_r has a full column rank, the pseudoinverse is of the form $\mathrm{X}_r^+ = (\mathrm{X}_r^*\mathrm{X}_r)^{-1}\mathrm{X}_r^*$ and the matrix $\tilde{C}$ is

$$\begin{aligned}
\tilde{C} &= \mathrm{X}_r^+ \mathrm{Y}_r \\
&= (\mathrm{X}_r^*\mathrm{X}_r)^{-1}\mathrm{X}_r^*\mathrm{Y}_r \\
&= \left(\frac{1}{n}\mathrm{X}_r^*\mathrm{X}_r\right)^{-1}\left(\frac{1}{n}\mathrm{X}_r^*\mathrm{Y}_r\right) \\
&= \tilde{G}^{-1}\left(\frac{1}{n}\mathrm{Y}_r\mathrm{X}_r^*\right). \qquad (***)
\end{aligned}$$

Here $\tilde{G}$ denotes the numerical Gramian matrix whose elements are equal to

$$\begin{aligned}
\tilde{G}_{ij}(\omega, \mathrm{x}) &= \frac{1}{n}\langle \mathrm{f}_n^{i-1}(\mathrm{x}), \mathrm{f}_n^{j-1}(\mathrm{x})\rangle \\
&= \frac{1}{n}\sum_{k=0}^{n-1} (\mathcal{K}^{i-1} f)(T^k(\omega, \mathrm{x}))(\mathcal{K}^{j-1} f^*)(T^k(\omega, \mathrm{x})) \\
&= \frac{1}{n}\sum_{k=0}^{n-1} (\mathcal{K}^{i-1} f)(\mathcal{K}^{j-1} f^*)(T^k(\omega, \mathrm{x})), \quad i, j = 1, \ldots, r.
\end{aligned}$$

Now, by using (**), we conclude that

$$\lim_{n\to\infty} \tilde{G}_{ij}(\omega, \mathrm{x}) = \langle \mathcal{K}^{i-1}, \mathcal{K}^{j-1} f\rangle_{\mathcal{H}}, \quad i, j = 1, \ldots, r, \text{for a.e. x}, \tag{3.3.9}$$

with respect to the measure μ. From (***) we get that the last column

$$\tilde{c} = (\tilde{c}_0(\omega, \mathrm{x}), \tilde{c}_1(\omega, \mathrm{x}), \ldots, \tilde{c}_{r-1}(\omega, \mathrm{x}))^T \tag{3.3.10}$$

of $\tilde{C}$ is equal to

$$\tilde{c} = \tilde{G}^{-1} \frac{1}{n} \left(\langle \mathrm{f}_n^0(\mathrm{x}), \mathrm{f}_n^r(\mathrm{x}) \rangle, \langle \mathrm{f}_n^1(\mathrm{x}), \mathrm{f}_n^r(\mathrm{x}) \rangle, \ldots, \langle \mathrm{f}_n^{r-1}(\mathrm{x}), \mathrm{f}_n^r(\mathrm{x}) \rangle \right)^T . \tag{3.3.11}$$

Since

$$\lim_{n\to\infty} \langle \mathrm{f}_n^{j-1}(\mathrm{x}), \mathrm{f}_n^r(\mathrm{x}) \rangle = \langle \mathcal{K}^{j-1} f, \mathcal{K}^r f \rangle_{\mathcal{H}}, \quad j = 1, \ldots, r, \text{ for a.e. x}, \tag{3.3.12}$$

with respect to the measure μ, and the matrix $\tilde{G}^{-1}$ converges to the inverse of the true Gramian matrix G^{-1}, it follows that the components of $\tilde{c}$ converge to the components of $\tilde{c}$ (3.3.7). As in [2, Proposition 3.1], we use the fact that the eigenvalues of the matrix $\tilde{C}$ are the roots of a polynomial having coefficients $\tilde{c}_i$. Therefore, the eigenvalues of $\tilde{C}$ are continuously dependent on these coefficients, which implies that the eigenvalues of the companion matrix $\tilde{C}$ converge to the eigenvalues of the exact companion matrix, i.e., to the exact eigenvalues of the stochastic Koopman operator. The eigenvalues and eigenvectors provided by the DMD algorithm are computed from the eigenvalues and eigenvectors of the matrix S_r, which is similar to the companion matrix $\tilde{C}$ (see [11, Proposition 3.1]), thus the conclusion follows.

3.4 Numerical examples

3.4.1 Discrete RDS

Noisy rotation on the circle

To analyze the noisy rotation on the circle using the DMD algorithm, focusing on the computation of the Koopman operator's spectral properties.
The dynamical system under consideration is described as in Example 2, where the evolution of the state variable is subject to both deterministic rotation and additive noise. To analyze this system, we define a set of real-valued observable functions, which serve as the input to the DMD algorithm:

$$\begin{aligned} f_j(x) &= \cos(j2\pi x), \\ g_j(x) &= \sin(j2\pi x), \quad \text{where } j = 1, \ldots, n-1 \end{aligned}$$

These observables are arranged into a vector-valued observable:

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_{n-1} \\ g_1 \\ \vdots \\ g_{n-1} \end{pmatrix}$$

This vector captures the essential harmonics of the system and provides a suitable subspace for spectral approximation. The input data matrices X_m and Y_m,used for the application of the DMD-RRR (Refined Rayleigh–Ritz) algorithm, are constructed as in Equation((3.1.5)), using snapshots of the observable vector evaluated along a trajectory of the system.
Analysis:
The analysis considers both deterministic and stochastic cases of the system.

- **Deterministic Case:**
  - According to Tu et al. [40], for $m \gg n$, the DMD algorithm approximates the associated Koopman operator in a least-squares sense.
  - Due to ergodicity and the law of large numbers, as $m \to \infty$, the DMD algorithm's results converge to the eigenvalues and eigenvectors of the finite-dimensional approximating matrix of the Koopman operator.
- **Stochastic Case:**
  - According to Takeishi et al. [39], the DMD algorithm also converges to the eigenvalues and eigenfunctions of the stochastic Koopman operator.

- **Data Acquisition:**
  - To recover the spectrum of the stochastic Koopman operator in the ergodic system, the matrices X_m and Y_m are constructed using values of the observable vector from a single, sufficiently long trajectory:

$$X_m = \Big(f(x_0), f(T(\omega, x_0)), \ldots, f(T^{m-1}(\omega, x_0))\Big)$$
$$Y_m = \Big(f(T(\omega, x_0)), f(T^2(\omega, x_0)), \ldots, f(T^m(\omega, x_0))\Big)$$

  - where x_0 is the initial state.
- **Numerical Experiment Setup:**
  - The number of observable functions is set to $n_1 = 150$, resulting in $n = 300$ observable functions.
  - The number of sequential snapshots used to determine the matrices X_m and Y_m is $m = 5000$.
  - Parameters for the flow are set to $\vartheta = \pi/320$ and $\delta = 0.01$.
  - The DMD RRR algorithm is applied to obtain numerical results.
- **Results:**
  - In the deterministic case, the eigenvalues lie on the unit circle, as expected.
  - The real parts of the numerically recovered eigenfunctions $\phi_i(x)$, $i = 1, 2, 3$, closely match the theoretically established eigenfunctions given by equation (2.6.12).
  - The numerically captured eigenvalues of the stochastic Koopman operator closely coincide with the theoretical eigenvalues given by equation (2.6.13).
  - In the stochastic case, the real parts of the eigenfunctions corresponding to the first three eigenvalues λ_i^S, $i = 1, 2, 3$, are presented; these eigenfunctions are given by equation (2.6.12).
- **Remarks:**
  - The deterministic and stochastic Koopman operators commute in this case, so the eigenfunctions are the same, and the addition of noise does not perturb them (consistent with Giannakis et al. [19]).
  - The computed eigenvalues approximate the eigenvalues of the stochastic Koopman operator restricted to the subspace spanned by the components of the vector observable.
  - Since this subspace is invariant under the Koopman operator, the eigenvalues are computed with high accuracy.
- **Alternative Observables:**

– Similar results were obtained using observable functions:

$$f_j(x) = e^{j2\pi ix} \quad \text{and} \quad g_j(x) = e^{-j2\pi ix}, j = 1, \ldots, n_1.$$

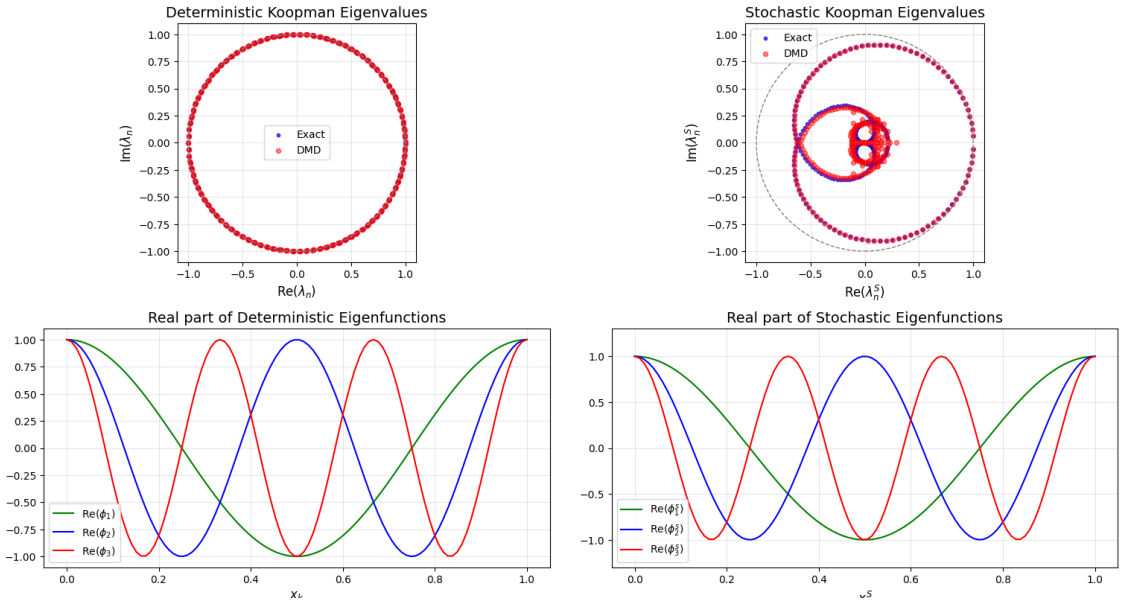

Figure 3.1: Rotation on circle

Discrete linear RDS

The discrete-time linear Random Dynamical System (RDS) under consideration is defined by a mapping as introduced in Equation (2.3.1). The evolution of the system is governed by a random matrix $A(\omega)$, explicitly given by

$$A(\omega) = \begin{pmatrix} 0 & \pi(\omega) \\ -\pi(\omega) & 0 \end{pmatrix}, \tag{3.4.1}$$

where the function π is defined in Equation (2.6.5). The random parameter $\omega \in \Omega$ evolves according to the shift transformations $\theta = (\theta(t))_{t\in\mathbb{Z}^+}$, as specified in Equation (2.6.4). Each coordinate ω_i of ω is generated by an independent and identically distributed (i.i.d.) random variable, satisfying the probability distribution

$$P(\omega_i = 1) = p_1 \quad \text{and} \quad P(\omega_i = 2) = 1 - p_1, \quad 0 < p_1 < 1.$$

Theoretical Background:

- As proved in Proposition 1, the principal eigenvalues of the stochastic Koopman operator are equal to the eigenvalues of the matrix $\hat{A} = \mathbb{E}[A(\omega)]$.
- The associated eigenfunctions are given by equation (2.3.2).

Numerical Experiment Setup:

- Set $p_1 = 0.75$.
- Select $N = 10^4$ initial points uniformly distributed over $[0, 1] \times [0, 1]$.
- For every chosen initial point $x_{j,0}$, $j = 1, \ldots, N$, determine the random trajectory $x_{j,k}$, $k = 1, 2, \ldots$, where $x_{j,k}$ denotes the state value at the k-th step.

DMD Application:

- Apply the DMD RRR algorithm to the full-state observables by taking states on each trajectory separately.
- Define the input matrices for the DMD RRR algorithm as:

$$\begin{aligned} X_{m,j} &= (x_{j,1}, x_{j,2}, \ldots, x_{j,m-1}) \\ Y_{m,j} &= (x_{j,2}, x_{j,3}, \ldots, x_{j,m}), \quad j = 1, \ldots, N \end{aligned}$$

Eigenvalue Approximation and Error Estimation:

- In each computation associated with the initial value $x_{j,0}$, obtain the eigenvalue pair $\lambda_{1,2}^{(j)}$, which approximates the principal Koopman eigenvalues $\hat{\lambda}_{1,2}$.
- For fixed m and the chosen set of initial conditions, obtain samples of approximating eigenvalues $\lambda_{1,2}^{(j)}$, $j = 1, \ldots, N$.
- To estimate the error, evaluate the L_1, L_2, and L_∞ norms of the difference between the exact eigenvalues and the computed eigenvalues for the obtained sets of approximating eigenvalues.
- Present the norms determined for different values of parameter m (as shown in Figure 3.2).

Observations and Conclusions:

- The accuracy of the obtained eigenvalues increases monotonically with the number of snapshots m used in the computations.
- The error is $O(\frac{1}{\sqrt{m}})$, as expected for a random process of this type.
- The norm of the state values $\|x_{j,m}\|$ increases with m and can become quite large for large values of m.
- The condition number of the input matrices of the DMD RRR algorithm can be very large.
- The scaling applied in the enhanced DMD RRR algorithm prevents instabilities that could arise if the standard DMD algorithm without scaling is applied.

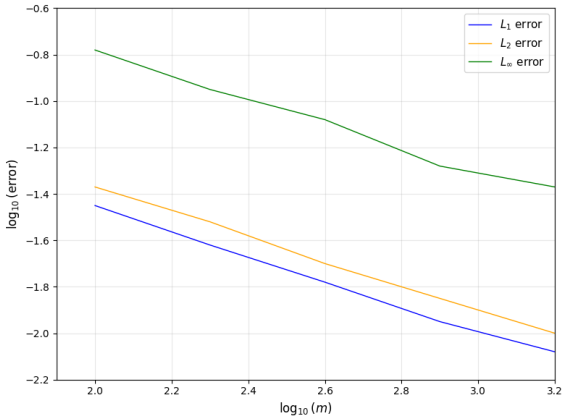

Figure 3.2: Discrete linear RDS defined by (3.4.1). L_1, L_2, and L_∞ errors of approximated Koopman eigenvalues.

3.4.2 Stochastic differential equations examples

Linear scalar SDE

The system under consideration is governed by a linear scalar stochastic differential equation (SDE) of the form:

$$dX = \mu X\, dt + \sigma\, dW_t, \tag{3.4.2}$$

where $\mu < 0$ and $\sigma > 0$. This equation defines a one-dimensional Ornstein–Uhlenbeck process, as described in [33].

In the deterministic case ($\sigma = 0$), the Koopman eigenvalues are given by

$$\lambda_n = n\mu, \quad n = 0, 1, 2, \ldots$$

and the corresponding eigenfunctions are the monomials

$$\phi_n(x) = x^n.$$

In the stochastic case, the eigenvalues of the stochastic Koopman generator remain the same as in the deterministic case. However, the eigenfunctions take the form

$$\phi_n(x) = a_n H_n(\alpha x), \quad \alpha = \sqrt{\frac{|\mu|}{\sigma^2}}, \quad n = 0, 1, 2, \ldots$$

where H_n denotes the Hermite polynomials and a_n are normalization constants, as detailed in [33, 3].

DMD Application in Deterministic Case:

- Observable functions: $f_j(x) = x^j$, $j = 1, \ldots, n$.
- Satisfactory results for the first n eigenvalues were obtained for moderate values of n (e.g., $n = 10$).
- Significant numerical errors arise for larger values of n due to the ill-conditioned system caused by the evolution of higher-order monomials ($e^{n\mu t}x^n \to 0$).
- With $n = 10$ and $m = 2000$ snapshots, the ten leading eigenvalues are determined with accuracy greater than 0.01 (see Figure 3.3(b)).
- The standard DMD algorithm (without scaling) was more sensitive, capturing at most three eigenvalues with accuracy greater than 0.01 on the same data.

DMD Application in Stochastic Case (Approach 1):

- Observable functions: $f_j(x) = x^j$, $j = 1, \ldots, n$.
- $m = 100$ initial points from the interval $[-1, 1]$ are chosen.
- n-dimensional observable vector is used.
- Input matrices $X_m, Y_m \in \mathbb{R}^{m \times n}$ are defined by equation (3.1.4), with Y_m determined for $k = 100$ and using $N = 1000$ trajectories for each initial condition.
- Eigenvalues computed using this approach lead to very accurate results for the Koopman operator (see Figure 3.3(d) of the source document), with accuracy similar to the deterministic case.
- Accuracy depends on the number of computations N, as well as the number and distribution of initial points.

DMD Application in Stochastic Case (Approach 2):

- Observable functions: $f_j(x) = x^j$, $j = 1, \ldots, n$.
- For one chosen initial point, $N = 1000$ trajectories are generated.
- These trajectories are used to determine approximations of the expected values of the observable functions.
- These expected values are used in equation (3.1.5) to form input matrices X_m and Y_m for $m = 2000$.
- The number of eigenvalues captured with satisfactory accuracy decreases (only four with accuracy greater than 10^{-2} in the presented case - see Figure 3.3(e)).
- The eigenfunctions associated with these four eigenvalues are captured with satisfactory accuracy (Figure 3.3(f)).
- The lower accuracy could be due to errors introduced when approximating expected values by averaging along trajectories.
- Increasing the number of computed trajectories should improve accuracy due to the law of large numbers.

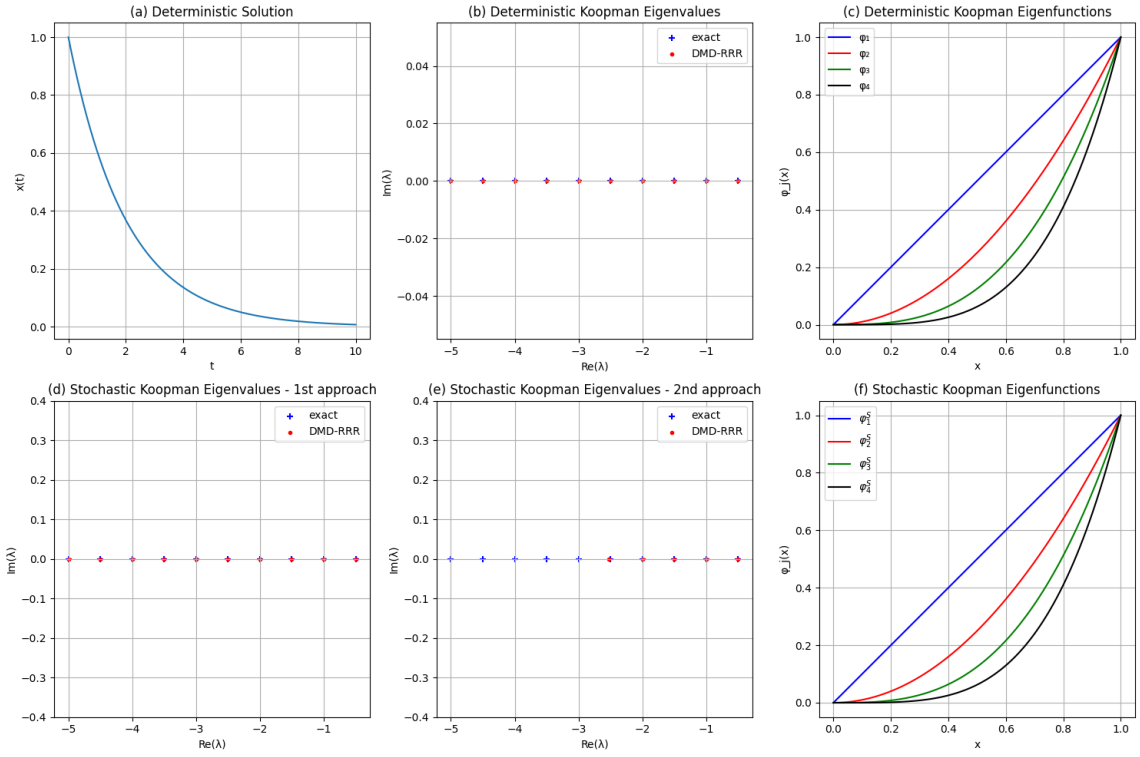

Figure 3.3: Linear scalar equation. Deterministic and Stochastic cases with μ = -0.5, σ = 0.001

Nonlinear scalar SDE

The nonlinear scalar stochastic differential equation (SDE) under consideration is:

$$dX = (\mu X - X^3)\,dt + \sigma\,dW_t. \tag{3.4.3}$$

The qualitative behavior of the solutions depends on the sign of the parameter μ, as discussed in [17]. In this analysis, we focus on the case where $\mu < 0$ is fixed.

Theoretical Koopman Spectrum (Deterministic Case, $\sigma = 0$):

- The eigenvalues of the adjoint of the Liouville operator (which is the Koopman operator for the deterministic equation) are:

  $$\lambda_n = n\mu, \quad n = 0, 1, 2, \ldots,$$

  (from [17]).

- The associated eigenfunctions are:

  $$\phi_n(x) = \left(\frac{x}{\sqrt{x^2 + |\mu|}} \right)^n,$$

  (from [17]).

Theoretical Koopman Spectrum (Stochastic Case, $\sigma > 0$):

- Eigenvalues and eigenfunctions of the generator of the stochastic Koopman family can be evaluated by solving the associated backward Kolmogorov equation, which can be transformed into the Schrödinger equation (see [17]).
- For small values of σ, the eigenvalues are very similar to the deterministic case.
- Eigenvalues are approximated numerically by solving the Schrödinger equation using the finite difference method.

DMD Application in Deterministic Case ($\sigma = 0$):

- The choice of observable functions is crucial for the accuracy of the results.
- **Using Analytical Eigenfunctions or Linear Combinations as Observables:**
  - The subspace spanned by these functions is closed under the action of the Koopman operator.

  - The restriction of the Koopman operator on this subspace can be exactly described by a finite-dimensional matrix (the key matrix for DMD RRR).
  - As a consequence, eigenvalues and eigenfunctions are computed with high accuracy (see Figure 3.4).

- **Using Monomials ($f_j(x) = x^j$) as Observables:**
  - Results were less accurate.
  - Only a few leading eigenvalues and eigenfunctions were computed with satisfactory accuracy.

DMD Application in Stochastic Case ($\sigma > 0$):

- The DMD RRR algorithm is applied using the same set of observable functions as in the deterministic case.
- **Approach 1 (Multiple Initial Conditions):**
  - $m = 100$ initial conditions are chosen from an unspecified interval.
  - Expected values of the observable functions are determined over $N = 1000$ trajectories and after $k = 100$ time steps.
  - Results in the same number of accurately computed eigenvalues as in the deterministic case (see Figure 3.4(d)).

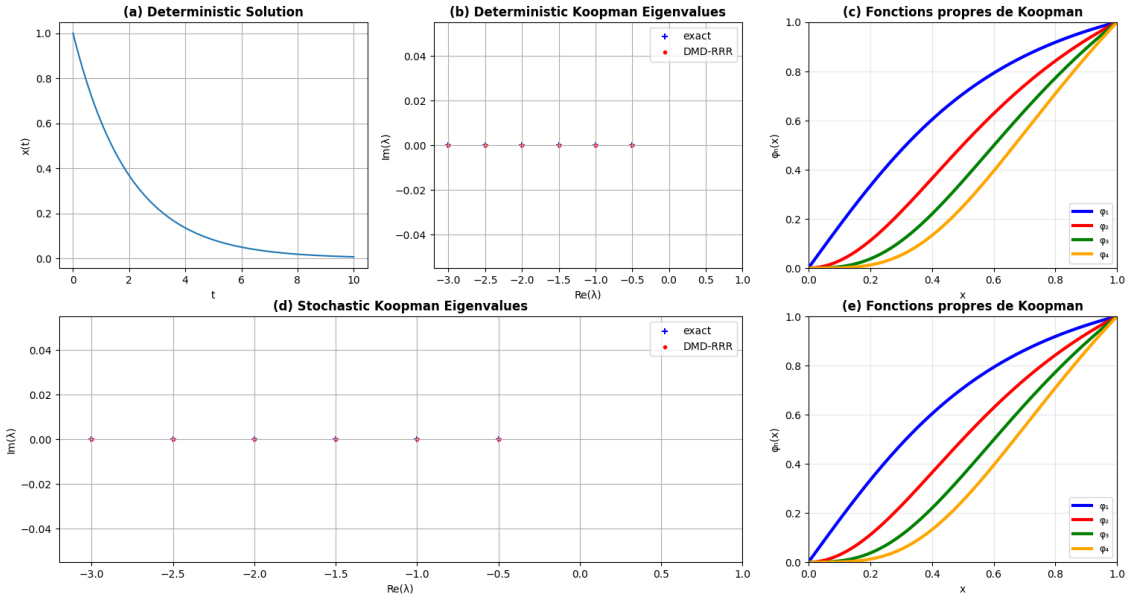

Figure 3.4: Nonlinear scalar equation. Deterministic and Stochastic cases with μ = -0.5, σ = 0.001

Noisy van der Pol Oscillator

The system is described by the following two-dimensional stochastic differential equation (SDE):

$$\begin{aligned} dX_1 &= X_2\, dt, \\ dX_2 &= \left(\mu(1 - X_1^2)X_2 - X_1\right) dt + \sqrt{2\varepsilon}\, dW_t, \end{aligned} \tag{3.3.3}$$

where W_t denotes a one-dimensional Wiener process.

The existence of a solution to this SDE system has been established in [4]. Furthermore, the convergence of numerical approximations computed via the Euler–Maruyama method has been demonstrated in [21].

Deterministic Case ($\varepsilon = 0$):

- The dynamics converge to an asymptotic limit cycle whose basin of attraction is $\mathbb{R}^2$.
- The base frequency of the limit cycle is approximately $\omega_0 = 1 - \frac{1}{16}\mu^2 + O(\mu^3)$ [37].
- Applying the Hankel DMD RRR algorithm with the observable function

  $$f(x_1, x_2) = x_1^2 + x_2^2 + \sqrt{x_1^2 + x_2^2}$$

  yields a computed base frequency of 0.994151 for the chosen initial state.
- Eigenvalues computed with standard DMD and DMD RRR (with residual threshold $\eta_0 = 10^{-3}$) are compared (see Figure 3.5).
- The number of eigenvalues/eigenvectors obtained is 250 (for a Hankel matrix with $r = 250$ columns), but only a few satisfy the residual threshold in DMD RRR.
- DMD RRR eigenvalues form a lattice structure consistent with theoretical results $\{k\omega_0 - \mu + k\omega_0 : k \in \mathbb{Z}\}$ [31].
- Standard DMD computes many spurious eigenvalues.
- Solution and DMD RRR eigenvalues (with residual threshold) are presented in Figure 3.5 .

Stochastic Case ($\varepsilon > 0$):

- For large t, the solution is "smeared out" around the deterministic limit cycle, forming a stochastic asymptotic limit cycle (see Figure 3.5).
- This statistical equilibrium can be determined by solving the associated Fokker-Planck equation.

- Energy bounds for the noisy van der Pol oscillator were determined in [14].
- For small to moderate noise, the base frequency of the averaged limit cycle remains similar to the deterministic case.
- Noise parameter is set to $\epsilon = 0.005$.

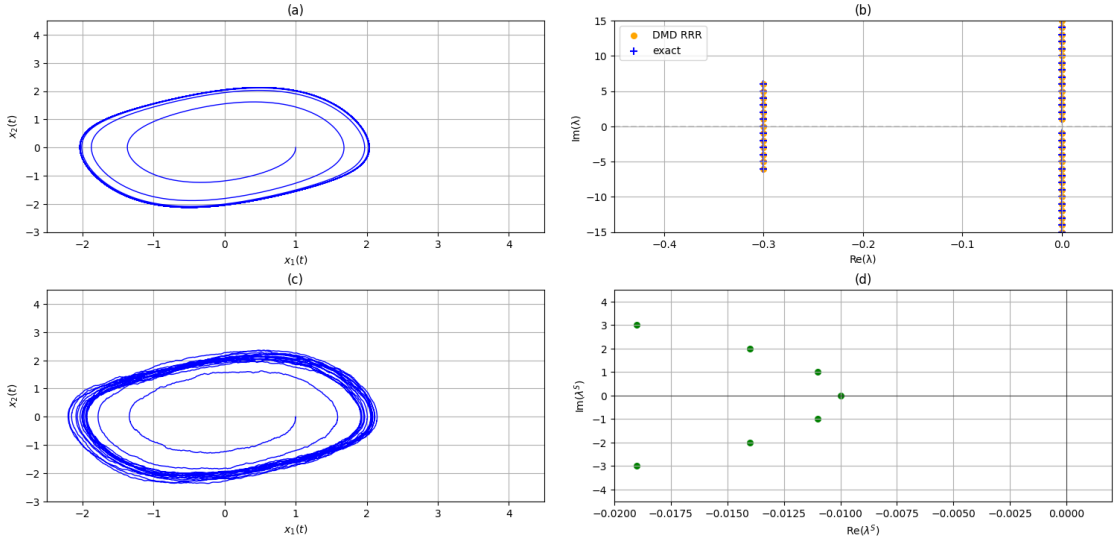

Figure 3.5: Van der Pol oscillator. Deterministic case: (a) solution; (b) Koopman eigenvalues.Stochastic case: (c) solution; (d) stochastic Koopman eigenvalues

Conclusion

This thesis thoroughly explores the framework of the stochastic Koopman operator, extending the classical Koopman operator theory to random dynamical systems. Given the intrinsic complexity of such systems, often shaped by inherent uncertainties, the objective was to provide theoretical and numerical tools for their linear analysis and the prediction of their observables.

The theoretical foundations of the stochastic Koopman operator were first established. This included the formalization of its action on observables across various classes of random systems, ranging from deterministic systems perturbed by noise to stochastic differential equations (SDEs). This groundwork facilitated the characterization of the operator's fundamental properties, such as linearity and its infinite-dimensional nature, thereby enabling subsequent applications and approximations.

Building on this theoretical basis, numerical approximation methods for the stochastic Koopman operator were developed and assessed. Particular emphasis was placed on adapting Dynamic Mode Decomposition (DMD) algorithms originally designed for deterministic systems [34, 35] to enhance robustness under stochastic influences. This adaptation allowed for the extraction of stochastic Koopman modes and eigenvalues, yielding critical insights into the underlying dynamics of random systems. The effectiveness and relevance of these numerical approaches were rigorously validated through a series of examples, including both discrete cases (e.g., noisy rotation on the circle) and continuous ones (e.g., linear and nonlinear SDEs, noisy Van der Pol oscillator). These examples demonstrated the capacity of the methods to capture observable dynamics in the presence of noise.

Finally, these tools were applied to practical problems to illustrate their usefulness in tasks such as long-term prediction of observables and the identification of coherent structures in complex random systems. These applications underscored the added value of the stochastic Koopman framework over traditional methods for analyzing random systems, particularly in terms of simplifying complex dynamics and enhancing predictive capabilities.

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