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Fractional Calculus & Fractional Stochastic Processes: Theory and Applications

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Contents

Abstract

THe main goal of this master thesis, is to give by means of the examples chosen a little glance on fractional calculus and fractional processes and discuss how are used in modeling some real phenomena, We begin by giving some preliminary background on stochastic calculus. Then we give an overview on the theory of fractional calculus and its application to the respiratory system.

We wanted after to widens the notion of the fractional paradigm from calculus to stochastic processes by studying one of self-similar, long-range dependence, Gaussian fractional processes called weighted fractional Brownian motion (wfBm), which depends on two real parameters a, b. It includes fractional Brownian motion when $a = 0$, standart Brownian motion when $a = b = 0$. Then we will give some properties of this process. These properties, which are analogous to those of fBm, are self-similarity, path continuity, behavior of increments and long-range dependence. $B^{a,b}$ is neither a semi-martingale nor a Markov process unless $b = 0$. Although, the wfBm $B^{a,b}$ has not stationary increments in general. wfBm widens the scope of behaviour of fBm, it may be useful in some applications.

Key words: Standard Brownian motion. Stochastic differential equations. Fractional calculus. Fractional Brownian motion. Weighted fractional Brownian motion.

Résumé

L'objectif principal de ce travail est de donner à l'aide de quelques exemples un aperçu sur le calcul fractionnaire et les processus fractionnaires et leurs utilisation dans la modélisation de certains phénomènes réels. Nous commençons par donner quelques notions préliminaires sur le calcul stochastique. Ensuite, nous donnons un aperçu de la théorie du calcul fractionnaire avec une application au système respiratoire.

Nous avons ensuite voulu élargir le paradigme fractionnaire du calcul aux processus stochastiques en étudiant l'un des processus fractionnaires Gaussiens autosimilaires et qui posséde la proprieté de la longue memoire appelé mouvement brownien fractionnaire pondéré (wfBm), qui dépend de deux paramètres réels a, b. Il comprend le mouvement brownien fractionnaire lorsque $a = 0$, le mouvement brownien standard lorsque $a = b = 0$. Ensuite, nous donnerons quelques propriétés de ce processus analogues aux celles de fBm, telle que l'auto-similarité, le comportement des incréments et la dépendance à longue terme. $B^{a,b}$ n'est ni une semi-martingale ni un processus de Markov sauf que pour $b = 0$. généralemnt les incréments du wfBm $B^{a,b}$ ne sont pas stationnaires. wfBm élargit la portée du comportement de fBm, cela peut être utile dans certaines applications.

Mots clés: Mouvement Brownien standard. Equation différentielle stochastique. Calcul fractionnaire. Mouvement Brownien fractionnaire. Mouvement Brownien fractionnaire pondérée.

Introduction

Fractional calculus is a generalization of ordinary calculus. Calculus proved to be a key tool for modern science because it allows the writing of differential equations that link variables and their rates of change. Fractional calculus has a long history. It is seen as the generalization of ordinary differentiation and integration to non-integer orders. Over the last few years, fractional calculus was found to play an important role in the modeling of a considerable number of real-life or physical phenomena see [18, 11] and references therein.

The fractional paradigm applies not only to calculus but also to stochastic processes, used in many applications in finance and economics such as modeling volatility and interest rates and modeling high frequency data. The key features of fractional processes that make them interesting to these disciplines include the following: Long-range memory, non Markovian properties and self-similarity. The definition and properties of fractional processes look very different from those of fractional calculus; actually, fractional Brownian motion (fBm) seems to be the simplest fractional process. It was introduced in the pioneer paper[8], and has been widely studied due to some compact properties such as long/short range dependency, self-similarity, stationary increments and Hölder's continuity, and also due to its numerous applications in various scientific areas. Some surveys and complete literatures on fractional Brownian motion could be found in Alos and other.[1], Biagini and other.[2], Decreusefond and Üstüunel[8], Embrechts and Maejima[10], Mishura [15], Nourdin[16], Nualart [17], Samorodnitsky [20], Taqqu[24] and Tudor[23]. The fractional nature of these processes appears in some parameters that characterize autocorrelations, namely the so called Hurst-exponent H, which might assume fractional values as opposed to integer values.

However, fBm alone cannot serve as an adequate model in all the fields of applications, and more complex fractional random processes are needed to model real phenomena. In the same context, the fractional Brownian motion, which is characterized by a single parameter, namely the Hurst index, cannot serve as a good model where there are several levels of fractionality. It is for this, and as an extension of the fractional Brownian motion, Bojdecki and other. [7] introduced and studied a rather special class of self-similar Gaussian processes which preserve many properties of the fractional Brownian motion. This process is called weighted fractional Brownian motion (weighted-fBm).

In contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes such as the weighted fractional Brownian motion, for which some works for weighted-fBm can be found in Bojdecki and other [7, 4, 5, 6], Yan-An [25, 26]. Fractional calculus and fractional Brownian motion are studied in detail both in the sense of theory and in terms of applications. In this master thesis, we wanted to give an idea of both fractional calculus, and some other extension of fBm more precisely the weighted fractional Brownian motion with application in finance.

The subject matter presented in this master thesis has been divided into three chapters.

The first chapter gives an introduction to the topic of our study and a brief survey of the stochastic calculus. We start this chapter with the general definition of stochastic process and study the relation between stochastic processes and finite dimensional distribution of a stochastic process. In particular, we present some important class of stochastic process such as stationary, martingales, Markov and Gaussian processes. Then we discuss some properties of a very particular class of stochastic processes witch is the Brownian motion. We conclude the chapter with a section about stochastic integration.

The second chapter gives a brief description of some fractional differintegral operators. The Grünwald-Letnikov, Riemann-Liouville and Caputo approaches will be explored. These are the most frequently used differintegrals fractional operators, and we will discover if some basic properties, such as linearity, Leibniz's rule and composition, still apply to differintegrals fractional operators. An application to human respiratory system is given in the last of this chapter.

The third chapter is devoted to fractional stochastic processes, we shall first introduce the notions of self-similarity, the long range dependency and the connection between them. Second we will define the fractional Brownian motion , study its essential properties and some of its several representations also. Then we will study pathwise integration with respect to fractional Brownian motion. the two last sections of this chapter are devoted to the weighted fractional Brownian motion and its application in finance.

Chapter 1

Introduction to stochastic calculus

In this chapter we collect the basic notions of stochastic process and study some class of stochastic process such as stationary, martingales, Markov and Gaussian processes. Then we discuss some properties of the Brownian motion, we conclude this chapter by stochastic integration and stochastic differential equation. For more reference on this chapter we refer the reader to [24, 14].

1.1 Basic concepts of stochastic processes

P.A. Meyer and **C. Dellacherie** have created the so called general theory of stochastic processes, which consists of a number of fundamental operations on either real valued stochastic processes indexed by $[0, \infty)$, or random measures on $[0, \infty)$, relative to a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. So throughout this master thesis, we assume we are given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ that satisfies the usual conditions, that is $(\mathcal{F}_t)$ is a right continuous filtration of $(\mathcal{F}, \mathbb{P})$, complete sub-$\sigma$-fields of $\mathcal{F}$.

1.1.1 Propaedeutic notions

Definition 1.1.1. ***(of Stochastic process)***.

We define real valued (one-dimensional) stochastic process as a family of random variables

$\{X_t\}_{t\in T}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$:

$$X_t : \Omega \longrightarrow \mathbb{R}, t \in T \subseteq \mathbb{R}_+.$$

A stochastic process could be a discrete time or a continuous time process, according to the set T is countable or continuous.

Definition 1.1.2. ***(of Trajectory)**.*

For each element $w \in \Omega$ the mapping $t \mapsto X_t(w)$ defined on the parameter set T, is called a realization (or a trajectory, a sample path) or sample function of the stochastic process $\{X_t\}_{t\in T}$.

Definition 1.1.3. ***(Finite-dimensional marginal distribution)***

Let $\{X_t; t \in T\}$, be a real-valued stochastic process and $\{t_1 < ... < t_n\} \subset T$, then the probability distribution $\mathbb{P}_{t_1,...,t_n} = \mathbb{P} \circ (X_{t_1}, ..., X_{t_n})^{-1}$ of the random vector $(X_{t_1}, ..., X_{t_n}) : \Omega \to \mathbb{R}^n$ is called a finite-dimensional marginal distribution of the process $\{X_t; t \in T\}$.

Definition 1.1.4. ***Equivalence of stochastic processes***

Let $X = \{X_t, t \in T\}$ and $Y = \{Y_t, t \in T\}$ be two stochastic processes. Then X and Y are:

1. *Equivalent if they have the same finite dimensional distributions.*

2. *Modification if $\mathbb{P}\{X_t = Y_t\} = 1$, for every $t \in T$.*

3. *Indistinguable if $\mathbb{P}\{X_t = Y_t$, for every $t \in T\} = 1$.*

Definition 1.1.5. ***(Continuity concept)***

Fix $p > 1$. Let $\{X_t, t \in T\}$ be a real-valued stochastic process, such that $\mathbb{E}(|X_t|^p) < \infty$, for all $t \in T$. The process $\{X_t, t \in T\}$ is said to be continuous in mean of order p if

$$\lim_{s\to t} \mathbb{E}(|X_t - X_s|^p) = 0.$$

Definition 1.1.6. ***(Filtration)***

*A **filtration** on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $(\mathcal{F}_t)_{t\in T}$, of sub -σ-field of $\mathcal{F}$.*

A measurable space $(\Omega, \mathcal{F})$ endowed with a filtration $(\mathcal{F}_t)_{t\in T}$ is said to be a filtered space.

Definition 1.1.7. ***(of adapted stochastic process)***
A stochastic process $(X_t)_{t\in T}$ *is adapted to the filtration* $(\mathcal{F}_t)_{t\in T}$ *if* $\forall t \in T$, X_t *is* $\mathcal{F}_t-$ *measurable.*

1.1.2 Main Classes of stochastic process

1.1.2.1 Processes with independent increments

Definition 1.1.8. *The process* X *is said to have independent increments if for any finite subset* $\{t_0 < \ldots < t_n\} \subset T$, *the increments* $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$, *are independent random variables.*

1.1.2.2 Markov processes.

Definition 1.1.9. X *is a Markov process if for any* t *and* $s > 0$ *the conditional distribution of* $X(t+s)$ *given* $\mathcal{F}_t$ *is the same as the conditional distribution of* $X(t+s)$ *given* $X(t)$, *that is,*

$$\mathbb{P}(X(t+s) \leq y \backslash \mathcal{F}_t) = \mathbb{P}(X(t+s) \leq y \backslash X(t)), a.s.$$

The above well-known formulation of the Markov property states that given the current state of X at time t, the future of X is independent of the $\sigma-$algebra $\mathcal{F}_t$ of events including, alternative and useful statements of the Markov property.

1.1.2.3 Stationary processes.

Definition 1.1.10. *A stochastic process* $(X_t)_{t\geq 0}$ *is said a stationary process if any collection* $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ *has the same distribution of* $(X_{t_{1+\tau}}, X_{t_{2+\tau}}, ..., X_{t_{n+\tau}})$, *for each* $\tau \geq 0$. *That is,*

$$(X_{t_1}, X_{t_2}, ..., X_{t_n}) \stackrel{d}{=} (X_{t_{1+\tau}}, X_{t_{2+\tau}}, ..., X_{t_{n+\tau}}).$$

1.1.2.4 Processes with stationary increment

Definition 1.1.11. *A stochastic process* $(X_t)_{t\geq 0}$ *is said a stationary increments, if for any* $h \geq 0 :$ *finite subset* $\{t_0 < ... < t_n\} \subset T$ *the increments*

$$(X_{t+h} - X_h)_{t\geq 0} \stackrel{d}{=} (X_t - X_0)_{t\geq 0}.$$

1.1.2.5 Gaussian process

Definition 1.1.12. *A stochastic process* $(X_t)_{t\geq 0}$ *is Gaussian if every finite linear combination of* $(X_t)_{t\geq 0}$ *is a gaussian r.v i.e*

$$\forall n, \forall t, 1 \leq i \leq n, \forall a, \sum_{i=1}^{n} a_i X_{t_i}.$$

is a gaussian r.v.

1.1.2.6 Martingale

Definition 1.1.13. *A stochastic process* $(M_t)_{t\in T}$ *is a martingale with respect to the filtration* $(\mathcal{F}_t)_{t\in T}$ *if :*

(a) $(M_t)_{t\in T}$ *is adapted to the filtration* $(\mathcal{F}_t)_{t\in T}$;

(b) $\mathbb{E}(|M_t|) < \infty, \forall t \in T.$

(c) $\mathbb{E}(M_t \backslash \mathcal{F}_s) = M_s, \forall s \leq t, s, t \in T.$

Definition 1.1.14. ***(Stopping Times)***
A $\mathcal{F}_t$*-stopping times is r.v.* $\mathrm{T} : \Omega \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ *such that* $\{\mathrm{T} \leq t\} \in \mathcal{F}_t, \quad \forall t \in \mathbb{R}_+.$

Definition 1.1.15. ***(Local-martingale)***
A process X *is a* ***local martingale*** *if there exists a sequence of stopping times* T_n *with* $T_n \nearrow \infty$ *a.s.,* $T_n < T$ *a.s. on* $\{T > 0\}$*, and* $\lim_{n\to\infty} T_n = T$ *a.s. and moreover* $X_{t\wedge T_n}$ *is a martingale for each* n

Definition 1.1.16. ***(Semi-martingale)***
A stochastic process is called a ***semimartingale*** *if it can be written in the form*

$$X_t = X_0 + M_t + A_t,$$

where $(M_t)_{t\in\mathbb{R}_+}$ *is a local martingale vanishing at* 0 *and* $(A_t)_{t\in\mathbb{R}_+}$ *is a right−continuous* $\mathcal{F}_t-$ *adapted process of finite variation vanishing at* 0.

1.2 Brownian Motion and stochastic integration

In this section we are going to introduce Brownian motion and stochastic integration with respect to it. We shall see that the paths of the Brownian motion are not of finite variation. Therefore, it is not possible to give a pathwise definition of the integral with respect to the Brownian motion.

1.2.1 Brownian motion

In 1827 Robert Brown (botanist, 1773-1858) observed the erratic and continuous motion of plant spores suspended in water. Later in the 20's Norbert Wiener (mathematician, 1894-1964) proposed a mathematical model describing this motion, the Brownian motion (also called the Wiener process).

Definition 1.2.1. *A Gaussian, continuous process characterized by mean value* $m(t) = 0$ *and covariance function* $\varphi := \min(s,t) = s \wedge t$*, for any* $s, t \in [0, T]$*, is called a Brownian motion.*

We will often talk about a Wiener process following the characterization just stated, but it is appreciable to notice that next definition can provide a more intuitive description of the fundamental properties of a process of this kind.

Definition 1.2.2. *A stochastic process* $(B_t)_{t \in \mathbb{R}_+}$ *is called a* ***standard Brownian motion*** *if it satisfies the following conditions:*

1. $\mathbb{P}(w \in \Omega : B_t(w) = 0) = 1$.
2. $\forall n, \forall t_i, 0 \leq t_0 \leq t_1 \leq ... \leq t_n$, *the r.v* $(B_{t_n} - B_{t_{n-1}}, ..., B_{t_1} - B_{t_0}, B_{t_0})$ *are independent.*
3. *For any* $s \leq t, B_t - B_s$ *is a centered real valued r.v. normally distributed with variance* $t - s$*, i.e.* $B_t - B_s \sim \mathcal{N}(0, t - s)$.
4. $\mathbb{P}(w \in \Omega : t \to B_t(w)$ *is continuous* $) = 1$.

1.2.1.1 Properties of Brownian motion

From the description of BM as the motion of a pollen grain-and even more from its derivation as the limit of a random walk, it should be clear why the following result holds.

Proposition 1.2.1. *Let $B(t)$ be a standard Brownian motion in dimension $d \geq 1$.*

- ***Isometry.*** *If ϕ is a linear isometry of $\mathbb{R}^d$, then $\phi(B_t)$ is a Brownian motion.*
- ***Translation.*** *For every $s \geq 0$, the process $B_t^{(s)} = B_{s+t} - B_s$ is a Brownian motion.*
- ***Time reversal.*** *The process $(B_1 - B_{1-t})_{t\in[0,1]}$ is distributed as $(B_t)_{t\in[0,1]}$.*
- ***Scale invariance.*** *For every $a > 0$, the process $(\frac{1}{a}B_{a^2t} : t \geq 0)$ is a Brownian motion.*

Proof: All the processes considered are continuous, Gaussian, centered (mean zero) and the covariance functions are easily seen to coincide with that of Brownian motion. ■

Theorem 1.2.1. *Brownian Motion is a martingale.*

Proof:

By definition, $B(t) \sim \mathcal{N}(0, t)$, so that $B(t)$ is integrable with $\mathbb{E}(B(t)) = 0$. Let $\mathcal{F}_t = \sigma(B(s) : s \leq t)$. Then

$$\begin{aligned}
\mathbb{E}[B(t+s)\backslash\mathcal{F}_t] &= \mathbb{E}[B(t+s) - B(t) + B(t)\backslash\mathcal{F}_t] \\
&= \mathbb{E}[B(t+s) - B(t)\backslash F_t] + \mathbb{E}[B(t)\backslash\mathcal{F}_t] \\
&= \mathbb{E}[B(t+s) - B(t)] + \mathbb{E}[B(t)\backslash\mathcal{F}_t] \\
&= \mathbb{E}[B(t)\backslash\mathcal{F}_t] \\
&= B(t).
\end{aligned}$$

So B is an $\{\mathcal{F}_t\}$-martingale. ■

Theorem 1.2.2. *The standard Brownian motion $B(t)$ possesses Markov property.*

Proof:

It is seen by using the moment generating function that the conditional distribution of $B(t+s)$ given $\mathcal{F}_t$ is the same as that given $B(t)$. Indeed, let as define

$$T(s)f(x) = \mathbb{E}[f(x + B(s))],$$

and note that

$$\begin{aligned}\mathbb{E}[B(t+s)\backslash\mathcal{F}_t] &= \mathbb{E}[f(B(t+s)-B(t)+B(t))\backslash\mathcal{F}_t] \\ &= T(s)f(B(t)) \\ &= \mathbb{E}[f(B(t+s))\backslash B(t)] \qquad \blacksquare\end{aligned}$$

Definition 1.2.3. ***Quadratic Variation of Brownian Motion.***
The quadratic variation of Brownian motion $[B,B](t)$ is defined as

$$[B,B](t) = [B,B]([0,t]) = \lim \sum_{i=1}^{n} |B^n_{t_i} - B^n_{t_{i-1}}|^2, \tag{1.1}$$

where the limit is taken over all shrinking partitions of [0,t], with $\sigma_n = \max_i(X^n_{t_{i+1}} - X^n_{t_i}) \longrightarrow 0$ as $n \to \infty$. It is remarkable that although the sums in the definition (1.1) are random, their limit is non-random, as the following result shows.

Theorem 1.2.3. *[14] Quadratic variation of a Brownian motion over $[0,t]$ is t.*

- **Properties of Brownian paths**

$B(t)$'s as functions of t have the following properties. Almost every sample path $B(t), 0 < t < T$

1. is a continuous function of t,
2. is not monotone in any interval, no matter how small the interval is,
3. is not differentiable at any point,
4. has infinite variation on any interval, no matter how small it is,
5. has quadratic variation on $[0,t]$ equal to t, for any t.

Theorem 1.2.4. *For every t_0,*

$$\limsup_{t\to t_0} \left| \frac{B(t)-B(t_0)}{t-t_0} \right| = \infty \qquad a.s.,$$

which implies that for any t_0, almost every sample $B(t)$ is not differentiable at this point.

Proof. We refer the reader to ([14]).

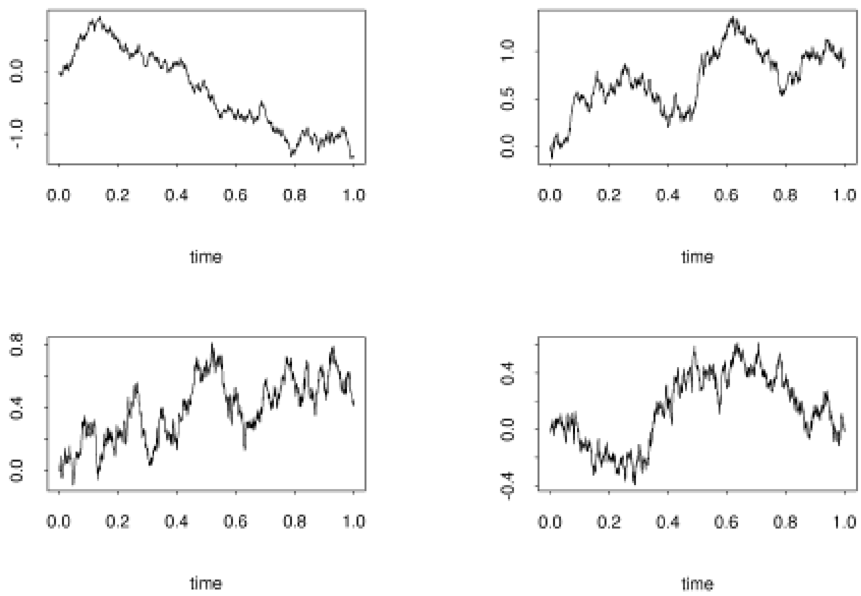

Figure 1.1: Four realizations or paths of Brownian motion

1.2.2 Stochastic diffierential equations and It$\hat{o}$ integrals

Definition 1.2.4. *(Itô integral).*

Let $\{B_t, \quad 0 \leq t \leq T\}$*, be the standard Wiener process (Bm) adapted to the filtration* $\mathcal{F}_t$*. Let* φ_t *is also an* $\mathcal{F}_t-$*adapted stochastic process such that*

$$\mathbb{E}\left[\int_0^T \varphi_t dt\right] < \infty. \tag{1.2}$$

Let $\phi_n = \{0 = t_0, t_1, ..., t_n = T\}$ *be the partitioning set and the norm* $\|\phi_n\|$ *is defined as*

$$\|\phi_n\| = \max_{k=1,2,...,n-1}(t_{k+1} - t_k)$$

The Itô integrals I_T *is then obtained by the following expression*

$$I_T = \int_0^T \varphi_t dB_t := \lim_{\|\phi_n\| \to 0} \sum_{i=0}^{n-1} \varphi_{t_i}(B_{t_{i+1}} - B_{t_i}) \tag{1.3}$$

I_T defined above is a random variable and the following theorem lists some of its properties.

Theorem 1.2.5. *Let* $T > 0$*, let* φ_t *and* ψ_t*,* $0 \leq t \leq T$ *, be* $\mathcal{F}_t$*-adapted stochastic processes that satisfy* (1.2)*. Itô integral It defined by* (1.3) *has the following properties:*

1. ***Adaptivity:*** *For each t,* I_t *is* $\mathcal{F}_t-$*measurable.*

2. ***Continuity:*** *The sample paths of* I_t *are continuous.*

3. ***Itô isometry:*** $\mathbb{E}[I_t^2] = \mathbb{E}\int_0^t \varphi_s^2 ds.$

4. ***Linearity:***$\int_0^t (a\varphi_s + b\psi_s)dB_s = a\int_0^T \varphi_s dB_s + b\int_0^T \psi_s dB_s.$ *for* $a, b \in \mathbb{R}$.

5. ***Martingale:***$\mathbb{E}[I_t] = 0$ *and* I_t *is a martingale.*

Proof: See [14].

Itô integrals are of key importance for the stochastic differential equations (SDE's). In fact, they appear in what is defined as the strong solution of an SDE. Let us start with the definition of this class of differential equations.

Definition 1.2.5. *(SDE).Let* $\mathcal{F}_t$ *be a filtration generated by a Wiener process* B_t*, let* T *be a positive constant. Also let* $\mu : \mathbb{R} \times [0, T] \to \mathbb{R}, \sigma : \mathbb{R} \times [0, T] \to \mathbb{R}$ *be measurable functions and* X_0 *be* $\mathcal{F}_0$*-measurable random variable. A stochastic differential equation can be expressed in the form of stochastic differential*

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t, \quad t \in [0, T] \tag{1.4}$$

$$X_0 = x_0 \in \mathbb{R}. \tag{1.5}$$

Moreover, if $\int_0^T \mu(X_t, t)dt < \infty$ *and* $\int_0^T \sigma^2(X_t, t)dt < \infty$ *then a continuous* $\mathcal{F}_t$*-adapted process* X_t *satisfying*

$$X_t = x_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s, t \in [0, T], \tag{1.6}$$

is said to be a (strong) solution of the SDE $1.4 - 1.5$

Remark 1.2.1. *The first integral on the right hand side of* (1.6) *is a Lebesgue integral and the second one is an Itô integral. The functions* $\mu(X_t, t), \sigma(X_t, t)$ *are called the drift and diffusion respectively.* X_t *defined as* (1.6) *is also known as an Itô process.*

Definition 1.2.6. *(Pathwise uniqueness). Let* $X_t, 0 \leq t \leq T$*, and* $Y_t, 0 \leq t \leq T$*, are adapted stochastic processes defined on the same probability space with filtration* $\mathcal{F}_t$*. If both processes satisfy* $1.4 - 1.5$ *then the solution of* $1.4 - 1.5$ *is (pathwise) unique, if any two processes satisfying the SDE with respect to the same initial condition are indistinguishable.*

Theorem 1.2.6. *If the following conditions are satisfied*

1. *Coefficients are locally Lipschitz in* x *uniformly in* t*, that is, for every* T *and* N*, there is a constant* K*depending only on* T *and* N *such that for all* $|x|, |y| \leq N$ *and all* $0 \leq t \leq T$

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| < K|x - yl$$

2. *Coefficients satisfy the linear growth condition*

$$|\mu(x,t)| + |\sigma(x,t)| \leq K(1 + |x|)$$

3. $X(0)$ *is independent of* $(B(t), 0 \leq t \leq T)$*, and* $\mathbb{E}X^2(0) < \infty$*.*

Then there exists a unique strong solution $X(t)$ *of the SDE* $1.4-1.5$*.* $X(t)$ *has continuous paths, moreover* $\mathbb{E}(\sup_{0 \leq t \leq T} X^2(t)) < C(1 + \mathbb{E}(X^2(0)))$*, where constant* C *depends only on* K *and* T*.*

Proof :

The proof can be found in the original paper [14]

Chapter 2

Essentials of fractional calculus

In this second chapter, we briefly introduced some of the useful definitions in fractional calculus theory, such as Grünwald-Letnikov, Riemann-Liouville and Caputo approach and we will discover if some their basic properties, some examples are explored and application. All definitions and results recalled below are very standard in the literature and are mostly extracted from the main reference who is based around them this chapter are [18, 11, 15].

2.1 Special functions

In this section we give a brief review of some important functions for the fractional calculus theory. These functions are: the gamma, Beta Mittag-Leffler and Weighted functions.

2.1.1 The Gamma function

A comprehensive definition of The Gamma function $\Gamma(x)$, is that provided by the Euler limit

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)...(x+n)}, \tag{2.1}$$

but the integral transform definition

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0. \tag{2.2}$$

is often more useful, although it is restricted to positive x values.

An integration by parts applied to the definition (2.2) leads to the recurrence relationship

$$\Gamma(x+1) = x\Gamma(x) \tag{2.3}$$

which is the most important property of the gamma function. The same result is a simple consequence of the Euler limit definition. Since $\Gamma(1) = 1$, this recurrence shows that for a positive integer n

$$\Gamma(n+1) = n\Gamma(n) = n[n-1]\Gamma(n-1) = ... = n[n-1]...2.1.\Gamma(l) = n!$$

2.1.2 The Beta function

The Beta function β, is defined by Euler integral of the first Kind

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, x, y \in \mathbb{R}. \tag{2.4}$$

This function is connected to the Gamma function by the relation :

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

2.1.3 The Mittag-Lefler Function

The **Mittag-Leffler** function is named after a Swedish mathematician who defined and studied it in 1903. The function is a direct generalization of the exponential function, it plays a major role in fractional calculus. Firstly, we introduce one parameter function by using series, namely

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \tag{2.5}$$

Then, we define the **Mittag-Leffler** function with two parameters, as:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \tag{2.6}$$

Some examples of **Mittag-Leffler** function, are given:

$$E_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp(x),$$
$$E_{2,1}(x^2) = \sum_{k=0}^{\infty} \frac{x^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh(x) \qquad .$$
$$E_{2,2}(x^2) = \sum_{k=0}^{\infty} \frac{x^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{x(2k+1)!} = \frac{\sinh(x)}{x}$$

2.1.4 The Weighted function

Weights give more weight to some elements in a set. The weight function gives weights to data.

The weight function has many uses, including:

- Compensating for bias(error).
- Giving some data points more, or less, influence. For example, you can adjust for outliers.
- Calculating integrals.

Mathematically, a weight is a positive measure such as $w(x)\,dx$ on some domain Ω, which is typically a subset of a Euclidean space $\mathbb{R}^n$, for instance Ω could be an interval $[a, b]$, dx is Lebesgue measure and $w\colon \Omega \to \mathbb{R}^+$ is a non-negative measurable function. In this context, the weight function $w(x)$ is sometimes referred to as a density.

Definition 2.1.1. *If $f : \Omega \to \mathbb{R}$ is a real-valued function, then the unweighted integral*

$$\int_\Omega f(x)\,dx,$$

can be generalized to the weighted integral

$$\int_\Omega f(x)w(x)\,dx.$$

Remark 2.1.1. *Note that one may need to require f to be absolutely integrable with respect to the weight $w(x)\,dx$ in order for this integral to be finite.*

2.2 The basic fractional derivatives approaches

Definitions of the fractional order derivative are not unique and there exist several definitions, including Grünwald-Letnikov, Riemann-Liouville, and the Caputo and more other definitions. This section is dedicated to basic recalls about G-L, R-L and Caputo fractional operators.

2.2.1 Riemann-liouville definition, 1982-1847

At first, we define the Cauchy's formula

$$I^n f(x) = \int_a^x \int_a^{x_1} ... \int_a^{x_{n-1}} f(x_n) dx_n dx_2 dx_1 = \frac{1}{(n-1)!} \int_a^x \frac{f(t)}{(x-t)^{1-n}} dt. \tag{2.7}$$

Definition 2.2.1. *Suppose that* $f \in L_1([a, b])$ *and* $a \leq x \leq b$. *Then we have*

$$D_{a_+}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt.$$

$$D_{b_-}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt.$$

Is named ***the Riemann-Liouville*** *fractional integral of order* $0 < \alpha < 1$.

Definition 2.2.2. *Suppose that* $f \in L_1([a, b])$ *and* $a \leq x \leq b$. *Then we have*

$$D_{a_+}^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha+1-m}} dt & n-1 < \alpha < n \in \mathbb{N}, \\ \frac{d^m}{dx^m} f(x), & \alpha = m \in \mathbb{N}. \end{cases}$$

$$D_{b\_}^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(t-x)^{\alpha+1-m}} dt & n-1 < \alpha < n \in \mathbb{N}, \\ \frac{d^m}{dx^m} f(x), & \alpha = m \in \mathbb{N} \end{cases}$$

Is named the ***Riemann-Liouville*** *fractional derivative of order* α

2.2.1.1 Basic Properties

1- Linearity of the operator:

Let f and g are functions for which the given derivatives or integrals operator are defined and $\lambda, \mu \in \mathbb{R}$ are real constants.

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x) \tag{2.8}$$

Proof:

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1}(\lambda f(t) + \mu g(t))dt$$

$$= \frac{\lambda}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1} f(t)dt + \frac{\mu}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1} g(t)dt$$

$$= \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x)$$

2.2.1.2 Some examples

1. **The Power Function**

$$D^{\alpha} t^m = \frac{\Gamma(m+1)}{\Gamma(m-\rho+1)} t^{m-\rho} \quad , j-1 < \alpha < j \, , m > -1 \, , m \in \mathbb{R}$$

2. **The Exponential Function** Now, we consider the exponential function $e^{\lambda t}$

$$D^{\alpha}(e^{\lambda t}) = t^{\alpha} E_{1,1-p}(\lambda t), \; n-1 \leq \alpha < n, \; \alpha \in \mathbb{R}, \; \lambda \in \mathbb{C}$$

2.2.2 Grünwald-Letnikove definition, 1867-1868

Grünwald-Letnikov derivative is a basic extension of the natural derivative to fractional one. It was introduced by A. Grünwald in 1867, and then by A. Letnikov in 1868.

The G-L definition of a fractional derivative can be viewed as a derivative which finds its roots in the definition of a first derivative in terms of a limit:

$$f'(x) = \lim_{h_1 \to 0} \frac{f(x) - f(x-h_1)}{h_1} \approx \frac{f(x) - f(x-h_1)}{h_1}, \quad if \quad 0 < h_1 < 1 \tag{2.9}$$

In a similar way, we can define the second derivative of a function:

$$f''(x) = \lim_{h_2 \to 0} \frac{\frac{f(x)-f(x-h_1)}{h_1} - \lim\limits_{h_1 \to 0} \frac{f(x-h_2) - f(x-h_1-h_2)}{h_1}}{h_2} \tag{2.10}$$

It is clear that one can extend this derivation to third, fourth and higher integer derivatives as well. These formulas in ordinary calculus may be used in the construction of approximations, i.e., numerical discretizations of derivatives in terms of finite differences. If we take $h = h_1 = h_2$ in (2.10), we see that we obtain for the ordinary second derivative:

$$f''(x) = \lim_{h\to 0} \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2} \approx \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}, \quad if \quad 0 < h < 1. \tag{2.11}$$

Using the principle of mathematical induction one can extend this idea to the n th derivative in the following way:

$$f^{\alpha}(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\alpha} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t - jh), n \in \mathbb{N}$$

Definition 2.2.3. *Let $\alpha \in (0,1)$ be fixed and let $f : \mathbb{R} \to \mathbb{R}$ be a given function. The Right-G-L derivative of order α of f is defined as:*

$$D_{+}^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t - jh)$$

We recall that the binomial coefficients can be defined as: $\begin{pmatrix} \alpha \\ j \end{pmatrix} = \frac{\alpha!}{j!(\alpha-j)!}$

Next, replacing the terms with values in terms of the Gamma function, we define the G-L fractional derivative:

$$D_{GL}^{\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha - j + 1)} f(x - jh) \tag{2.12}$$

2.2.2.1 Basic Proprties

Lemma 2.2.1. *[18] Let $y \in C[0,\infty)$ be a continuous function and $\alpha, \beta > 0$. Then, for any $x > 0$*

$${}_0D_x^{-\alpha}\{{}_0D_x^{\beta} f(x)\} ={}_0 D_x^{-(\alpha+\beta)} f(x) ={}_0 D_x^{-\beta}\{{}_0D_x^{-\alpha} f(x)\}$$

Linearity of the operator:

As for integer-order differentiation, fractional differentiation defines a linear operator as well:

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x) \tag{2.13}$$

2.2.2.2 Some Examples

1. **The constant Function**

Corollary 2.2.1. *[18] If f is a constant function (i.e. $f(x) = C$ for all $t \in [a-L, b]$), then ${}_{L}^{MG}D_x^{\alpha} f$ is a constant function for all $t \in [a, b]$. Furthermore we have*

$$ {}_{L}^{MG}D_t^{\alpha} f = \frac{C}{L^{\alpha}\Gamma(1-\alpha)} \tag{2.14} $$

2. **The Power Function**

Proposition 2.2.1. *[18] Let $\alpha > 0, L > 0$, m an integer such that $m - 1 < \alpha < m$ and $f(t) = t^n$, then we have*

$$ {}_{L}^{MG}D_x^{\alpha}(x^n) = \sum_{j=0}^{n} \frac{n! L^{j-\alpha}(x-L)^{n-j}}{(n-j)!\Gamma(j-\alpha+1)} \tag{2.15} $$

3. **The Exponential Function**

Proposition 2.2.2. *[18] Let $\alpha > 0, L > 0$ and m an integer such that $m - 1 < \alpha < m$, then we have*

$$ {}_{L}^{MG}D_x^{\alpha}(e^x) = \frac{E_{1,1-\alpha}(L)}{L^{\alpha}e^{L}} e^x. \tag{2.16} $$

2.2.3 Caputo fractional derivatives definition, 1969

Since R-L fractional derivatives failed in the description and modeling of some complex phenomena, Caputo derivative was introduced in 1967.

Definition 2.2.4. *Suppose that $\alpha > 0, x > \alpha, a, x \in \mathbb{R}$. The fractional caputo operator has the form*

$$ D_*^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha+1-m}} dt, m - 1 < \alpha < m \in \mathbb{N}, \\ \\ \frac{d^m}{dx^m} f(x), \alpha = m \in \mathbb{N}, \end{cases} $$

Remark 2.2.1. *The difference between Caputo and R-L formulas for the fractional derivatives leads to the following differences:*

a) *Caputo fractional derivative of a constant equals zero, while R-L fractional derivative of a constant does not equal zero.*

b) *The non-commutation, in Caputo fractional derivative we have:*

$$D_*^{\alpha} D^m f(t) = D_*^{\alpha+m} f(t) \neq D_*^{\alpha} f(t)$$

In general, the R-L derivative has also the non-commutation propriety :

$$D^m D_{R-L}^{\alpha} f(t) = D^{\alpha+m} f(t) \neq D_{R-L}^{\alpha} D^m f(t)$$

Where $\alpha \in (n-1, n), n \in \mathbb{N}, m \in \mathbb{N}^*$.

2.2.3.1 Basic Properties

Lemma 2.2.2. *[18] Let* $n-1 < \alpha < n, n \in \mathbb{N}, \alpha \in \mathbb{R}$ *and f(t) be such that* $D^{\alpha} f(t)$ *exists. Then the following properties for the Caputo operator hold*

$$\lim_{\alpha \to m} D^{\alpha} f(x) = f^{(m)}(x),$$

$$\lim_{\alpha \to m-1} D^{\alpha} f(x) = f^{(m-1)}(x) - f^{(m-1)}(0),$$

Proof: The proof uses integration by Parts

$$\begin{aligned}
D^{\alpha} f(x) &= \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^m(t)}{(x-t)^{\alpha+1-m} dt} \\
&= \frac{1}{\Gamma(m-\alpha)} (-f^m(t) \frac{(x-t)^{m-\alpha}}{m-\alpha} |_{t=0}^x - \int_0^x -f^{m+1}(t) \frac{(x-t)^{m-\alpha}}{m-\alpha} dt) \\
&= \frac{1}{\Gamma(m-\alpha)} (f^m(0) x^{m+\alpha} + \int_0^x f^{m+1}(t)(x-t)^{m-\alpha} dt
\end{aligned}$$

Now, by taking the limit for $\alpha \to m$ and $\alpha \to m-1$, respectively , it follows

$$\lim_{\alpha \to m} D^{\alpha} f(x) = (f^m(0) + f^m(t)|_{t=0}^x = f^m)$$

And

$$\begin{aligned}
\lim_{\alpha \to m-1} D^{\alpha} f(x) &= f^m(0)t + f^m(t)(x-t)|_{t=0}^x - \int_0^x -f^m(t) dt \\
&= f^{m-1}(t)|_{t=0}^x \\
&= f^{m-1}(x) - f^{m-1}(0).
\end{aligned}$$

Linearity of the operator:

Let $m-1 < \alpha < m, m \in \mathbb{N}, \alpha, \lambda \in \mathbb{C}$ and let the two functions f and g such as ${}^cD^{\alpha} f(x)$ and ${}^cD^{\alpha} g(x)$ exist. Caputo fractinary derivation is an operator linear :

$${}^cD^{\alpha}(\lambda f(x) + g(x)) = \lambda {}^cD^{\alpha} f(x) +^c D^{\alpha} g(x)$$

2.2.3.2 Some Examples

1. **The constant Function**

Theorem 2.2.1. *[18] For the Caputo Fractional derivative it holds* $D^{\alpha}C = 0, C =$ *const*

2. **The Power Function**

Theorem 2.2.2. *[18] The Caputo fractional derivative of the power function satisfies*

$$D^{\alpha}t^{\rho} = \begin{cases} \frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}t^{\rho-\alpha} = D^{\alpha}t^{\rho}, \quad n-1 < \alpha < n, \quad \rho > n-1, \quad \rho \in \mathbb{R} \\ \\ 0, \qquad n-1 < \alpha < n, \quad \rho \leq n-1, \rho \in \mathbb{N} \end{cases}$$

3. **The Exponential Function**

Theorem 2.2.3. *[18] Let* $\alpha \in \mathbb{R}, m-1 < \alpha < m, m \in \mathbb{N}, v \in \mathbb{C}$ *Then the Caputo fractional derivative of the exponential function has the form :*

$$D^{\alpha}e^{vx} = v^{m}x^{m-\alpha}E_{1,m-\alpha+1}(vx),$$

2.3 Application

2.3.1 Fractional Calculus and Its Application to the Respiratory System

Of all applications in biology, linear viscoelasticity is certainly the most popular field, for their ability to model hereditary phenomena with long memory. Viscoelasticity has been shown to be the origin of the appearance of fractional models in biological tissues [12].

Viscoelasticity of the lungs is characterized by compliance, expressed as the volume increase in the lungs for each unit increase in alveolar pressure or for each unit decrease of pleural pressure. The most common representation of the compliance is given by the pressure-volume (PV) loops. The initial steps under-taken to characterize the pressure-volume relationship in the lungs by means of exponential functions suggested a new interpretation of mechanical properties in lungs. Hildebrandt [12] used similar concepts to assess the viscoelastic properties of a rubber balloon [12] as a model of the lungs. He obtained similar static pressure-volume curves by stepwise infiation in steps of 10 ml (volume) increments in a one minute time interval. He then points out that the curves can be represented by means of a power-law function.

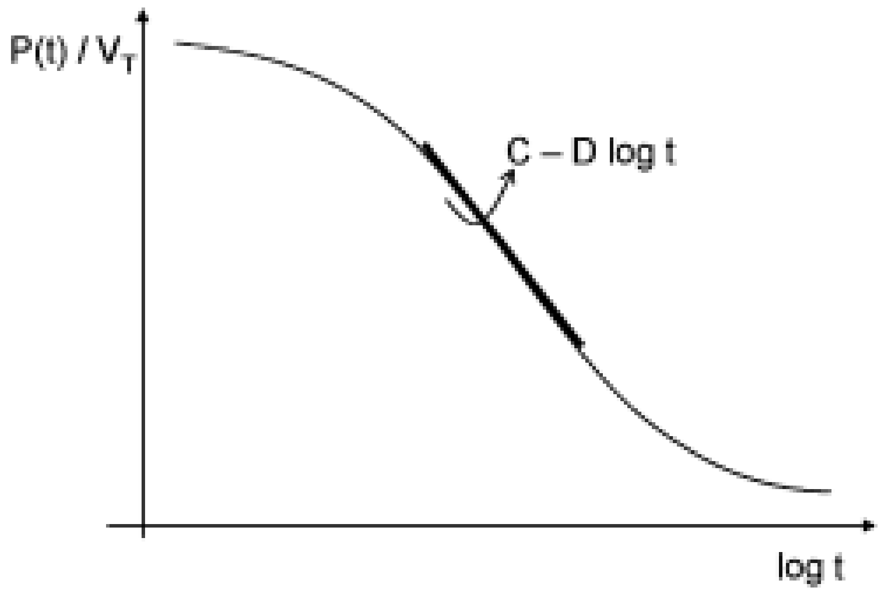

Figure 2.1: Schematic representation dependence of the pressure-volume ratio with the logarithm of time

Instead of deriving the compliance from the PV curve, Hildebrandt suggests to apply sinusoidal inputs instead of steps and he obtains the frequency response of the rubber balloon. The author considers the variation of pressure over total volume displacement also as an exponentially decaying function:

$$\frac{P(t)}{V_T} = At^{-\alpha} + B \quad , \frac{P(t)}{V_T} = C - Dlog(t) \tag{2.17}$$

with A, B, C, D arbitrary constants, V_T the total volume, t the time, and α the power-law constant. The transfer function obtained by applying Laplace to this stress relaxation curve is given by

$$\frac{P(t)}{V_T} = A\frac{\Gamma(1-\alpha)}{s^{1-\alpha}} + \frac{B}{s} \tag{2.18}$$

with Γ the Gamma function. If the input is a step $v(t) = V_T u(t)$., then $V(s) = V_T/s$ and the output is given by $P(s) = T(s)V_T/s$ with T(s) the unknown transfer function. Introducing this into (2.18) one obtains

$$T(s) = \frac{P(s)}{V(s)} = As^n\Gamma(1-\alpha) + B \tag{2.19}$$

By taking into account the mass of air introduced into the balloon, an extra term appears in the transfer function equation:

$$T(s) = \frac{P(s)}{V(s)} = As^\alpha\Gamma(1-\alpha) + B + L_r s^2 \tag{2.20}$$

with L_r the inductance. The equivalent form in frequency domain is given by

$$T(jw) = A\Gamma(1-\alpha)w^\alpha\cos(\frac{\alpha\pi}{2}) - L_r w^2 + B + j\left[A\Gamma(1-\alpha)w^\alpha\sin(\frac{\alpha\pi}{2})\right] \tag{2.21}$$

This function describes the behavior of the balloon in a plethysmograph, while undergoing sinusoidal forced oscillations.

Chapter 3

Fractional stochastic processes

In this chapter we shall first introduce self-similarity and the long range the fractional Brownian motion , study its essential properties and some representations are given. Then we will study then pathwise and Wick-Ito integration theories with respect to fBM. the weighted fractional Brownian motion and its application in finance. good monographs on this subject are [10, 24, 7, 4, 5, 6, 25]

3.1 Self-similarity and long range dependency

Self-similar processes with long-range dependence have attracted much attention recently for their applications and their intrinsic mathematical interest (see [24]). Therefore it seems important and worthwhile to know and address the various new processes that are available in them.

3.1.1 Self-similarity

First we introduce stochastic processes that are invariant in distribution under suitable scaling of time and space. These processes can be used to model many space- time scaling random phenomena observed in physics, biology and other fields.

Definition 3.1.1. *A stochastic process* $(X_t)_{t\geq 0}$ *is called self-similar if there exists a real number* $H > 0$ *such that for any* $c > 0$ *the processes* $(X_{ct})_{t\geq 0}$ *and* $(c^H X_t)_{t\geq 0}$ *have the* *same finite dimensional distributions.*

Remark 3.1.1.

- *A self-similar process satisfies $X_0 = 0$ almost surely.*
- *The self-similar processes with stationary increments all have the same covariance.*

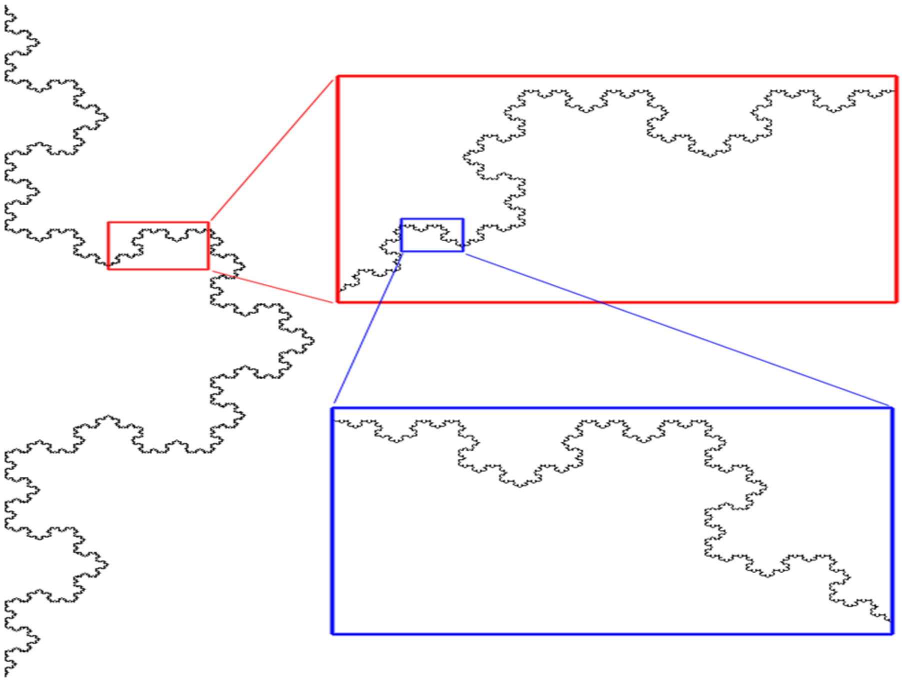

Figure 3.1: Self-Similarity

Theorem 3.1.1. *Let $(X_t)_{t\geq 0}$ be a non-trivial $H-$self-similar process with stationary increments such that $\mathbb{E}X_1^2 < \infty$. Then*

$$\mathbb{E}X_tX_s = \frac{1}{2}\mathbb{E}X_1^2(t^{2H} + s^{2H} - |t-s|^{2H})$$

Proof: Let $s \leq t$. Writing

$$X_tX_s = \frac{1}{2}(X_t^2 + X_s^2 - (X_t - X_s)^2)$$

we get

$$\begin{aligned} \mathbb{E}(X_tX_s) &= \frac{1}{2}\mathbb{E}(X_t^2 + X_s^2 - (X_t - X_s)^2) \\ &= \frac{1}{2}(\mathbb{E}[X_t^2] + \mathbb{E}[X_s^2] - \mathbb{E}(X_t - X_s)^2) \\ &= \frac{1}{2}\mathbb{E}X_1^2(t^{2H} + s^{2H} - |t-s|^{2H}).\blacksquare \end{aligned}$$

Proposition 3.1.1. *[23] Let $(X_t)_{t\geq 0}$ be a non-trivial $H-$self-similar process with stationary increments. Then*

1. *if $\mathbb{E}|X_1| < \infty$, then $0 < H \leq 1$.*
2. *if $\mathbb{E}|X_1| < \infty$, $H = 1$ then $\quad X_t = tX_1$.*
3. *if $\mathbb{E}|X_1|^\alpha < \infty \quad$ for some $\alpha \leq 1$, then $H < \frac{1}{\alpha}$.*

Proposition 3.1.2. *Let $(X_t)_{t\geq 0}$ be a non-trivial $H-$self-similar process with stationary increments such that $\mathbb{E}X_1^2 < \infty$. Define, for any integer $n \geq 1$*

$$r(n) = \mathbb{E}(X_1(X_{n+1} - X_n)).$$

Then, if $H \neq \frac{1}{2}$, as $n \to \infty$

$$r(n) \sim H(2H-1)n^{2H-2}\mathbb{E}X_1^2.$$

Proof: From Proposition 3.1.1

$$r(n) = \frac{1}{2}\mathbb{E}X_1^2((n+1)^{2H} + (n-1)^{2H} - 2n^{2H})$$

and it suffices to study the asymptotic behavior of the sequence on the right-hand side above when $n \to \infty$.

Remark 3.1.2. *If $H = \frac{1}{2}$ then $r(n) = 0$ for any $n \geq 1$.*

3.1.2 Long-range dependency

Long-range dependence is a phenomenon that may arise in the analysis of time series data. It relates to the rate of decay of statistical dependence. Mathematically long-rang dependence is defined as:

Definition 3.1.2. ***(Long-range dependence)***
We say that a process X exhibits long-range dependence (or it is a long-memory process) if

$$\sum_{n\geq 0} r_n = \infty$$

where $r(n) = \mathbb{E}(X_1 - X_0)(X_{n+1} - X_n)$. *Otherwise, if*

$$\sum_{n\geq 0} r_n < \infty$$

we say that X *is a short-memory process.*

Remark 3.1.3. *From Proposition 3.1.2 and Definition 3.1.2 we conclude that if* $(X_t)_{t\geq 0}$ *is a non- trivial* H*-self-similar process with stationary increments and with* $\mathbb{E}(X_1)^2 < \infty$ *then* X *is with long-range dependence for* $H > \frac{1}{2}$ *and with short-memory if* $H < \frac{1}{2}$.

There are fundamental connections between self-similarity and fractional calculus, which is an area of real analysis. These connections are explored in the context of fractional Brownian motion (fBm).

3.2 Fractional Brownian motion

The very first article about fractional Brownian motion (fBm) was published in 1940, by Andrey Nikolaevich Kolmogorov (1903-1987), a Soviet Russian mathematician. He introduced continuous time Gaussian processes with stationary increments and with the self-similarity property. Kolmogorov named such processes as Wiener spirals. However, that was Benoît B. Mandelbrot (1924-2010), a French mathematician and also best known as the father of fractal geometry. He considered an integral representation for fBm via a classical Brownian motion (Bm), and named the process as fractional Brownian motion.

Definition 3.2.1. *The fractional Brownian motion (fBm) with Hurst index* $H \in (0,1)$ *is a centered Gaussian process* $B^H = (B^H_t)_{t\in[0,T]}$ *with* $B_0 = 0$

$$\mathbb{E}[B^H_t B^H_s] = Cov[B^H_t B^H_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \tag{3.1}$$

Remark 3.2.1. *It follows from* (3.1) *that:*

1. *For* $H = 1/2$, *the covariance function is* $\mathbb{E}[B^{1/2}_t B^{1/2}_s] = t \wedge s$ *this mean that* $B^{1/2} = B$, *a standard Wiener process(BM).*

2. *The covariance of increments of fBm is easily given by*

$$\mathbb{E}[(B^H_{t_1} - B^H_{s_1})(B^H_{t_2} - B^H_{s_2})] = \frac{1}{2}(|t_1 - s_2|^{2H} + |t_2 - s_1|^{2H} - |t_2 - t_1|^{2H} - |s_2 - s_1|^{2H}) \tag{3.2}$$

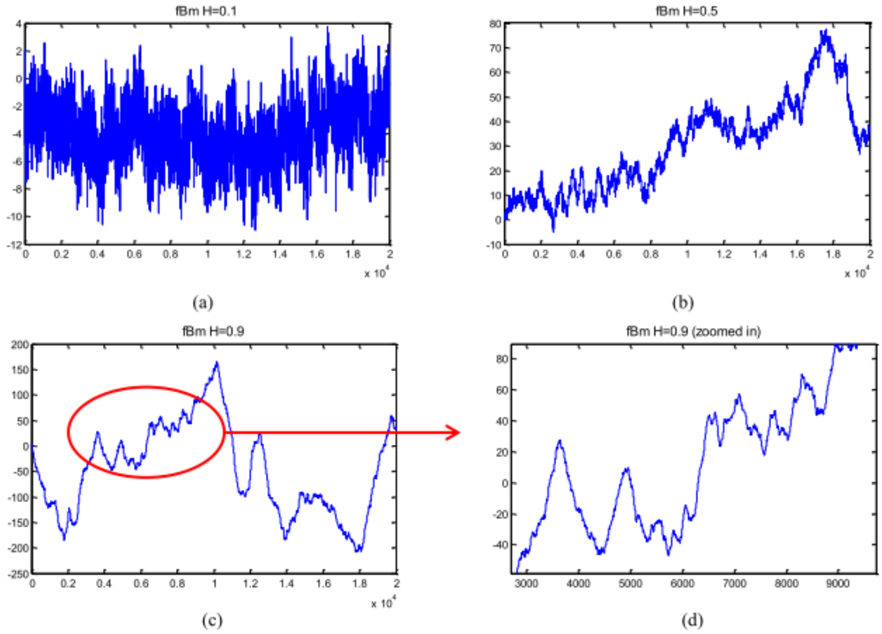

Figure 3.2: Sample paths of a fractional Brownian motion for different H.

3.2.1 The basic properties of fractional Brownian motion

3.2.1.1 Stationarity of the increments

Theorem 3.2.1. *Fractional Brownian motion has stationary increments.*

Proof :

Take a fixed $t \geq 0$ and consider the process $Y_t = B^H_{t+s} - B^H_s, t \geq 0$. It follows from (3.2) that the covariance function of Y is the same as that of B^H. Since the both processes are centered Gaussian, the equality of covariance functions implies means that Y has the same distribution as B^H . Thus, the incremental behavior of B^H at any point in the future is the same, for this reason B^H is said to have stationary increments.■

3.2.1.2 Self-similarity property of fBm

The fBm with Hurst parameter H is up to a constant, the only $H-$self-similar Gaussian process with stationary increments.

Theorem 3.2.2. *Fractional Brownian motion is $H-$self-similar.*

Proof :

Consider centred Gaussian processes $X_t = B^H_{at}$ and $Y_t = a^H B^H_t$. By applying the covariance structure(3.1) of fractional Brownian motion it is straightforward to see that X and Y have the same covariance functions from which the result follows.■

3.2.1.3 Hölder continiuity

Theorem 3.2.3. *(Kolmogorov-Chentsov continuity theorem, [14]) Assume that for a stochastic process $\{X_t, t \geq 0\}$ there exist such $K > 0, p > 0, \beta > 0$ such that for all $t \geq 0, s \geq 0$*

$$\mathbb{E}[|X_t - X_s|^p] \leq K|t-s|^{1+\beta}.$$

Then the process X has a continuous modification $(\tilde{X})$. Moreover, for any $\gamma \in (0, \beta/p)$ and $T > 0$ the process $\tilde{X}$ is γ-Hölder continuous on $[0,T]$, i.e.

$$\sup_{0 \leq s < t \leq T} \frac{|X_t - X_s|}{(t-s)^\gamma} < \infty.$$

Corollary 3.2.1. *The fractional Brownian motion B^H has continuous modification. Moreover, for any $\gamma \in (0, H)$ this modification is γ-Hölder continuous on each finite interval.*

Proof :

Since $B^H_t - B^H_s$ is centered Gaussian with variance $|t-s|^H$, we have

$$\mathbb{E}[|B^H_t - B^H_s|] = K_p|t-s|^{pH}.$$

Therefore, taking any $p > 1/H$, we get the existence of continuous modification. We also get the Hölder continuity of the modification with exponent $\gamma \in (0, H - 1/p)$. Choosing p sufficiently large, we arrive at the desired statement.■

3.2.1.4 Differentiability

As in the Brownian case, the fBm is a.s. nowhere differentiable. Effectively, we have the following proposition.

Proposition 3.2.1. *For $H \in (0,1)$, the fBm sample paths $B^{(H)}$ are non differentiable. Indeed, for every $t_0 \in [0, \infty]$,*

$$\mathbb{P}\left(\limsup_{t \to t_0} \left|\frac{B^{(H)}_t - B^{(H)}_{t0}}{t - t_0}\right| = \infty\right) = 1.$$

Proof :

Let us denote by $\mathfrak{B}_{t,t_0} = \frac{B_t^{(H)} - B_{t_0}^{(H)}}{t-t_0}$. Using the self-similarity property, we have:

$$\mathfrak{B}_{t,t_0} \stackrel{d}{=} (t-t_0)^{H-1} B_1^{(H)}.$$

Let us define $\mathfrak{U}(t,w) = \left\{ \sup_{0\leq s\leq t} \left| \frac{B_s^H}{s} \right| > d \right\}$. Then, for any $(t_n)_{n\in\mathbb{N}} \searrow 0$, we have

$$\mathfrak{U}(t_{n+1}, w) \subseteq \mathfrak{U}(t_n, w).$$

$$\mathbb{P}\left(\lim_{n\to\infty} \mathfrak{U}(t_n) \right) = \lim_{n\to\infty} \mathbb{P}(\mathfrak{U}(t_n)) \tag{3.3}$$

and

$$\mathbb{P}(\mathfrak{U}(t_n)) \geq \mathbb{P}\left(\left| \frac{B_{t_n}^{(H)}}{t_n} \right| > d = \mathbb{P}(|B_1^{(H)}| > t_n^{1-H} d) \right) \stackrel{n\to\infty}{\longrightarrow} 1 \tag{3.4}$$

■.

3.2.1.5 Not a semimartingale

Theorem 3.2.4. *$\{B^H(t) : t \geq 0\}$, for $H \neq 1/2$, is not semimartingale.*

Proof :

In fact, it is sufficient to compute p-variation of $B^{(H)}$. More precisely, we asserts that the index of p-variation of a fBm is $\frac{1}{H}$. Indeed, let us consider for fixed $p > 0$,

$$Y_{n,p} = \sum_{i=1}^{n} \left| B_{\frac{i}{n}}^{(H)} - B_{\frac{i-1}{n}}^{(H)} \right|^p n^{pH-1}$$

Now if we consider

$$\tilde{Y}_{n,p} = \sum_{i=1}^{n} \left| B_i^{(H)} - B_{i-1}^{(H)} \right|^p \frac{1}{n},$$

we have, by the self-similar property of the fBm, that $Y_{n,p} \stackrel{d}{=} \tilde{Y}_{n,p}$. Besides, remark that the sequence $(B_n^{(H)} - B_{n-1}^{(H)})_{n\in\mathbb{Z}}$ is stationary and ergodic. Therefore, we can use the ergodic theorem [2] and obtain that

$$\tilde{Y}_{n,p} \stackrel{L_1}{\to} \mathbb{E}(|B_1^{(H)}|^p) a.s., \qquad as \qquad n \to \infty$$

So that $Y_{n,p} \stackrel{D}{\to} \mathbb{E}(|B_1^{(H)}|^p)$ which implies, $Y_{n,p} \stackrel{\mathbb{P}}{\to} \mathbb{E}(|B_1^{(H)}|^p)$ Therefore,

$$V_{n,p} = \sum_{i=1}^{n} \left| B_{\frac{i}{n}}^{(H)} - B_{\frac{i-1}{n}}^{(H)} \right|^p \stackrel{\mathbb{P}}{\to} \begin{cases} 0 & if \quad pH > 1, \\ & \\ \infty & if \quad pH < 1, \end{cases} \qquad as \quad n \to \infty.$$

Then we showed that the index of p-variation is $\frac{1}{H}$. However, for a semimartingale, the index must be either in [0,1] either equal to 2, i.e. $\frac{1}{H} \in [0,1] \cup \{2\}$. But since $H \in (0,1), H^{-1} \in [0,1]$. Therefore, the fBm is a semimartingale only for $H = \frac{1}{2}$

3.2.1.6 Long-Range Dependence

The next property of fBm is the long-rang dependency, which is determined by correlation of increments. Recall that the mathematical definition of the long-rang dependence is:

Definition 3.2.2. *A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits long-range dependence if*

$$r_H(n) := Cov(X_k, X_{k+n})$$

satisfy

$$\lim_{n\to\infty} \frac{r_H(n)}{cn^{-\alpha}} = 1.$$

for some constant c and $\alpha \in (0,1)$.

The covariance function of fBm. It is defined as follow:

$$r_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),\ s,t \geq 0$$

Use Taylor expansion on

$$cov(B_s^H - B_{s-1}^H, B_{s+n}^H - B_{s+n-1}^H)$$

gives:

$$r_H(n) = \frac{1}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^2H] \sim H(2H-1)n^{2H-2}, \quad |n| \to \infty.$$

And therefor

1. *For* $H \in (0, \frac{1}{2}), \quad \sum_{n=1}^{\infty} |r_H(n)| < \infty.$
2. *For* $H \in (\frac{1}{2}, 1), \quad \sum_{n=1}^{\infty} |r_H(n)| = \infty.$

The fBm have a long-range dependence property when $H \in (\frac{1}{2}, 1)$, since

$$\lim_{n\to\infty} \frac{r_H(n)}{H(2H-1)n^{2H-2}} = 1.$$

3.2.2 Integral representation of fractional Brownian motion fractionaire

Now we show that the fractional Brownian motion can be represented as a stochastic integral

$$B_t^H = C \int K_H(t,u) dB_u$$

Where C is a standardized constant.

3.2.2.1 Representations of the FBm on $\mathbb{R}$

- **Moving Average Representation**

  The standard fractional Brownian motion as introduced by Mandelbrot and Van Ness is defined by the following moving average representation:

$$B^H(t) = \frac{1}{\Gamma(H+1/2)} \left\{ \int_{-\infty}^{0} [(t-u)^{H-1/2} - (-u)^{H-1/2} dB(u) + \int_0^t (t-u)^{H-1/2} dB(u) \right\} \tag{3.5}$$

  where $B(t)$ is the standard Brownian motion, Γ is the gamma function. Equation(3.5) can be written more compactly as

$$B^H(t) = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{\infty} [(t-u)_+^{H-1/2} - (-u_+)^{H-1/2}] dB(u). \tag{3.6}$$

- **Harmonizable Representation**

  There is another representation of the real-valued fBm using the complex-valued Brownian motion. In fact, for a fBm $(B_t^H)_{t\in\mathbb{R}}$, we obtain

$$B_t^H = \frac{1}{C_1(H)} \int_{\mathbb{R}} \frac{e^{itx}-1}{ix} \mid x \mid^{-(H-\frac{1}{2})} d\tilde{B}_x, \qquad t \in \mathbb{R},$$

  where $(\tilde{B}_t)_{t\in\mathbb{R}}$ is a complex Brownian measure and

$$C_1(H) = \left(\frac{\Pi}{H\Gamma(2H)\sin(H\Pi)} \right)^{1/2}.$$

Remark 3.2.2. *The complex Brownian measure on $\mathbb{R}$ can be splitted as $\tilde{B} = B_1 + iB_2$ and is such that $B_1(A) = B_1(-A), B_2(A) = -B_2(-A)$ and $\mathbb{E}(B_1(A))^2 = \frac{|A|}{2}, \forall A \in \mathcal{B}(\mathbb{R})$. We also call this representation, the spectral representation.*

3.2.2.2 Representions of FBm on a finite interval

- **Levy-Hida Representation**

The fBm admits a third representation as a Wiener integral of the form

$$B^H = \int_0^t K_H(t,s)dB_s, \qquad t \in [0,T]$$

Where $B = (B_t)_{t\in T}$ is a Wiener process, and $K_H(t,s)$ is the kernel

$$K_H(t,s) = C^H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}}F_1\left(\frac{t}{s}\right),$$

with C_H is a constant and

$$F_1(z) = C_H\left(\frac{1}{2}-H\right)\int_0^{z-1}\theta^{H-\frac{3}{2}}(1-(\theta+1)^{H-\frac{1}{2}})d\theta.$$

If $H > \frac{1}{2}$, the kernel K_H has the simpler expression

$$K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du.$$

where $t > s$ and $c_H = \left(\frac{H(H-1)}{\mathbf{B}(2-2H,H-\frac{1}{2})}\right)^{\frac{1}{2}}$.

3.2.3 Stochastic integration with respect to fractional Brownian motion

As same as the classical Brownian motion case, a definition is needed for

$$\int_0^t f(s,w)dB_s^H(w). \tag{3.7}$$

There are several ways to reach the goal. This mean that several ways of introducing a stochastic calculus with respect to the fBm are defined such as Wiener, divergence-type integral, Wick-Itô, Wick-Itô-Skorohod and Fractional Wick-Itô-Skorohod integrals for fBm. We refer the reader to [2] for further details. From the view of simulation, the pathwise integration is which makes most sense.

3.2.3.1 Pathwise integral for fractional Brownian motion

As we have just proved fBM is not a semimartingale, hence one cannot use standard Itô integration theory. A natural way to introduce a stochastic integral with respect to the fBM is to consider the Riemann sums:

$$\sum_{i=1}^{n} f(t_i)[B^H(t_{i+1} - B^H(t_i)]$$

where $0 = t_1 < ... < t_n = T$ is a partition of $[0,T]$ and then to investigate if the convergence holds at least in probability. For a more thorough study of different types of integration see [2]

Definition 3.2.3. *By the forward integral of a process $f : [0,T] \times \Omega \to \mathbb{R}$ with respect to B^H we mean:*

$$\int_0^T f(s)dB_s^H = \lim_{|\pi|\to 0} \sum_{i=1}^{n} f(t_i)[B^H(t_{i+1} - B^H(t_i)] \tag{3.8}$$

The fundamental question is when (upon what conditions) this integral exists. Young proved that the Riemann-Stieltjes integral can be extended to functions that are together smooth in the $p-$variation sense:

Theorem 3.2.5 ([9], Theorem 2.1)**.** *Suppose $f \in \mathcal{W}_p, g \in \mathcal{W}_q$ for some p and q such that $\frac{1}{p} + \frac{1}{q} > 1$ and have no common discontinuities. Then the Riemann-Stieltjes integral exists.*

With $\mathcal{W}_p$, is the Banach space of all functions with bounded $p-$variation equipped with the norm $\|f\|_{[p]} = \|f\|_{(p)} + \|f\|_\infty$ where $\|f\|_{(p)} = v_p(f)^{\frac{1}{p}}$ and $\|f\|_\infty = \sup_{t\in[0,T]} |f(t)|$. We may apply this theorem to the fBM and obtain:

Proposition 3.2.2 ([22], Theorem 6.2)**.** *Let $f : [0,T] \times \Omega \to \mathbb{R}$ be a stochastic process with sample paths in $\mathcal{W}_q$ a.s. with $q < \frac{1}{1-H}$. Then the integral*

$$\int_0^T f(s)dB_s^H$$

exists a.s.

Since we defined pathwise integrals in the Riemann-Stieltjes style, we have the classical change of variables formula:

Proposition 3.2.3 ([22], Theorem 6.4)**.** *Let $F \in C^{1,1}([0,T] \times \mathbb{R})$ such that the mapping $[0,T] \ni t \mapsto \frac{\partial F}{\partial x}(t, B_t^H) \in \mathbb{R}$ is in $\mathcal{W}_q$ for some $q < \frac{1}{1-H}$. Then the equation*

$$F(t.B_t^H) - F(s, B_s^H) = \int_s^t \frac{\partial F}{\partial x}(u.B_u^H)dB_u^H + \int_s^t \frac{\partial F}{\partial t}(u.B_u^H)du$$

holds a.s. for all $s, t \in [0,T]$.

The above can be regarded as an analogue of Ito's formula.

3.3 The Weighted Fractional Brownian Motion

As an extension of the Brownian motion, Bojdecki et al. [7] introduced and studied a rather special class of self-similar Gaussian processes which preserve many properties of the fractional Brownian motion. This process is called weighted fractional Brownian motion (weighted-fBm). More works for weighted-fBm can be found in Bojdecki [4, 5, 6], Yan-An [26, 25] and references therein.

3.3.1 Definition and basic properties

Definition 3.3.1. *The weighted-fBm* $B^{a,b} = \{B_t^{a,b}, 0 \leq t \leq T\}$ *with indices* a *and* b *is a mean zero Gaussian processes such that* $B_0^{a,b} = 0$ *and*

$$\mathbb{E}[B_t^{a,b} B_s^{a,b}] = Q(t,s) := \int_0^{s \wedge t} u^a[(t-u)^b + (s-u)^b]du \tag{3.9}$$

for $s, t \geq 0$.

Remark 3.3.1. 1. *For* $a = 0$ *the weighted-fBm* $B^{a,b}$ *reduces to the usual fractional Brownian motion with Hurst parameter* $\frac{1}{2}(b+1)$.

2. *For* $a = b = 0$, $B^{a,b}$ *reduces to the Brownian motion for (up to a multiplicative constant).*

Let us discussing conditions under which the function Q (symmetric, continuous) is the covariance of a stochastic process.:

$$Q(s,t) = \int_0^{s \wedge t} u^\alpha((t-u)^b + (s-u)^b)du, \qquad s, t \geq 0. \tag{3.10}$$

Theorem 3.3.1. *The function* Q *defined by* (3.10) *is positive-definite if and only if* a *and* b *satisfy the condition*

$$a > -1, \quad -1 < b \leq 1, \quad |b| \leq 1 + a. \tag{3.11}$$

Proof :

Firstly, we prove positive definiteness of Q in the case

$$a > -1, \quad -1 < b \le 0, \quad a + b + 1 \ge 0 \tag{3.12}$$

(see(3.11).) We have from (3.10)

$$Q(s,t) = Q_1(s,t) + Q_2(s,t),$$

where

$$Q_1(s,t) = \int_0^{s\wedge t} u^a((s\wedge t) - u)^b du, \qquad Q_2(s,t) = \int_0^{s\wedge t} u^a((s\vee t) - u)^b du.$$

It suffices to show that both Q_1 and Q_2 are positive-definite.

Q_1 can be written as

$$Q_1(s,t) = (s\wedge t)^{1+a+b} \int_0^1 u^a(1-u)^b du,$$

so it is positive definite by (3.12). Next, since $b \le 0$ we can write Q_2 as

$$Q_2(s,t) = \int_0^\infty u^a[(s-u)^b \wedge (t-u)^b]\mathbf{1}_{[0,s]}(u)\mathbf{1}_{[0,t]}(u)du,$$

hence positive definiteness of Q_2 follows easily.

Now assume that

$$a > -1, \quad 0 < b \le 1, \quad b \le 1 + a. \tag{3.13}$$

From (3.11), Q can be transformed into

$$Q(s,t) = b\int_0^s \int_0^t (u\wedge r)^a |u-r|^{b-1} dr du, \tag{3.14}$$

hence it is clear that Q is positive-definite for $b = 1$ (note that (3.13) implies $a \ge 0$ in this case). Assume $b < 1$; then, from (3.14),

$$Q(s,t) = b\int_0^s \int_0^t (u\wedge r)^a (u\vee r)^{b-1} \left(1 - \frac{u\wedge r}{u\vee r}\right)^{b-1} dudr.$$

Using the power series expansion of $(1-.)^{b-1}$ we obtain

$$Q(s,t) = \sum_{n=0}^{\infty} b\frac{\Gamma(1-b+n)}{\Gamma(1-b)n!} \int_0^s \int_0^t (u\wedge r)^a (u\vee r)^{b-1} \frac{(u\wedge r)^n}{(u\vee r)^n} dudr$$

Each summand is positive definite, since

$$(u\wedge r)^a(u\vee r)^{b-1}(u\wedge r)^n(u\vee r)^{-n}=u^ar^a(u^{b-1-a}\wedge r^{b-1-a})(u\wedge r)^n(u^{-n}\wedge r^{-n})$$

(we have used that $b-1-a\leq 0$, by(3.13)). Hence Q is positive-definite in this case.

We will show that for the remaining values of parameters Q is not positive-definite (recall that Q is infinite if either a or b is ≤ -1). More precisely, we will prove that there exists $t>0$ such that the covariance inequality

$$Q(1,t)\leq (Q(1,1)Q(t,t))^{1/2} \tag{3.15}$$

does not hold. Indeed, for $-1<b<0, a>-1, a+b+1<0$ and $t\searrow 0$, the left-hand side of (3.15) is of order t^{1+a+b} while the right-hand side is of order $t^{(1+a+b)/2}$. For $a>-1, b>a+1$ and $t\nearrow\infty$ the left-hand side of (3.15) is of order t^b and the right-hand side is of order $t^{(1+a+b)/2}$.

It remains to consider the case $1<b\leq a+1$. We show that (3.15) does not hold for $t=1+\varepsilon$, $\varepsilon\searrow 0$. Using convexity of the function x^b for $b>1$, we have

$$\begin{aligned}
Q(1,1+\varepsilon) &\geq 2\int_0^1 u^a\left(1+\frac{\varepsilon}{2}-u\right)^b du\\
&= 2\left(1+\frac{\varepsilon}{2}\right)^{a+b+1}\int_0^{(1+\varepsilon/2)^{-1}} u^a(1-u)^b du\\
&\geq 2\left(1+\varepsilon+\frac{\varepsilon^2}{4}\right)^{(a+b+1)/2}\left[\int_0^1 u^a(1-u)^b du-\left(\frac{\varepsilon}{2}\right)^{b+1}\frac{1}{b+1}\right].
\end{aligned}$$

This implies

$$\begin{aligned}
&Q(1,1+\varepsilon)-(Q(1,1)Q(1+\varepsilon,1+\varepsilon))^{1/2}\\
=\;& Q(1,1+\varepsilon)-2(1+\varepsilon)^{(a+b+1)/2}\int_0^1 u^a(1-u)^b du\\
\geq\;& 2\left[\left(\left(1+\varepsilon+\frac{\varepsilon^2}{4}\right)^{(a+b+1)/2}-(1+\varepsilon)^{(a+b+1)/2}\right)\int_0^1 u^a(1-u)^b du\right]\\
&-\frac{1}{2^b(b+1)}\left(1+\varepsilon+\frac{\varepsilon^2}{4}\right)^{(a+b+1)/2}\varepsilon^{b+1}\geq A\varepsilon^2-B\varepsilon^{b+1}
\end{aligned} \tag{3.16}$$

for some positive constants A, B. In the last estimate we have used the fact that $a+b+1 \geq 2b > 2$.

The right-hand side of (3.16) is strictly positive for e sufficiently small, since $b > 1$, so (3.15) does not hold for such ε.■

In the next theorem we collect the main properties of wfBm.

- **The basic properties of wfBm.**

Theorem 3.3.2. *The weighted fractional Brownian motion $B^{a,b}$ with parameters a and b has the following properties:*

1. ***Self-similarity:***

$$(B^{a,b}_{ct})_{t\geq 0} \stackrel{d}{=} (c^{1+a+b)/2} B^{a,b}_t)_{t\geq 0} \qquad for\, each \qquad c > 0.$$

2. ***Second moments of increments:*** *for* $0 \leq s < t$

$$\mathbb{E}(B^{a,b}_t - B^{a,b}_s)^2 = 2\int_s^t u^a (t-u)^b du, \tag{3.17}$$

- $$\mathbb{E}(B^{a,b}_t - B^{a,b}_s)^2 \leq C|t-s|^{b+1}, \tag{3.18}$$

  if $a \geq 0,\ s\,, t \leq T$ *for any* $T > 0$ *with* $C = C(T)$*, and also if* $a < 0, s, t \geq \varepsilon > 0$ *for any* $\varepsilon > 0$*, with* $C = C(\varepsilon)$*;*

- $$\mathbb{E}(B^{a,b}_t - B^{a,b}_s)^2 \leq C|t-s|^{1+a+b}, \qquad s, t \geq 0, \tag{3.19}$$

  if $a < 0, \quad 1 + a + b > 0$*;*

- $$\mathbb{E}(B^{a,b}_t - B^{a,b}_s)^2 \geq C|t-s|^{1+b} \tag{3.20}$$

  if $a > 0, s, t \geq \varepsilon > 0$ *for any* $\epsilon > 0$ *with* $C = C(\varepsilon)$ *and also if* $a \leq 0, s, t \leq T$ *for any* $T > 0$ *with* $C = C(T)$

3. ***Covariance of increments:*** *For* $0 \leq r < v \leq s < t$,

$$Q(r,v,s,t) = \mathbb{E}((B_t^{a,b} - B_s^{a,b})(B_v^{a,b} - B_r^{a,b})) = \int_r^v u^a[(t-u)^b - (s-u)^b]du, \quad (3.21)$$

hence

$$Q(r,v,s,t) = \begin{cases} > 0 & if \quad b > 0 \\ = 0 & if \quad b = 0 \\ < 0 & if \quad b < 0 \end{cases}$$

4. ***Asymptotic homogeneity:*** *The finite-dimensional distributions of the process* $(T^{-a/2}(B_{t+T}^{a,b} - B_T^{a,b}))_{t\geq 0}$ *converge as* $T \to \infty$ *to those of fBm with Hurst paramter* $(1+b)/2$, *multiplied by* $(2/(1+b))^{1/2}$.

5. ***Short and long-time asymptotics:***

$$\lim_{\varepsilon \searrow 0} \varepsilon^{-b-1}\mathbb{E}(B_{t+\varepsilon}^{a,b} - B_t^{a,b})^2 = \frac{2}{b+1}t^a, \quad (3.22)$$

$$\lim_{T\to\infty} T^{-1+a+b}\mathbb{E}(B_{t+T}^{a,b} - B_t^{a,b})^2 = 2\int_0^1 u^a(1-u)^b du. \quad (3.23)$$

Hence $B^{a,b}$ *has asymptotically stationary increments for long time intervals, but not for short time intervals.*

6. $B^{a,b}$ *is* ***not a semimartingale*** *if* $b \neq 0$.

7. $B^{a,b}$ *is* ***not Markov*** *if* $b \neq 0$.

Proof :

1. Self similarity follows from ((3.9), [7]).

2. Formula (3.17) is a direct consequence of (3.9), and (3.18) follows from (3.17). To prove (3.19), first observe that if $a < 0, b \geq 0$, (3.17) implies

$$\mathbb{E}(B_t^{a,b} - B_s^{a,b})^2 \leq \frac{2}{a+1}|t-s|^b|t^{a+1} - s^{a+1}| \leq \frac{2}{a+1}|t-s|^{1+a+b}.$$

Next assume that $a < 0, b < 0, 1+a+b > 0$. Fix any θ such that $|a|/(1+b) < \theta < 1$, and put $p = \theta/|a|$ and $q = \theta/(\theta - |a|)$. For $s < t$, the Hölder inequality applied to

(3.17) yields

$$\begin{aligned}\mathbb{E}(B_t^{a,b}-B_s^{a,b})^2 &\leq 2\left(\int_s^t u^{ap}du\right)^{1/p}\left(\int_s^t (t-u)^{bq}du\right)^{1/q}\\ &\leq C(t^{ap+1}-s^{ap+1})^{1/p}(t-s)^{(bq+1)/q}\\ &\leq C(t-s)^{1+a+b}\end{aligned}$$

since $0<ap+1<1$. The inequality (3.20) follows from (3.17).

3. Formula (3.21) follows from (3.9).

4. For $0\leq s\leq t$, by (3.17) and (3.21) we have

$$\begin{aligned}\frac{1}{T^a}E((B_{t+T}^{a,b}-B_T^{a,b})(B_{s+T}^{a,b}-B_T^{a,b})) &= \frac{1}{T^a}\int_T^{s+T} u^a((t+T-u)^b+(s+T-u)^b)du\\ &= \int_0^s\left(\frac{u}{T}+1\right)^a((t-u)^b+(s-u)^b)du\\ &\to \frac{1}{b+1}(t^{b+1}+s^{b+1}+s^{b+1}-(t-s)^{b+1})\end{aligned}$$

as $T\to\infty$ hence the assertion follows because $B^{a,b}$ is Gaussian.

5. For the proofs of (3.22) and (3.23), see [7].

6. The non-semimartingale property follows from (3.18), (3.20) and ([7], Corollary 2.1).

7. For $b\neq 0$ the covariance (3.10) does not have the triangular property, so $B^{a,b}$ is not Markovian (see [13], Proposition 13.7).

- **Hölder continuity**

Theorem 3.3.3. *Under the condition* $a>-1,\quad |b|<1,\quad |b|<1+a,$ *we have*

$$\mathbb{E}[B_t^{a,b}-B_s^{a,b})^2]\asymp(t\vee s)^a|t-s|^{b+1}$$

for $s,t\geq 0$. *In particular, we have*

$$\mathbb{E}[B_t^{a,b}-B_S^{a,b})^2]\leq C_{a,b}|t-s|^{a+b+1}$$

for $a\leq 0$

where,

- $x \vee y := \max\{x, y\}$

- $F \asymp G$ with the meaning that there are positive constants c_1 and c_2 so that $c_1 G(x) \leq F(x) \leq c_2 G(x)$ in the common domain of F and G.

Proof: For all $t > s > 0$ we have

$$Q(t,s) := [(B_t^{a,b} - B_s^{a,b})^2] = 2\int_s^t u^a(t-u)^b du$$

$$= 2t^{a+b+1}\int_{\frac{s}{t}}^1 T^a(1-r)^b dr$$

Consider the function

$$x \to f(x) = \int_x^1 r^a(1-r)^b dr, \quad x \in [0,1]$$

for all $a, b > -1$. We have

$$\lim_{x \to 1} \frac{f(x)}{(1-x)^{1+b}} = \frac{1}{1+b}$$

for all $a, b > -1$, which gives

$$\int_x^1 r^a(1-r)^b dr \asymp (1-x)^{1+b}. \quad x \in [0,1]$$

In particular, for $a \leq 0$ we have $(1-x)^{1+b} = (1-x)^{1+a+b}$.

Thus, Kolmogorov's continuity criterion implies that weighted-fBm is Hölder continuous of order d for any $\delta < 1 + b$.■

- **Long/short-range dependency:**

Theorem 3.3.4. *Let $B^{a,b}$ be a weighted-fBm with $a > -1, \quad -1 < b < 1$ and $|b| < 1 + a$*

- *If $b > 0$, then $B^{a,b}$ is long-range dependence;*

- *If $b < 0$, then $B^{a,b}$ is short-range dependence.*

Proof:

For any $\alpha > 0$ and $n \geq \alpha + 1$ we have

$$\rho_n(\alpha) = \mathbb{E}[(B_{\alpha+1}^{a,b} - B_\alpha^{a,b})(B_{n+1}^{a,b} - B_n^{a,b})].$$

$$= \int_{\alpha}^{\alpha+1} u^a[(n+1-u)^b - (n-u)^b]du.$$

If $b > 0$, we have

$$0 < (n+1-u)^b - (n-u)^b = (n+1-u)^b \left[1 - \left(1 - \frac{1}{n+1-u}\right)^b\right].$$

$$\sim (n+1-u)^{b-1},$$

for all $\alpha \leq u \leq \alpha + 1$, and

$$\rho_n(\alpha) \sim \int_{\alpha}^{\alpha+1} u^a(n+1-u)^{b-1}du \geq \frac{1}{a+1}((\alpha+1)^{1+a} - \alpha^{1+a})(n+1-\alpha)^{b-1},$$

which deduces:

$$\sum_{n\geq\alpha} \rho_n(\alpha) = \infty,$$

if $b < 0$, we have

$$0 < (n-u)^b - (n+1-u)^b = (n-u)^b \left[1 - \left(1 + \frac{1}{n-u}\right)^b\right] \sim (n-u)^{b-1},$$

for all $\alpha \leq u \leq \alpha + 1$,and

$$|\rho_n(\alpha)| \sim \int_{\alpha}^{\alpha+1} u^a(n-u)^{b-1}du \leq \frac{1}{a+1}((\alpha+1)^{1+a}) - \alpha^{1+a}(n-\alpha-1)^{b-1},$$

which deduces the following sum

$$\sum_{n\geq\alpha} |\rho_n(\alpha)| < \infty.$$

This completes the proof.■

3.4 Application

Since a financial system is a complex system with great flexibility, investors do not make their decisions immediately after receiving the financial information, but rather wait until information reaches to its threshold limit value. This behavior can lead to the features of "asymmetric leptokurtic" and "long/short memory". The weighted fractional Brownian motion may be a useful tool for capturing this phenomenon.

3.4.1 A Weighted-fractional (Merton weighted fractional) model to European option pricing

In this section we consider the following dynamics for V :

$$dV_t = \mu V_t dt + \sigma V_t dB_t^{a,b}, \tag{3.24}$$

where $B_t^{a,b}$ denotes a weighted fractional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Stochastic integration in (3.24) is of divergence-type [21]. Let $\Omega = C_0(0, T; \mathbb{R})$ be the Banach space of a real-valued continuous function on $[0, T]$ with the initial value zero and the super norm.

In what follows we model long-range dependence of financial assets under the assumption $b > 0$, and denote by $\phi(.)$ the cumulative probability distribution function of a standard normal random variable:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{1}{2}u^2) du$$

and by $\varphi(.) = \phi'(.)$ the density function.

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Consider a financial market in which we have two securities: a bond (Security 1) with (instantaneous) interest rate which is also interpreted as the risk-free rate of interest, and a stock (Security 2) which is described by the stochastic price process (pay-out) V_t at time t. A time interval $[0, T]$ considered with 0 being the initial or present time and T being the terminal time. The price of Security 2 is denoted by V_0. We are interested in calculating the pricing of a European call option $C(K, T)$, say, written on Security 2 with strike price K and time to maturity T .

Definition 3.4.1. *The value $\{V_t\}$ results in an expected (instantaneous) rate of return μ and T is defined as*

$$e^{\mu T} = \frac{\mathbb{E}[V_T]}{V_0}$$

Since nothing has been assumed about the process $\{V_t\}$, μ will in general depend on T.

Lemma 3.4.1. *(M. Bladt, [3])*

The fair premium, and hence the call option price, $C(K,T)$, of a European call option with time to maturity T and strike price K is given by

$$C(K,T) = \mathbb{E}[(e^{-\mu T}V_T - e^{-rT}K)\mathbf{1}_{\{e^{-\mu T}V_T > e^{-rT}K\}}]$$

and the put option price, $P(K,T)$, of a European put option with time to maturity T and strike price K is given by

$$P(K,T) = \mathbb{E}[(e^{-rT}K - e^{-\mu T}V_T)\mathbf{1}_{\{e^{-\mu T}V_{T<e^{-rT}K}}\}].$$

According to Alos et al [1] (see also Yan-An [25]), we have the following.

Lemma 3.4.2. *[21]The solution to Equation* (3.24) *is given by*

$$V_t = V_0 exp\left(\mu t - \sigma^2 \int_0^t u^a(t-u)^b du + \sigma B_t^{a,b}\right). \tag{3.25}$$

Theorem 3.4.1. *The fair premium, and hence the call option price, $C(K,T)$, of a European call option with time to maturity T and strike price K, is given by*

$$C(K,T) = V_0\phi(d_1) - Ke^{-rT}\phi(d_2),$$

where

$$d_1 = \frac{ln\frac{V_0}{K} + rT + \sigma^2 \int_0^T u^a(T-u)^b du}{\sigma\sqrt{2\int_0^T u^a(T-u)^b du}}$$

and

$$d_2 = \frac{ln\frac{V_0}{K} - rT - \sigma^2 \int_0^T u^a(T-u)^b du}{\sigma\sqrt{2\int_0^T u^a(T-u)^b du}}$$

Proof:

Fix $T > 0$, for $t \in [0,T]$, the weighted fractional Brownian motion $B_t^{a,b}$ is a centered Gaussian process with variance $2\int_0^T u^a(T-u)^b du$. According to (3.25), we have

$$\frac{V_t}{V_0} = exp\left(\mu t - \sigma^2 \int_0^t u^a(t-u)^b du + \sigma B_t^{a,b}\right).$$

Then

$$log\frac{V_t}{V_0} = \mu t - \sigma^2 \int_0^t u^a(t-u)^b du + \sigma B_t^{a,b}.$$

which means $log\frac{V_t}{V_0}$ is a Gaussian process with mean $\mu s - \sigma^2 \int_0^t u^a(t-u)^b du$ and variance $2\sigma^2 \int_0^t u^a(t-u)^b du$. The distribution of V_T at T is in fact the only thing we need since only the price at the terminal date matters. Then noticing that $e^{-\mu T}V_T > e^{-rT}K$ is equivalent to

$$S_T > \frac{log\frac{K}{V_0} + \sigma^2 \int_0^T u^a(T-u)^b du - rT}{\sigma}$$

From the Lemma 3.4.1, the call option price, $C(K,T)$, of a European call option with time to maturity T and strike price K is given by

$$C(K,T) = \mathbb{E}[(e^{-\mu T}V_T - e^{-rT}K)\mathbf{1}_{\{e^{-\mu T}V_T > e^{-rT}K\}}].$$

First get that with

$$y = \frac{log\frac{k}{V_0} + \sigma^2 \int_0^T u^a(T-u)^b du - rT}{\sigma},$$

$$\begin{aligned}
&\mathbb{E}[e^{\mu T}V_T\mathbf{1}_{\{e^{-\mu T}V_T > e^{-rT}K\}}] \\
&= e^{-\mu T}\int_y^\infty V_0 e^{\mu T - \sigma^2\int_0^T u^a(T-u)^b du + \sigma x}\frac{1}{\sqrt{4\pi\int_0^T u^a(T-u)^b du}}e^{-\frac{x^2}{4\int_0^T u^a(T-u)^b du}}dx \\
&= V_0\int_y^\infty \frac{1}{\sqrt{4\pi\int_0^T u^a(T-u)^b du}}e^{-\frac{(x-2\sigma\int_0^T u^a(T-u)^b du)^2}{4\int_0^T u^a(T-u)^b du}}dx \\
&= V_0\mathbb{P}(Z > y),
\end{aligned}$$

where $Z \sim \mathcal{N}(2\sigma\int_0^T u^a(T-u)^b du, 2\int_0^T u^a(T-u)^b du)$. Furthermore

$$\mathbb{P}(Z > y) = \phi\left(\frac{ln\frac{V_0}{k} + rT + \sigma^2\int_0^T u^a(T-u)^b du}{\sigma\sqrt{2\int_0^T u^a(T-u)^b du}}\right)$$

On the other hand

$$\mathbb{E}[e^{\mu T}V_T\mathbf{1}_{\{e^{-\mu T}V_T > e^{-rT}K\}}] = e^{-rt}K\phi\left(\frac{ln\frac{V_0}{k} + rT - \sigma^2\int_0^T u^a(T-u)^b du}{\sigma\sqrt{2\int_0^T u^a(T-u)^b du}}\right)$$

Then the proof of this theorem is complete.■

Corollary 3.4.1. *The put option price, $P(K,T)$ of an European put option with time to maturity T and strike price K is given by*

$$P(K,T) = Ke^{-rT}\phi(-d_2) - V_0\phi(-d_1).$$

Proof: The proof can be found in [21]

Conclusion

Our interest through this work has been focused on the fractional calculus and a class of fractional stochastic processes, we begin by giving a very brief overview on stochastic calculus theory in the first chapter. The second chapter presented some frequently used differintegral operators in fractional calculus theory with application to the respiratory system.

We extend the fractional paradigm from calculus to stochastic processes in the last chapter by studying the weighted fractional Brownian motion (wfBm). After recalling some basic notions such as the self-similarity, the long range dependency, we have studied the fractional Brownian motion by giving its essential properties, representations and stochastic integration with respect to it. We have concluded this master thesis by introducing the weighted fractional Brownian motion (wfBm), we showed that this process is made up of self self-similar, long-range dependence, Gaussian fractional processes which depends on two real parameters a, b. It includes fractional Brownian motion when $a = 0$, standart Brownian motion when $a = b = 0$. Then some of the basic properties of this process were discussed. The most ones, which are analogous to those of fBm, are self-similarity, path continuity, behavior of increments and long-range dependence. $B^{a,b}$ is neither a semi-martingale nor a Markov process unless $b = 0$. Although, the wfBm $B^{a,b}$ has not stationary increments in general. wfBm widens the scope of behavior of fBm, it may be useful in some domain of applications such as finance as we have seen, more precisely the option pricing problem in the weighted fractional Brownian motion model.

This can be extended to several directions in the future. Other fractional Gaussian/non gaussian processes can be studied, stochastic integration, stochastic differential equations driven by one of them and more of their application can be viewed in the future.

Bibliography

[1] E. Alos, O. Mazet, D. Nualart, Stochastic calculus with respect to Gaussian processes, *Ann. Probab.* 29(2001)766-801.

[2] F. Biagini, Y. Hu, B. Øksendal and T. Zhang, *Stochastic Calculus for fBm and Applications* , Springer, 2008.

[3] M. Bladt, T.H. Rydberg, An actuarial approach to option pricing under the physical measure and without market assumptions, Insurance: *Mathematics and Economics*, 22 (1998) 65-73.

[4] T. Bojdecki, L.G. Gorostiza, A. Talarczyk, Fractional Brownian density process and its self-intersection local time of order k, J. *Theor. Probab*, 17 (2004) 717-793.

[5] T. Bojdecki, L. G. Gorostiza, A. Talarczyk, Occupation time limits of inhomoge- neous Poisson systems of independent particles, *Stoch. Proc. Appl*, 118 (2008) 28-52.

[6] T. Bojdecki, L. G. Gorostiza, A. Talarczyk, Self-similar stable processes arising from high density limits of occupation times of particle systems, *Potential Anal*, 28(2008) 71-103.

[7] T. Bojdecki, L. G. Gorostiza, A. Talarczyk, Some extensions of fractional Brownian motion and sub-fractional Brownian motion related to particle systems, *Elect. Comm. Probab*, 12 (2007) 161-172.

[8] L. Decreusefond, A. S.$\ddot{U}$st$\ddot{u}$nel, Stochastic analysis of the fractional Brownian motion, *Potential Anal*, 10 (1999) 177-214.

[9] R. M. Dudley, R. Norvaisa, An introduction to $p-$variation and Young integrals. *MaPhySto*, 1998.

[10] P. Embrechts and M. Maejima, *Self-Similar Processes*, Princeton Univ. Press, 2002.

[11] H.A. Fallahgoul, S. M. Focardi, F. J. Fabozzi , *Fractional calculus and fractional processes with applications to financial economics*, Elsevier , 2017.

[12] C. M. Ionescu, The Human Respiratory System, Springer, 2013

[13] O. Kallenberg, *Foundations of Modern Probability*, Springer, 2002.

[14] F. Klebaner, *Introduction to Stochastic Calculus with Applications*, Imperial College Press, 2012. measure structure, *J. Funct. Anal.* 249 (2007) 92-142.

[15] Y. S. Mishura, *Stochastic calculus for fractional Brownian motion and related processes*, Springer, 2008.

[16] I. Nourdin, *Selected Aspects of fractional Brownian motion,* Springer, 2012.

[17] D. Nualart, *Malliavin Calculus and Related Topics,* Springer, 2006.

[18] K. Oldham , J. Spanier *The fractional calculus.* Academic Press, 1974.

[19] F. Russo, P. Vallois, Stochastic calculus with respect to a finite quadratic variation process, *Stochastics stochastics rep.* 70 (2000) 1-40.

[20] G. Samorodnitsky, M. Taqqu, *Stable Non-Gaussian Random Variables,* Chapman and Hall, 1994.

[21] X. Sun, L. Yan, H. Jing, A Weighted-fractional model to European option pricing, *Theoretical Mathematics & Applications*, 2(2012) 87-99.

[22] T. Sottinen, Fractional Brownian motion in finance and queueing, Ph.D. Thesis, Helsinki, 2003

[23] C. A. Tudor, *Analysis of Variations for Self-Similar Processes,* Springer, 2013.

[24] M .S. Taqqu, *Self-similarity and long-range dependence*, IMPA, 2006.

[25] L. Yan, L. An, The Itô formula for weighted fractional Brownian motion, submitted 2012.

[26] L. Yan, Z. Wang, H. Jing, Some path properties of weighted fractional Brownian motion, *Journal of Probability and Stochastic Processes,* 86 (2014) 721-758.