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par

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Thème :

On some fractional derivatives

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Dedications

I dedicate this modest work

♡ To my grandmother

♡ To my mother

♡ To my father

♡ To my husband

♡ To my mother-in-law

♡ To my sister

♡ To my aunt

♡ To my fetus

♡ To all my big family

Fatima

Contents

0.1 Introduction

Fractional calcus is a new power full tool which has been recently employed to model complex systems with non-linear in spite of its complicated mathemathical background, fractional calculus came into being of some simple questions which were related to the derivation concept; such question as while the first order derivative represent the slope of a function reveal about it? Finding answers to such question, scientists managed to open a new window of opportunity to mathematical and real world, which has arisen many new question and intriguing result. For example, the fractional order derivative of a constant function, unlik the ordinary derivative, is note always zero.[16]

history of Fractional Calculus

Fractional Calculus was born on 30^{th}september,1695 by question of G.F.A de $L'Hospital(1661-1704)$ to G.W $Leibniz(1646-1716)$ for the derivative "Wath if the order will be $n=\frac{1}{2}$",In that year Leibniz in a letter replied: "It will lead to a paradox, from which one day useful consequences will be drawn." The issue raised by Leibniz for a fractional derivative (semi-derivative, to be more precise) was an ongoing topic in decades to come Following L'Hopital's and Liebniz's first inquisition, fractional calculus was primarily a study reserved for the best mathematical minds in Europe. Euler wrote in 1730:

> "When n is a positive integer and P function of x,$P=P(x)$,The ratio of $\frac{d^p}{dx^n}$ can be always be expressed algebraically. But what kind of ratio can then be made if n be a fraction?"

Subsequent references to fractional derivatives were made by Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, Weyl in 1919, and other. During 19^{th} century the theory of fractional calculus was developed primarily in this way, trough insight and genius of great mathematicians. Namely, in 1819 Lacroix, gave the correct answer to the problem raised by Leibnitz and L'Hospital for the first time. Lacroix developed the formula for n-th derivative of $y=x^m$, with m being a positive integer and he Replaced the factorial symbol by Gamma function

$$D_x^\alpha x^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}.$$

In particular, Lacroix calculated

$$D_x^{1/2}x = \frac{\Gamma(2)}{\Gamma(3/2)}x^{1/2}.$$

Surprisingly, the previous definition gives a nonzero value for the fractional derivative of a constant function
($\beta = 0$), since

$$D_x^{\alpha}1 = D_x^{\alpha}x^0 = \frac{1}{\Gamma(1-\alpha)}x^{-\alpha} \neq 0.$$

Using linearity of fractional derivatives, the method of Lacroix is applicable to any analytic function by term-vise differentiation of its power series expansion. Unfortunately, this class of functions is too narrow in order for the method to be considered general.[15]

Over the years, many mathematicians, using their own notation and approach, have found various definitions that fit the idea of a non-integer order integral or derivative. One version that has been popularized in the world of fractional calculus is the Riemann Liouville definition. It is interesting to note that the Riemann-Liouville definition of a fractional derivative gives the same result as that obtained by Lacroix in equation . Since most of the other definitions of fractional calculus are largely variations of the Riemann-Liuoville version.[12]

The subject of fractional calculus has applications in diverse and widespread fields of engineering and science such as electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics, and signals processing. It has been used to model physical and engineering processes that are found to be best described by fractional differential equations. The fractional derivative models are used for accurate modelling of those systems that require accurate modelling of damping. In these fields, various analytical and numerical methods including their applications to new problems have been proposed in recent years. This special issue on "Fractional Calculus and its Applications in Applied Mathematics and Other Sciences" is devoted to study the recent works in the above fields of fractional calculus done by the leading researchers. The papers for this special issue were selected after a careful and studious peer-review process.

Mathematical modelling of real-life problems usually results in fractional differential equations and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one or more variables. In addition, most physical phenomena of fluid dynamics, quantum mechanics, electricity, ecological systems, and many other models are controlled within their domain of validity by fractional order PDEs. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving fractional order PDEs and the implementations of these methods.

At present, the use of fractional order partial differential equation in real-physical systems is commonly encountered in the fields of science and engineering. The efficient computational tools are required for analytical and numerical approximations of such physical models. The present issue has addressed recent trends and developments regarding the analytical and numerical methods that may be used in the fractional order dynamical systems. Eventually, it may be expected that the present special issue would certainly helpful to explore the researchers with their new arising fractional order problems and elevate the efficiency and accuracy of the solution methods for those problems in use nowadays.

This memory contains three chapters in which we summaries the fractional differentiation.

The first chapter will be devoted to the basic elements of fractional calculus, Some special function his propriéties and exemple used in this work.
the second chapter we present some approaches to fractional integrals and fractional derivatives illustrated by examples.
the third chapter is dedicated to the different extensions of the gamma and betta functions with applications; and extension of the fractional derivative of Riemman-Liouville

Chapter 1

Some special function

Special Function are found to be of particular importance in mathematical analysis or in other applications. This definition possibly vague, since there is no consensus a general definition of special, but there exists a common agreement on a large amount of them, like those reported here, appear anther as integrals of some elementary function or as solution of deferential equation.The following description of some kinds of special function is limited to those used in this subject.

1.1 The Gamma function

The gamma function was first introduction by the Swiss mathematician Leonhard Euler (1707−1783) in his goal to generalize the factorial to non integer values. Later, because of its great importance, it was studied by other eminent mathematician like Adrien-Marie Legendre(1752−1833), Carl Friedrich Gauss(1777 − 1855), Christoph Gudermann(1798 − 1852), Joseph Liouville (1822 − 1897), Charler Hermite(1822 − 1901), as well as many others.[22]

Definition 1.1.1. [*22*]

The most basic interpretation of the Gamma function is simply the generalization of the factorial for all real numbers. Its definition is given by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x \in C, (Re(x) > 0).$$

Some properties of the Gamma function:

1. Finite Difference Formula

$$\Gamma(x+1) = x\Gamma(x), \quad x \in R.$$

2.
$$\Gamma(n+1) = n!, \quad n \in N.$$

3.
$$\Gamma(0) = \infty.$$

4.
$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n n!}, \quad n \in N.$$

proof .1.

1. *For* $x \in R$ *we have*

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty e^{-t}t^x dt \\ &= \lim_{p\to\infty} \int_0^p e^{-t}t^x dt \\ &= \lim_{p\to\infty} \left[(-t^x \exp^{-t}) + \int_0^p xt^{x-1}e^{-t}dt\right] \\ &= \lim_{p\to\infty} \left[(-p^x \exp^{-p}) + x\int_0^p t^{x-1}e^{-t}dt\right] \\ &= x\int_0^\infty t^{x-1}e^{-t}dt \\ &= x\Gamma(x)\end{aligned}$$

2. *we have* $\Gamma(1) = 1$
For $n = 1$, $\Gamma(1+1) = 1\Gamma(1) = 1!$
For $n = 2$, $\Gamma(2+1) = 2\Gamma(2) = 2!$
The hypothese $\Gamma(x+1) = x!$
proof that $\Gamma(x+2) = (x+1)!$
we have

$$\begin{aligned}\Gamma(x+2) &= (x+1)\Gamma(x+1) \\ &= (x+1)x! \\ &= (x+1)!\end{aligned}$$

3. *We have* $\Gamma(1) = 1$ *and* $\Gamma(n+1) = n\Gamma(n)$
So, $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

put, $n = 0$

$$\begin{aligned}\Gamma(0) &= \frac{\Gamma(0+1)}{0} \\ &= \frac{1}{0} \\ &= \infty.\end{aligned}$$

4. *definition by recurrence:*

   (a) *for* $n = 0$

$$\Gamma(0+\frac{1}{2}) = \frac{(0)!\sqrt{\pi}}{4^0 0!} = \sqrt{\pi}.$$

   so, for $n = 0$ *is true*

   (b) $\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$ *is true proof about* $(n+1)$

$$\Gamma(n+1+\frac{1}{2}) = (n+\frac{1}{2})\Gamma(n+\frac{1}{2})$$

$$= (n+\frac{1}{2})\frac{(2n)!}{4^n n!}\sqrt{\pi}$$

$$= \frac{(2n+1)(2n)!}{2*4^n n!}\sqrt{\pi}$$

$$= \frac{(2n+1)!}{2*4^n n!}\sqrt{\pi}$$

$$= \frac{(2(n+1)-1)!}{2*4^n n!}\sqrt{pi}$$

$$= \frac{(2(n+1))!}{(2(n+1))2*4^n n!}\sqrt{pi}$$

$$= \frac{(2(n+1))!}{4^{n+1}(n+1)!}\sqrt{pi}.$$

   So, by recurrence $\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$ *is true*

Example 1.1.1.

$$\Gamma(1) = 1.$$

$$\Gamma(1) = \int_0^{+\infty} exp(-t)dt$$

$$\Gamma(1) = -exp(-t)|_0^{+\infty}$$

$$\Gamma(1) = 1$$

Example 1.1.2.

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} exp(-t)t^{(1/2-1)}dt$$

If we let $t = y^2$ *, then* $dt = 2ydy$ *and we now have*

$$\Gamma(\frac{1}{2}) = 2\int_0^{+\infty} exp(-y^2)dy \tag{1.1}$$

we can write:

$$\Gamma(\frac{1}{2}) = 2\int_0^{+\infty} exp(-x^2)dx \tag{1.2}$$

If we multiply together (1.1) and (1.2) we get

$$[\Gamma(\frac{1}{2})]^2 = 4\int_0^{+\infty}\int_0^{+\infty} exp(-x^2-y^2)dxdy$$

with polar coordinates :

$$[\Gamma(\frac{1}{2})]^2 = 4\int_0^{\frac{\pi}{2}}\int_0^{+\infty} exp(-r^2)drd\theta = \pi$$

So,

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Example 1.1.3.

$$\begin{aligned}\Gamma(\frac{3}{2}) &= \frac{\sqrt{\pi}}{4}\\ &= \Gamma(\frac{1}{2}+1)\\ &= \frac{1}{2}\Gamma(\frac{1}{2})\\ &= \frac{\sqrt{\pi}}{2}\end{aligned}$$

1.2 The Beta function:

The beta function is a unique function where it is classified as the first kind of Euler's integral. The beta function is defined in the domain of real number. The beta function is meant by B(p,q).

Definition 1.2.1. [22]
the Beta function is defined by a definite integral. Its definition is given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x,y \in C, (Re(x), Re(y) > 0.$$

Remark:

The Beta function can also be defined in terms of the Gamma function: for $(Re(x), Re(y) > 0)$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y \in C.$$

Lemma 1.2.1. [22]
The Beta function is symmetric which means the order of its parameters does not change the outcome of the operation.

$$B(x,y) = B(y,x)$$

proof .2. *we have*

$$\int_0^a f(x)dx = \int_0^a f(1-x)dx$$

so

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}dt \\ &= \int_0^1 (1-t)^{x-1}(1-(1-t))^{y-1}dt \\ &= \int_0^1 (1-t)^{x-1}(t)^{y-1}dt \\ &= B(y,x). \end{aligned}$$

Some properties of the Beta function:[22]

1. $B(x,y) = B(x,y+1) + B(x+1,y)$.

2. $B(x,y+1) = B(x,y).[\frac{y}{x+y}]$.

3. $B(x+1,y) = B(x,y).[\frac{x}{x+y}]$.

4. $B(x,y).B(x+y,1-y) = \frac{\pi}{x} sin(\pi y)$.

5. The important integrals of beta functions are:

   (a) $B(x,y) = \int_0^\infty \frac{t^{n-1}}{(1+t)^{x+y}} dt$

   (b) $B(x,y) = 2\int_0^{\frac{\pi}{2}} sin^{2x-1}\theta cos^{2y-1} d\theta$

Example 1.2.1.

$$\begin{aligned} B(\frac{1}{2},\frac{1}{2}) &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} \\ &= \frac{\sqrt{\pi}\sqrt{\pi}}{1} \\ &= \pi \end{aligned}$$

Example 1.2.2.

$$\begin{aligned} B(2,3) &= \int_0^1 t(1-t)^2 dt \\ &= \int_0^1 (t-2t+t^3) dt \\ &= \frac{1}{12} \end{aligned}$$

The Beta function can also be defined in terms of the Gamma function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, x,y \in R^+.$$

1.3 The Mittag-Leffler function

The Mittag-Leveler function is named after a Swedish mathematician who defined and studied it in 1903 [11]. The function is a direct generalization of the exponential function, and it plays a major role in fractional calculus. The one and two-parameter

Definition 1.3.1. *(Mittag-Leffler function to one parameter)*
The classical Mittag Leffler function to one parameter α, is defined by the expression

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}; \; z \in \mathbb{C}, \; Re(z) > 0.$$

Example 1.3.1. [*18*]

$E_1(z) = e^z$,

$E_2(z) = cosh(\sqrt{z})$.

Definition 1.3.2. *(Mittag-Leffler function to two parameters)*
The classical Mittag Leffler function to two parameters α, β is given by the expression:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}; \; z \in \mathbb{C}, \; \beta \in \mathbb{C}, \; Re(\alpha) > 0.$$

Example 1.3.2. [*9*]

1. $E_{\alpha,1}(z) = E_\alpha(z)$.
2. $E_{1,2}(z) = \dfrac{e^z - 1}{z}$.
3. $E_{2,2}(z) = \dfrac{sinh(\sqrt{z})}{\sqrt{z}}$.
4. $E_{\alpha,\beta}(z) + E_{\alpha,\beta}(-z) = 2E_{2\alpha,\beta}(z^2)$.
5. $E_{\alpha,\beta}(z) - E_{\alpha,\beta}(-z) = 2zE_{2\alpha,\alpha+\beta}(z^2)$.

1.4 The Mellin-Ross Function

The Mellin-Ross function $E_t(v, a)$, arises when finding the fractional integral of an exponential $e^{(at)}$. The function is closely related to both the incomplete Gamma and Mittag-Leffler functions. Its definition is given by

Definition 1.4.1.

$$E_t(v,a) = t^v e^{(at)}\Gamma^*(v,t)$$

We can also write,

$$E_t(v,a) = t^v \sum_{\infty}^{k=0} \frac{(at)^k}{\Gamma(k+v+1)}$$
$$= t^v E_{1,v+1}(at)$$

1.5 Macdonald Function

Modified cylinder function, Bessel function of imaginary argument
A function

$$K_s(z) = \frac{\pi}{2}\frac{I_{-S}(z) - I_S(z)}{sin(s\pi)}$$

Where s is an arbitrary non-integral real number and

$$I_s(z) = \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{s+2m}}{m!\Gamma(s+m+1)}$$

is a cylinder function with pure imaginary argument; They have been discussed by H.M. Macdonald [15]
If n is an integer, then

$$K_n(z) = \lim_{s\to n} K_s(z)$$

The Macdonald function $K_s(z)$ is the solution of the differential equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - (z^2 - s^2)y = 0.$$

Properties 1.5.1. [15]

1. *Series and asymptotic representations are:*

$$K_{n+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{r=0}^{n} \frac{(n+r)!}{r!(n-r)!(2z)^r}$$

Where n is a non-negative integer:

$$K_0(z) = -ln(\frac{z}{2})I_0(z) + \sum_{m=0}^{\infty}(\frac{z}{2})^{2m}\frac{1}{(m!)^2}\psi(m+1)$$

$$\psi(1) = -C, \quad \psi(m+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{m} - C$$

Where $C = 0.5772157\ldots$ *is the Euler constant;*

$$K_s(z) = \frac{1}{2}\sum_{m=0}^{s-1} \frac{(-1)^m (s-m-1)!}{m!(\frac{z}{2})^{s-2m}} + \tag{1.3}$$

$$+ (-1)^{s-1} \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{s+2m}}{m!(s+m)!} \left[ln(\frac{z}{2}) - \frac{\psi(m+1) + \psi(s+m+1)}{2} \right]. \tag{1.4}$$

where $n \geq 1$ *is integer,*

$$K_v \sim (\frac{\pi}{2z})^{\frac{1}{2}} e^{-z} \left[1 + \frac{4v^2 - 1^2}{1!8z} + \frac{(4v^2-1^2)(4v^2-3^2}{2!(8z)^2} + ... \right],$$

for large z *and* $|argz| < \frac{\pi}{2}$.

2. *Recurrence formulas:*

$$K_{v-1}(z) - K_{v+1}(z) = -\frac{2v}{z} K_v(z),$$

$$K_{v-1}(z) + K_{v+1}(z) = 2\frac{dK_v(z)}{dz}$$

1.6 hypergeometric functions

In this part, we give definitions and some properties of the hypergeometric functions, (see [24] for details).
The second order linear differential equation

$$z(1-z)\frac{d^2y}{dz^2} + [c - (a+b+1)z]\frac{dy}{dz} - aby = 0$$

where a, b and c are complex parameters, is called hypergeometric equation. The solutions (as series expansion) of the hypergeometric equation are valid in the neighborhood of $z = 0$, 1 or ∞. Thus, if c is not an integer, the general solution of differential equation is valid in a neighborhood of the origin and can be given by :

$$y = A_2\ F_1(a, b; c; z) + Bz^{1-c}{}_2F_1(a - c + 1, b - c + 1; 2 - c; z)$$

where A and B are arbitrary constants, and

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1+\frac{ab}{c.1}z+\frac{a(a+1)b(b+1)}{c(c+1).1.2}z^2+..... \\ &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c_n)}\frac{z^n}{n!} \end{aligned}$$

$$(c \neq 0,-1,-2,.....)$$

and $(\lambda)_v$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 \equiv 1 \; and \; (\lambda)_v := \frac{\Gamma(\lambda+v)}{\Gamma(\lambda)}$$

Hence

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

is called Gauss hypergeometric function. This serie is convergent for $|z|<1$ where $\Re_e(c)>\Re_e(b)>0$ and $|z|=1$ where $\Re_e(c-a-b)>0$.
The Gauss hypergeometric function can be given by Euler's integral representation as follows:

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

$$(|z|<1;\; \Re_e(c)>\Re_e(b)>0)$$

Replacing $z=\frac{z}{b}$ and by letting $|b|\to\infty$ in Gauss's hypergeometric equation, we have

$$z\frac{d^2y}{dz^2}+(c-z)\frac{dy}{dz}-ay=0$$

This equation has a regular singularity at $z=0$, The simplest solution of the equation is

$$\begin{aligned} \phi(a;c;z) &= 1+\frac{a}{c.1}z+\frac{a(a+1)}{c(c+1).1.2}z^2+..... \\ &= \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!} \end{aligned}$$

$$(c \neq 0,-1,-2,....)$$

Hence, we get

$$\phi(a;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}$$

which is called confluent hypergeometric function.
The confluent hypergeometric function can be given by an integral representation as follows:

$$\phi(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} exp(zt)dt$$

$$(\Re_e(c) > \Re_e(a) > 0)$$

A generalized form of the hypergeometric function is

$$_pF_q(\alpha_1, ..., \alpha_p; \gamma_1, ..., \gamma_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n.....(\alpha_p)_n}{(\gamma_1)_n.....(\gamma_q)_n} \frac{z^n}{n!} \tag{1.5}$$

$$(p, q = 0, 1, ...)$$

Setting $p = 2$, $q = 1$ in (1.5), we get the Gauss hypergeometric function,

$$F(\alpha_1, \alpha_2; \gamma_1; z) :=_2 F_1(\alpha_1, \alpha_2; \gamma_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n}{(\gamma_1)_n} \frac{z^n}{n!}$$

Setting $p = q = 1$ in (1.5) , we get confluent hypergeometric function

$$\phi(\alpha_1; \gamma_1; z) =_1 F_1(\alpha_1; \gamma_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n}{(\gamma_1)_n} \frac{z^n}{n!}$$

1.7 Whittaker function

In mathematics, a Whittaker function is a special solution of Whittaker's equation, a modified form of the confluent hypergeometric equation introduced by Whittaker (1903) to make the formulas involving the solutions more symmetric. More generally, Jacquet (1966, 1967) introduced Whittaker functions of reductive groups over local fields, where the functions studied by Whittaker are essentially the case where the local field is the real numbers and the group is $SL_2(R)$.

Whittaker's equation is:

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} + \mu^2}{z^2} w \right) = 0$$

It has a regular singular point at 0 and an irregular singular point at ∞. Two solutions are given by the Whittaker functions $M_{k,\mu}(z)$, $w_{k,\mu}(z)$, defined in terms of Kummer's confluent hypergeometric functions M and U by:

$$M_{k,\mu}(z) = exp(\frac{-z}{2}) z^{\mu+\frac{1}{2}} M(\mu - k + \frac{1}{2}, 1 + 2\mu, z)$$

$$W_{k,\mu}(z) = exp(\frac{-z}{2})z^{\mu+\frac{1}{2}}U(\mu - k + \frac{1}{2}, 1 + 2\mu, z)$$

The Whittaker functions $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ are the same as those with opposite values of μ, in other words considered as a function of μ at fixed k and z they are even functions. When k and z are real, the functions give real values for real and imaginary values of μ. These functions of μ play a role in so-called Kummer spaces.

1.8 Mellin Transform

Definition 1.8.1. *The Mellin transform of a function $f(t)$ defined by*

$$M\{f, s\} = \varphi(s) = \int_0^\infty t^{s-1} f(t)dt \tag{1.6}$$

and the inverse of the Mellin Transform defined as the following

$$\{M\varphi(s)\} = f(s) = \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty} t^{-s}\varphi(s)ds$$

Relation to Laplace and Fourier transformations

Mellin's transformation is closely related to an extended form of Laplace's. The change of variables defined by

$$t = e^{-x} \quad , dt = -e^{-x}dx$$

transforms the integral(1.6) into:

$$\varphi(s) = \int_{-\infty}^{+\infty} f(e^{-x})e^{-sx}dx \tag{1.7}$$

After the change of function:

$$g(x) = f(e^{-x})$$

one recognizes in (1.7) the two-sided Laplace $\mathcal{L}$ transform of g usually defined by:

$$\mathcal{L}[g, s] = \int_{-\infty}^{+\infty} g(x)e^{-sx}dx$$

This can be written symbolically as

$$M\{f, s\} = \mathcal{L}[f(e^{-x}), s]$$

The occurrence of a strip of holomorphy for Mellin's transform can be deduced directly from this relation. The usual right-sided Laplace transform is analytic in a half-plane $Re(s) > \sigma_1$. In the same way, one can define a left-sided Laplace transform analytic in the region $Re(s) < \sigma_2$. If the two half-planes overlap, the region of holomorphy of the two-sided transform is thus the strip $\sigma_1 < Re(s) < \sigma_2$obtained as their intersection.
To obtain Fourier's transform, write now $s = a + 2\pi j\beta$ in (1.7):

$$\varphi(s) = \int_{-\infty}^{+\infty} f(e^{-x})e^{-ax}e^{-2\pi j\beta}dx$$

The result is:

$$M\{f(t), a + 2\pi j\beta\} = \mathcal{F}[f(e^{-x})e^{-ax}, \beta]$$

where $\mathcal{F}$ represents the Fourier transformation defined by:

$$\mathcal{F}[f, \beta] = \int_{-\infty}^{+\infty} f(x)e^{-2\pi j\beta}dx$$

Thus for a given value of $Re(s) = a$ belonging to the definition strip, the Mellin transform of a function can be expressed as a Fourier transform.

Chapter 2

Fractional integrals and derivatives

Fractional calculus can be defined as a branch of mathematics that studies the properties and integrals of non-integrals orders(called fractional derivative and integrals). In particular, this specialization includes the idea and methods of solution Differential equations that include fractional derivatives of the unknown function.

2.1 Rimann-liouville fractional integral and derivative

In this section we define the first order-integral operator by:

$$I\left(f(x)\right) = \int_0^x f(s)ds.$$

Then we define the second order-integral by:

$$I^2\left(f(x)\right) = I(If(x)) = I(\int_0^x f(s)ds) = \int_0^x \int_0^y f(s)dsdy.$$

by calculat:

$$\int_0^x \int_0^y f(s)dsdy = \int_0^x f(s) \int_0^y dyds.$$

$$\int_0^x \int_0^y f(s)dsdy = \int_0^x f(s)[y+c]_0^y ds.$$

Replace $y = (x-s)$

$$\int_0^x \int_0^y f(s)dsdy = \int_0^x (x-s)f(s)ds.$$

So,

$$I^2\left(f(x)\right) = \int_0^x (x-s)f(s)ds.$$

Lemma 2.1.1. *[26]*
For any $n \in N$, if $f$is locally integrable

$$I^n\left(f(x)\right) = \int_0^x \frac{(x-s)^{n-1}}{(n-1)!}f(s)ds. \tag{2.1}$$

proof .3.
For $n = 1$we have:

$$I^1(f(x)) = \int_0^x \frac{(x-s)^{1-1}}{(1-1)!}f(s)ds.$$

$$I^1(f(x)) = \int_0^x f(s)ds.$$

So, Lemma true $(n = 1)$ assume Lemma true for n and we prove that true for $n + 1$

$$I^{n+1}\left(f(x)\right) = I(I^n f(x)) = I(\int_0^x \frac{(x-s)^{n-1}}{(n-1)!}f(s)ds).$$

$$I^{n+1}\left(f(x)\right) = \int_0^x \left(\int_0^y \frac{(y-s)^{n-1}}{(n-1)!}f(s)ds\right).$$

Changing order of integration
$\{0 \le y \le x, 0 \le s \le y\} = \{0 \le s \le x, s \le y \le x\}$, *we get*

$$\begin{aligned} I^k\left(f(x)\right) &= \int_x^0 \int_x^s \frac{(y-s)^{k-1}}{(k-1)!}f(s)dyds \\ &= \int_0^x f(s)\int_s^x \frac{(y-s)^{k-1}}{(k-1)!}dyds \\ &= \int_0^x f(s)\frac{(x-s)^k}{k(k-1)!}ds \\ &= \int_0^x \frac{(x-s)^k}{k!}f(s)ds. \end{aligned}$$

So, Lemma (2.1) is proved for any $n \in N$

2.1.1 Rimann-liouville fractional integral

In the same manner as Lemma used; then n-fold integrated integral is given by

$$I^n\left(f(x)\right) = \int_a^x dx_1 \int_a^{x_1} dx_2 \int_a^{x_{n-1}} f(x_n)dx_n.$$

$$I^n\left(f(x)\right) = \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} f(s)ds.$$

$$I^n\left(f(x)\right) = \frac{1}{(n-1)!}\int_0^x \frac{1}{(x-s)^{1-n}} f(s)ds.$$

Remark 2.1.1. *For generalization of intrgral of f of frational order remplace n by α; $\alpha \geqslant 0$; writing $(n-1)! = \Gamma(n)$*

Definition 2.1.1. [26]
The Riemann-Liouville fractional integral of order $\alpha \in R^+$ is given by

$$I_a^\alpha\left(f(x)\right) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{1}{(x-s)^{1-\alpha}} f(s)ds.$$

In the right hand we have :$(-\infty < a < x < +\infty)$

$$I_{a^+}^\alpha\left(f(x)\right) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{1}{(x-s)^{1-\alpha}} f(s)ds.$$

In the left hand we have :$(-\infty < x < b < +\infty)$

$$I_{b^-}^\alpha\left(f(x)\right) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{1}{(s-x)^{1-\alpha}} f(s)ds.$$

Properties 2.1.1. [26]

(a) *The Riemann - Liouville integral operator of order is a linear operator. That means*

$$I^\alpha(af(x) + bg(x)) = aI^\alpha f(x) + bI^\alpha g(x), \quad a, b \in R, \alpha \in R^+.$$

(b) *We have $I^k(I^l f(x)) = I^{k+l} f(x)$.*

(c) *We have $I^k[I^l f(x)] = I^l[I^k f(x)], \quad k, l \in R^+$.*

proof .4.

(a) *We using the definition of fractional integral operator, we get*

$$\begin{aligned} I^\alpha(af(x)+bg(x)) &= \frac{1}{\Gamma(\alpha)}\int_0^x \frac{af(x)+bg(x)}{(x-s)^{1-\alpha}}ds. \\ &= a\frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(x)}{(x-s)^{1-\alpha}}ds + b\frac{1}{\Gamma(\alpha)}\int_0^x \frac{g(x)}{(x-s)^{1-\alpha}}ds. \\ &= aI^\alpha f(x) + bI^\alpha g(x). \end{aligned}$$

(b) *Using the definition of fractional integral ,we get*

$$\begin{aligned} I^\alpha(I^\beta f(x)) &= I^\alpha\left[\frac{1}{\Gamma(\beta)}\int_0^x \frac{f(s)ds}{(x-s)^{1-\beta}}\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x \frac{1}{(x-y)^{1-\alpha}}\frac{1}{\Gamma(\beta)}\int_0^y \frac{f(s)ds}{(y-s)^{1-\beta}}dy. \end{aligned}$$

Changing the order of the integration, we get

$$\begin{aligned} I^\alpha(I^\beta f(x)) &= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^y \frac{1}{(x-y)^{1-\alpha}}\frac{1}{(y-s)^{1-\beta}}f(s)dsdy. \\ &= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_0^x\left[\int_s^x \frac{1}{(x-y)^{1-\alpha}}\frac{dy}{(y-s)^{1-\beta}}\right]f(s)ds. \end{aligned}$$

Now, we will obtain the integral

$$A(x,s) = \int_s^x \frac{1}{(x-y)^{1-\alpha}}\frac{dy}{(y-s)^{1-\beta}}$$

putting $y-s=t$*we get* $dy=dt$ *and*

$$A(x,s) = \int_0^{x-s} \frac{dt}{(x-s-t)^{1-\alpha}t^{1-\beta}}$$

Putting $t=(x-s)u$*, we get,* $dt=(x-s)du$*, then*

$$\begin{aligned} A(x,s) &= \int_0^1 \frac{(x-s)du}{(x-s)^{1-\alpha}(1-u)^{1-\alpha}(x-s)^{1-\beta}u^{1-\beta}} \\ &= \frac{1}{(x-s)^{1-(\alpha+\beta)}}\int_0^1 \frac{du}{(1-u)^{1-\alpha}u^{1-\beta}} \\ &= \frac{1}{(x-s)^{1-(\alpha+\beta)}}\int_0^1 (1-u)^{-1+\alpha}u^{-1+\beta}du \\ &= \frac{1}{(x-s)^{1-(\alpha+\beta)}}B(\alpha,\beta) \\ &= \frac{1}{(x-s)^{1-(\alpha+\beta)}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

$$\int_s^x \frac{dy}{(x-y)^{1-\alpha}(y-s)^{1-\beta}} = \frac{1}{(x-s)^{1-(\alpha+\beta)}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Then

$$\begin{aligned} I^{\alpha}(I^{\beta}f(x)) &= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_0^x \frac{1}{(x-s)^{1-(\alpha+\beta)}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}f(s)ds. \\ &= \frac{1}{\Gamma(\alpha+\beta)}\int_0^x \frac{f(s)ds}{(x-s)^{1-(\alpha+\beta)}} \\ &= I^{\alpha+\beta}(f(x)). \end{aligned}$$

(c) *Applying semigroup properties, we get*

$$\begin{aligned} I^{\alpha}(I^{\beta}f(x)) &= I^{\alpha+\beta}(f(x)) \\ &= I^{\beta+\alpha}(f(x)) \\ &= I^{\beta}(I^{\alpha}f(x)). \end{aligned}$$

Example 2.1.1. [*26*]

$$f(x) = 1$$

$$I^{\alpha}(1) = \frac{1}{\Gamma(1+\alpha)}x^{\alpha}, \quad for\ all\ \alpha > 0,\ x > 0.$$

Example 2.1.2. [*26*]

$$f(x) = x^{\beta}$$

$$I^{\alpha}(x^{\beta}) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{s^{\beta}}{(x-s)^{1-\alpha}}ds = x^{\beta+\alpha}\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

$$\begin{aligned} I^{\alpha}(x^{\beta} &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}s^{\beta}ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (1-\frac{s}{x})^{\alpha-1}x^{\alpha-1}s^{\beta}ds \end{aligned}$$

Putting $s = ux$, *we get* $dt = xdu$; *Then*

$$\begin{aligned} &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}x^{\alpha-1}(ux)^{\beta}xdu \\ &= \frac{1}{\Gamma(\alpha)}x^{\alpha+\beta}\int_0^1 (1-u)^{\alpha-1}u^{\beta}du \\ &= \frac{1}{\Gamma(\alpha)}x^{\alpha+\beta}B(\beta+1,\alpha) \\ &= x^{\beta+\alpha}\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}. \end{aligned}$$

In particular, if $\alpha = \frac{1}{2}$

$$I^{\frac{1}{2}} x^0 = \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}.$$

$$I^{\frac{1}{2}} x^1 = \frac{\Gamma(2)}{\Gamma(\frac{5}{2})} x^{\frac{3}{2}} = \frac{4}{3}\sqrt{\frac{x^3}{\pi}}.$$

$$I^{\frac{1}{2}} x^2 = \frac{\Gamma(3)}{\Gamma(\frac{7}{2})} x^{\frac{5}{2}} = \frac{16}{15}\sqrt{\frac{x^5}{\pi}}.$$

Example 2.1.3. [*26*]

$$f(x) = e^{\beta t}$$

$$I^{\alpha}(e^{\beta t}) = E_T(\alpha, \beta).$$

$$I^{\alpha}(e^{ax}) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{e^{as}}{(x-s)^{1-\alpha}} ds$$

Putting

$$y = x - s$$

, we get

$$I^{\alpha}(e^{ax}) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{e^{a(x-y)}}{y^{1-\alpha}} dy$$

According to the The Mellin-Ross Function, We get

$$I^{\alpha}(e^{ax}) = E_x(\alpha, a)$$

In particular, if $\alpha = \dfrac{1}{2}$,

$$I^{\frac{1}{2}}(e^{ax}) = E_x(\frac{1}{2}, a) = a^{-\frac{1}{2}} e^{ax} E_r f(ax)^{\frac{1}{2}}$$

Example 2.1.4. [*26*]

$$f(x) = te^{\beta t}$$

$$I^{\alpha}(te^{\beta t}) = tE_T(\alpha, \beta) - \alpha E_t(\alpha + 1, \beta).$$

2.1.2 Rimann-liouville fractional derivative

We will obtain the Riemann-Liouville fractional derivative by definition of Rimann-liouville fractional integral(2.1.1) if $f(x) \in C([a,b])$ and $a < x < b$ then:

$$I_a^\alpha (f(x)) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{1}{(x-s)^{1-\alpha}} f(s)ds.$$

If $\alpha \in [-\infty, +\infty]$ is called the Riemann-Liouville fractional integral of order α;
In the same fashion for $\alpha \in [0,1]$, We let

$$D_a^\alpha (f(x)) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{1}{(x-s)^\alpha} f(s)ds.$$

Which is called the Riemann-Liouville fractional derivative of order α

Definition 2.1.2. [*26*]
the Riemann-Liouville fractional derivative or the Riemann-Liouville fractional operator of order α
for $\alpha > 0, x > 0; \quad \alpha, x \in R$ *,Is given by*

$$D^\alpha (f(x)) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(s)}{(x-s)^{(1-n+\alpha)}} ds, & n-1 < \alpha < n; n \in N \\ \frac{d^n}{dx^n} f(x), & \alpha = n; n \in N. \end{cases}$$

Lemma 2.1.2. [*26*]
Let $n-1 < \alpha < n, n \in N, \alpha \in R$ *,and* $f(x)$*be such that* $D^\alpha (f(x))$ *exist.Then*

$$D^\alpha (f(x)) = D^{\lceil \alpha \rceil} I^{\alpha - \lceil \alpha \rceil} f(x).$$

This means the Riemann-Liouville fractional derivative is equivalent to $(\lceil \alpha \rceil - \alpha)$*-fold integration and*$\lceil \alpha \rceil$*-th order differential.*

Properties 2.1.2. [*26*]

1. *The Riemann-Liouville fractional differential operator of order is a linear operator. That means*

$$D^\alpha (af(x) + bg(x)) = aD^\alpha f(x) + bD^\alpha g(x) a, b \in R, \alpha \in R^+.$$

2. *The following non-semigroup properties hold*

$$D^\alpha D^\beta \neq D^{\alpha+\beta}, \alpha; \beta \in R^+.$$

3. *The following non-commutative properties hold suppose that* $n-1<\alpha<n, \quad n; m \in N$
Then in general:

$$D^m D^\alpha f(x) = D^{m+\alpha} f(x) \neq D^\alpha D^m f(x).$$

4. *For any constant C, the formulas hold*

$$D^\alpha(c) = \frac{1}{\Gamma(1-\alpha)} x^{-\alpha}$$

proof .5.

1. *Using the definition of* D^α *, we get*

$$\begin{aligned} D^\alpha(af(x)+bg(x)) &= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} \int_0^x \frac{af(s)+bg(s)}{(x-s)^{1-\lceil\alpha\rceil+\alpha}} ds. \\ &= \frac{a}{\Gamma(\lceil\alpha\rceil-\alpha)} \frac{d^{\lceil\alpha\rceil}}{dx^{\alpha}} \int_0^x \frac{f(s)}{(x-s)^{1-\lceil\alpha\rceil+\alpha}} ds \\ &+ \frac{b}{\Gamma(\lceil\alpha\rceil-\alpha)} \frac{d^{\lceil\alpha\rceil}}{dx^{\alpha}} \int_0^x \frac{bg(s)}{(x-s)^{1-\lceil\alpha\rceil+\alpha}} ds \\ &= aD^\alpha f(x) + bD^\alpha g(x) \end{aligned}$$

2. *Let* $\alpha = \frac{1}{2}, f(x)1, \beta = 1$ *using the definition of* D^α*, we get*

$$D^{\frac{1}{2}} D^1(1) = D^{\frac{1}{2}}(0) = 0.$$
$$D^{\frac{3}{2}}(1) = \frac{-1}{2\sqrt{\pi}} x \frac{-3}{2}$$
$$D^{\frac{1}{2}} D^1(1) = 0 \neq \frac{-1}{2\sqrt{\pi}} x^{\frac{-3}{2}} = D^{\frac{3}{2}}(1)$$

That means

$$D^{\frac{1}{2}} D^1(1) \neq D^{\frac{3}{2}}(1)$$

(non-semigroup)

3. *Let* $\alpha = \frac{1}{2}, f(x)1, \beta = 1$ *using the definition of* D^α*, we get*

$$\begin{aligned} D^1 D^{\frac{1}{2}}(1) &= D^1\left(\frac{1}{\sqrt{\pi}} x^{\frac{1}{2}}\right) \\ &= \frac{-1}{2\sqrt{\pi}} x^{\frac{-3}{2}} \end{aligned}$$
$$D^{\frac{1}{2}} D^1(1) = 0 \neq \frac{-1}{2\sqrt{\pi}} x^{\frac{-3}{2}}.$$

That means

$$D^1 D^{\frac{1}{2}}(1) \neq D^{\frac{1}{2}} D^1(1)$$

(non-commutative)

4. *Using the definition of* D^α*, we get*

$$= \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} \int_0^x \frac{c}{(x-s)^{1-\lceil\alpha\rceil+\alpha}} ds.$$
$$= \frac{c}{\Gamma(\lceil\alpha\rceil - \alpha)} \frac{d}{dx} \left[-\frac{(x-s)^{1-\alpha}}{(1-\alpha)} |_0^x \right]$$

Example 2.1.5. *The fractional derivative of* $f(x) = x^\mu$ *of order* v

$$\begin{aligned} D^v f(x) &= D^1[I^{(1-v)} f(x)] \\ &= D^1[I^{(1-v)} x^\mu] \\ &= D^1 \left[\frac{\Gamma(\mu+1)}{\Gamma((\mu-v+1)+1)} x^{(\mu-v+1)} \right] \\ &= (\mu-v+1) \frac{\Gamma(\mu+1)}{(\mu-v+1)\Gamma(\mu-v+1)} x^{\mu-v} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-v+1)} x^{\mu-v}. \end{aligned}$$

Examples for $1/2^{th}$*-order derivative:*

$$D^{1/2} x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-1/2+1)} x^{\mu-1/2}.$$
$$D^{1/2} x^1 = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = 2\sqrt{\frac{x}{\pi}}.$$
$$D^{1/2} x^2 = \frac{\Gamma(3)}{\Gamma(5/2)} x^{3/2} = \frac{8}{3}\sqrt{\frac{x^3}{\pi}}.$$

2.2 The Caputo fractional derivative

Definition 2.2.1. [*26*]
Suppose that $\alpha > 0, x > 0, \alpha, x \in R$*. The fractinal derivative*

$$D_*^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(s) ds}{(x-s)^{\alpha+1-n}}, n-1 < \alpha < n \in N \\ \frac{d^n}{dx^n} f(x), \qquad \alpha = n \in N. \end{cases}$$

Is called the Caputo fractional derivative or Caputo fractional differential operator of order α.

Properties 2.2.1.

1. *Caputo fractional differential operator D_*^α of order α is a linear operator. That means*

$$D_*^\alpha\left(af(x)+bg(x)\right)=aD_*^\alpha f(x)+bD_*^\alpha g(x).a,b\in R,\alpha\in R^+.$$

2. *The following non-semigroup properties hold*

$$D_*^\alpha D_*^\beta f(x)\neq D_*^{\alpha+\beta}f(x).\alpha,\beta\in R^+.$$

3. *For any constant properties hold $D_*^\alpha(c)=0$*

proof .6.

1. *Using the definition of D_*^α , we get*

$$\begin{aligned}D_*^\alpha\left(af(x)+bg(x)\right)&=\frac{1}{\Gamma(\lceil\alpha\rceil+\alpha)}\int_0^x\frac{1}{(x-s)^{1-\lceil\alpha\rceil+\alpha}}\frac{d^{\lceil\alpha\rceil}}{ds^\alpha}[af(s)+bg(s)]ds.\\&=a\frac{1}{\Gamma(\lceil\alpha\rceil+\alpha)}\int_0^x\frac{1}{(x-s)^{1-\lceil\alpha\rceil+\alpha}}\frac{d^{\lceil\alpha\rceil}}{ds^\alpha}f(s)ds+\\&+b\frac{1}{\Gamma(\lceil\alpha\rceil+\alpha)}\int_0^x\frac{1}{(x-s)^{1-\lceil\alpha\rceil+\alpha}}\frac{d^{\lceil\alpha\rceil}}{ds^\alpha}g(s)ds.\\&=aD_*^\alpha f(x)+bD_*^\alpha g(x).\end{aligned}$$

2. *Let $\alpha=1,\beta=\dfrac{1}{2},f(x)=x$. Then applying the definition, We get*

$$\begin{aligned}D_*^1D_*^{\frac{1}{2}}(x)&=D_*^1(D_*^{\frac{1}{2}}(x))\\D_*^{\frac{1}{2}}(x)&=\frac{1}{\Gamma(\frac{1}{2})}\int_0^x\frac{1}{(x-s)^{\frac{1}{2}}}\frac{d}{ds}(s)ds\\&=\frac{1}{\Gamma(\frac{1}{2})}\int_0^x\frac{1}{(x-s)^{1/2}}ds=\frac{2\sqrt{x}}{\sqrt{\pi}}.\\D_*^1(D_*^{\frac{1}{2}}(x))&=D_*^1\left(\frac{2\sqrt{x}}{\sqrt{\pi}}\right)=\frac{1}{\sqrt{\pi x}}.\end{aligned}$$

and

$$\begin{aligned} D_*^{1+\frac{1}{2}}(x) &= D_*^{\frac{3}{2}}(x) \\ &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^x \frac{1}{(x-s)^{\frac{1}{2}}}\frac{d^{\lceil 2\rceil}}{ds^{\lceil 2\rceil}}(s)ds \\ &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^x \frac{1}{(x-s)^{\frac{1}{2}}}\frac{d}{ds}(1)ds = 0. \end{aligned}$$

We see that:

$$D_*^1 D_*^{\frac{1}{2}}(x) = \frac{1}{\sqrt{\pi x}} \neq 0 = D_*^{1+\frac{1}{2}}(x).$$

3. *Using the definition, we get*

$$D_*^{\alpha}(c) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{1}{(x-s)^{\alpha+1-n}}\frac{d^n}{ds^n}(c)ds$$

We have $\frac{d^n}{ds^n}(c) = 0$ *, So* $D_*^{\alpha}(c) = 0$.

Example 2.2.1. *[26]*

$$D_*^{\alpha}(x^{\beta}) = \begin{cases} 0 & if \beta \in N^0 \quad and \beta < \lceil\alpha\rceil, \\ \dfrac{\Gamma(\beta+1)x^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)} & if \beta \in N^0 \quad and \beta \geq \lceil\alpha\rceil \quad or \beta \in N \quad and \beta > \lceil\alpha\rceil. \end{cases}$$

Here; $N^0 = N \bigcup \{0\}$.

1. *If* $\beta \in N^0$ *and* $\beta < \lceil\alpha\rceil$, *Than* $D_*^{\lceil\alpha\rceil}(u^{\beta}) = 0$, *and using this formula*

$$D_*^{\alpha} f(x) = \frac{1}{\Gamma(\lceil\alpha\rceil+\alpha)}\int_0^x \frac{1}{(x-u)^{1-\lceil\alpha\rceil+\alpha}} D^{\lceil\alpha\rceil} f(u)du; \qquad x, \alpha \in R^+ \quad (2.2)$$

We get
$D_*^{\alpha} f(x) = 0$

2. $If \beta \in N^0 and \beta \geq \lceil\alpha\rceil or \beta \in N and \beta > \lceil\alpha\rceil$, *Then* $D_*^{\lceil\alpha\rceil}(u^{\beta}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}u^{\beta-\alpha}$

Using formula(2.2), We get

$$\begin{aligned} D_*^{\alpha}(x^{\beta}) &= \frac{1}{\Gamma(\lceil\alpha\rceil+\alpha)}\int_0^x \frac{1}{(x-u)^{1-\lceil\alpha\rceil+\alpha}} D^{\lceil\alpha\rceil} u^{\beta} du \\ &= \frac{1}{\Gamma(\lceil\alpha\rceil+\alpha)}\int_0^x \frac{1}{(x-u)^{1-\lceil\alpha\rceil+\alpha}} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} u^{\beta-\alpha} du. \end{aligned}$$

Putting $u = xp$, we get $du = xdp$; Then

$$\begin{aligned}
D_*^{\alpha}(x^{\beta}) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)\Gamma(\lceil\alpha\rceil+\alpha)} \int_0^1 \frac{x}{[x(1-p)]^{1-\lceil\alpha\rceil+\alpha}} (xp)^{\beta-\lceil\alpha\rceil} dp \\
&= \frac{\Gamma(\beta+1)x^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)\Gamma(\lceil\alpha\rceil-\alpha)} \int_0^1 p^{\beta-\lceil\alpha\rceil}(1-p)^{\lceil\alpha\rceil-\alpha-1} dp \\
&= \frac{\Gamma(\beta+1)x^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)\Gamma(\lceil\alpha\rceil-\alpha)} \mathbb{B}(\beta-\lceil\alpha\rceil+1, \lceil\alpha\rceil-\alpha) \\
&= \frac{\Gamma(\beta+1)x^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)}.
\end{aligned}$$

2.3 Hadamard fractional integrals and fractional derivative

In this section, we present the definitions and the some properties of the Hadamard fractional integrals and Hadamard fractional derivatives[14],[11],[25],[23].

2.3.1 Hadamard fractional integrals

Definition 2.3.1. *(Hadamard fractional integral to left sided) Hadamard fractional integral to left sided of complex order α such that $Re(\alpha) \geq 0$ of a function f is given respectively by the expression:*

$$({}^{H}I_{\alpha}^{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(log\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t}, a < x < b \tag{2.3}$$

Definition 2.3.2. *(Hadamard fractional integral to right sided) Hadamard fractional integral to right sided of complex order α such that $Re(\alpha) \geq 0$ of a function is given respectively by the expression*

$$({}^{H}I_{\alpha}^{b-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(log\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t}, a < x < b \tag{2.4}$$

In particular, if $a = 0$ *and* $b = \infty$

$$({}^{H}I_{\alpha}^{0} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(log\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t} x > 0$$

$$({}^{H}I_{\alpha}^{\infty-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \left(log\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t} x > 0$$

2.3.2 Hadamard Fractional Derivatives

Definition 2.3.3. *(Hadamard fractional derivative to left sided)*
The left-sided Hadamard fractional derivative of order $\alpha \in \mathbb{C}, Re(\alpha) \geq 0)$*; in interval* (a, b)*, is defined by:*
we have $\delta = xD$ *and* $D = \dfrac{d}{dx}$

$$(D^{\alpha}_{a+}y)(x) = \delta^n(({}^H I^{n-\alpha}_{a+}y)(x)) = \left(x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(log\frac{x}{t}\right)_{(n-\alpha+a)} \frac{y(t)dt}{t}\,; (a < x < b)$$

Definition 2.3.4. *(Hadamard fractional derivative to right sided)*
The right-sided Hadamard fractional derivative of order $\alpha \in \mathbb{C}, Re(\alpha) \geq 0)$*; in interval* (a, b)*, is defined by we have* $\delta = xD$ *and* $D = \dfrac{d}{dx}$

$$(D^{\alpha}_{b-}y)(x) = -\delta^n(({}^H I^{n-\alpha}_{b-}y)(x)) = \left(-x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(log\frac{t}{x}\right)_{(n-\alpha+a)} \frac{y(t)dt}{t}\,; (a < x < b)$$

In particular, if $a = 0$ and $b = \infty$ $(D^{\alpha}_{0+}y)(x) = (D^{\alpha}_{-\infty}y)(x)$

Properties 2.3.1. *For* $Re(\alpha) > 0$, $Re(\beta) > 0$, *and* $0 < a < b < \infty$, *we have*

1.
$$({}^H I^{a+}_{\alpha}(log\frac{t}{a})^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(log\frac{x}{a})^{\beta+\alpha-1}.$$

2.
$$(D^{a+}_{\alpha}(log\frac{t}{a})^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(log\frac{x}{a})^{\beta-\alpha-1}.$$

3.
$$({}^H I^{b-}_{\alpha}(log\frac{b}{t})^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(log\frac{b}{x})^{\beta+\alpha-1}.$$

4.
$$(D^{b-}_{\alpha}(log\frac{b}{t})^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(log\frac{b}{x})^{\beta-\alpha-1}.$$

2.4 Caputo-Fabrizio fractional integrals and Fractional Derivative

Recently, a new derivative was launched by Caputo and Fabrizio and it was followed by some related theoretical and applied results , We recall that the existing fractional derivatives have been used in many real world problems with great success but still there are many thinks to be done in this direction. The Caputo-Fabrizio fractional derivative in the Caputo sense is defined by Let$f \in H^1(a,b), a < b, a \in (-\infty, t), 0 < \alpha < 1$;

$$({}^{CFC}D_a^\alpha f)(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds.$$

2.4.1 Caputo-Fabrizio fractional integral

Definition 2.4.1. [19]
The Caputo-Fabrizio fractional integral of order is defined by
Let $a, b, \alpha \in R, a < b,\ 0 < \alpha < 1,\ f \in H^1(a,b)$.

$$I_a^\alpha f(t) = (1-\alpha)f(t) + \alpha\int_a^t f(s)ds.$$

2.4.2 Caputo-Fabrizio fractional derivative

Definition 2.4.2. [19]
The Caputo-Fabrizio fractional derivative of orderα is defined by
Let $a, b, \alpha \in R,\ a < b,\ 0 < \alpha < 1,\ f \in H^1(a,b)$.

$$(D_a^\alpha f)(t) = \frac{1}{1-\alpha}\int_a^t f'(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds.$$

$$(D_a^\alpha f)(t) = \frac{1}{1-\alpha}\left(f(t) - e^{-\frac{\alpha}{1-\alpha}tf(a)}\right) - \frac{\alpha}{(1-\alpha)^2}\int_a^t f(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds.$$

Properties 2.4.1. [19]

1. *Let $f \in \mathcal{C}^1[a,b]$. Then $D_{at}^\alpha f(t) \in \mathcal{C}^1[a,b]$.*
2. *The operator $D_{at}^\alpha : \mathcal{C}^1[a,b] \to \mathcal{C}^1[a,b]$ is bounded and*

$$\|D_{at}^\alpha f\|_{\mathcal{C}^1[a,b]} \leq \frac{1}{\alpha}\left(1 - e^{-\frac{\alpha}{1-\alpha}(b-a)}\right)\|f\|_{\mathcal{C}^1[a,b]}. \tag{2.5}$$

3. *Let $f \in \mathcal{C}^1[a,b]$, Then $D_{at}^\alpha f(t) \in \mathcal{W}^{1.p}[a,b], 1 \leq p \leq \infty$.*
4. *The subspace $\mathcal{C}^1[a,b]^1$ is invariant with respect to the Caputo-Fabrizio operator D_{at}^α.*

proof .7.

1. *As the function*

$$y_\tau = \frac{1}{1-\alpha} e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f'(\tau).$$

Is continuous and integrable for all t,$\tau \in [a,b]$;We conclude that the function

$$F(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f'(\tau) d\tau$$

Is differentiable in $[a,b]$, This means that $D_{at}^{\alpha} f(t) \in C[a,b]$.

2. *We have,*

$$\begin{aligned}
\|D_{at}^{\alpha} f\|_{\mathcal{C}^1[a,b]} &= \|\frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} f'(\xi) d\xi\|_{\mathcal{C}^1[a,b]} \\
&\leq \|\frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} \mid f'(\xi) \mid d\xi\|_{\mathcal{C}^1[a,b]} \\
&\leq \|\frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} (\mid f(\xi) \mid + \mid f'(\xi) \mid) d\xi\|_{\mathcal{C}^1[a,b]} \\
&\leq \frac{1}{1-\alpha} \|f\|_{\mathcal{C}^1[a,b]} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} \\
&\leq \frac{1}{\alpha} \|f\|_{\mathcal{C}^1[a,b]} \left(1 - e^{-\frac{\alpha}{1-\alpha}(t-a)}\right) \\
&\leq \frac{1}{\alpha} \|f\|_{\mathcal{C}^1[a,b]} \left(1 - e^{-\frac{\alpha}{1-\alpha}(b-a)}\right)
\end{aligned} \tag{2.6}$$

The inequality (2.5) follows from (2.6)

3. *Let $f \in \mathcal{C}^1[a,b]$, Then $D_{at}^{\alpha} f(t) \in \mathcal{W}^{1.p}[a,b], 1 \leq p \leq \infty$. As $f \in \mathcal{C}^1[a,b]$, we obtain from property one that $D_{at}^{\alpha} f(t) \in \mathcal{C}^1[a,b], 1 \leq p \leq \infty$. We know that $\mathcal{C}^1[a,b] \subset \mathcal{W}^{1.p}[a,b], \forall p \geq 1$. Therefore $D_{at}^{\alpha} f(t) \in \mathcal{W}^{1.p}[a,b]$.*

4. *The subspace $\mathcal{C}^1[a,b]^1$ is invariant with respect to the Caputo-Fabrizio operator D_{at}^{α}. We want to show that for all $f \in \mathcal{C}^1[a,b]$, then $D_{at}^{\alpha} f \in \mathcal{C}^1[a,b]$, We know that $\mathcal{C}^1[a,b] \subset H^1$. Let $f \in \mathcal{C}^1[a,b]$. then using property 1 we conclude that $D_{at}^{\alpha} f \in \mathcal{C}^1[a,b]$.*

Chapter 3

Generalized incomplete gamma and Beta functions

In this chapter, we give denition and properties the generalized gamma function and incomplete gamma; Beta functions.

3.1 The generalized gamma function

Definition 3.1.1. [12]
The first generalized gamma function is defined by

$$\Gamma_c(s) = \int_0^\infty t^{s-1} e^{-t-c} dt.(Re(c) > 0; Re(s) > 0).$$

Notice that in the case $c = 0$ the function conclude with the classical gamma function.

Remark 3.1.1. *For $Re(c) > 0 and \mid arg(\sqrt{c}) \mid < \pi$,*

$$\Gamma_c(s) = 2c^{\frac{s}{2}} K_s(2\sqrt{c}).0$$

K_s is a Macdonald function

Properties 3.1.1. [*7*]

1. *The difirence formula*

$$\Gamma_c(s+1) = s\Gamma_c(s) + c\Gamma_c(s-1).$$

2. *Log-convex property*
   Let $1 < n < \infty$ *and* $(\frac{1}{n}) + (\frac{1}{m}) = 1$;
   Then

$$\Gamma_c(\frac{\alpha}{n} + \frac{\beta}{m}) \leq (\Gamma_c(\alpha))^{\frac{1}{n}} (\Gamma_c(\beta))^{\frac{1}{m}}, \ (c \geq 0, \ \alpha > 0, \ \beta > 0).$$

3. *The reection formula*
   For $R(c) > 0$;

$$c^s \Gamma_c(-s) = \Gamma_c(s)$$

4. *Product Formula*
   For, $c \geq 0$; $Re(q) > 0, \ Re(s) > 0$

$$\Gamma_c(s)\Gamma_s(q) = 2\int_0^\infty \tau^{2(s+q)} e^{-\tau^2} B(s, q; \frac{c}{\tau^2}) d\tau.$$

   Where

$$B(x, y; c) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{c}{t(1-t)}} dt.$$

   Is the Extended Beta Function.

3.2 The generalized incomplete gamma function

The lower incomplete gamma function is defined as

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt.$$

The upper incomplete gamma function is defined as

$$\Gamma(s, x) \int_x^\infty t^{s-1} e^{-t} dt.$$

The lower and upper Incomplete Gamma Functions were rst invastigated for $x \in \mathbb{R}$ by Legendre. Nevertheless, these functions give rise to some difficulties mostly in the neighborhood of 0. In order to overcome these problems, Chaudhry et al. [5], using an exponential regulazing term, Chaudhry *et al.* extended the incomplete gamma function as follow

Definition 3.2.1. [*5*]
The generalized incomplete gamma function; Chaudhry and Zubair introduced the definition of Generalized Incomplete Gamma functions as

$$\gamma(s,x;c) = \int_0^x t^{s-1} e^{-t-\frac{c}{t}} dt \tag{3.1}$$

and

$$\Gamma(s,x;c) = \int_x^\infty t^{s-1} e^{-t-\frac{c}{t}} dt \tag{3.2}$$

Found useful in a variety of heat conduction problems [6] The decomposition and extension of these functions were also found to be useful[5]
Miller and Moskowitz [21] found a representation of the generalized incomplete function (3.2) in terms of the Kampé de Fériet (KdF) functions and discussed its closed form representations. Miller [20] found several reduction formulae of the KdF functions in terms of the function (3.2) and discussed its relations with incomplete Weber integrals.
In this section we introduce a pair of functions :

$$\gamma(s,x;c,\beta) = \int_0^x t^{s-1} e^{-t-(\frac{c}{t^\beta})} dt \tag{3.3}$$

and

$$\Gamma(s,x;c,\beta) = \int_x^\infty t^{s-1} e^{-t-(\frac{c}{t^\beta})} dt \tag{3.4}$$

and call them extended incomplete gamma functions, we note that the generalized incomplete gamma functions (3.1),(3.2)are special cases of (3.3)(3.4) when $\beta = 1$

$$\gamma(s,x;c,1) = \gamma(s,x;c). \tag{3.5}$$

$$\Gamma(s,x;c,1) = \Gamma(s,x;c). \tag{3.6}$$

Theorem 3.2.1. *(Recurrence formula)* [11]*:*
The recurrence formula for the extended incomplete gamma function (3.4) naturally and simply extends the recurrence relation of the classical and generalized incomplete gamma functions.

$$\Gamma(s+1,x;c,\beta) = s\Gamma(s,x;c,\beta) + c\beta\Gamma(s-\beta,x;c,\beta) + x^s e^{-x-cx^{-\beta}} \tag{3.7}$$

proof .8.
Let us define

$$f(t) = e^{-t-ct^{-\beta}H(t-x)},(3.8)$$

Where

$$H(t-x) = \begin{cases} 1 & if t > x, \\ 0 & if t < x, \end{cases}$$

Is the Heaviside unit step function. The extended gamma function (3.4) is simply the Mellin transform of the function $f(t)$ *in[10]*

$$\Gamma(s, x; c, \beta) = M\{f(t) : t \to s\} =$$

The differentiation of (3.8) in the sense of distribution yields

$$\frac{d}{dt}\{f(t)\} = (-1 + c\beta t^{-\beta-1})f(t) + e^{-t-\beta t^{-\beta}}\delta(t-x),$$

Where

$$\delta(t-x) = \frac{d}{dt}(H(t-x)), \tag{3.9}$$

Is the Dirac delta function.
The Mellin transform of a function and its derivative are related via [10]

$$-(s-1)M\{f(t) : t \to s-1\} = M\left\{\frac{d}{dt}(f(t)); t \to s\right\} \tag{3.10}$$

From (3.9) and (3.10) we get

$$-(s-1)\Gamma(s-1, x; c, \beta) = -\Gamma(s, x; c, \beta) + c\beta\Gamma(s-\beta-1, x; c, \beta) + x^{s-1}e^{-x-cx^{-\beta}}$$

Which simplifies to give

$$\Gamma(s, x; c, \beta) = (s-1)\Gamma(s-1, x; c, \beta) + c\beta\Gamma(s-\beta-1, x; c, \beta) + x^{s-1}e^{-x-cx^{-\beta}} \tag{3.11}$$

Replacing s *by* $s+1$ *in (3.11) yields (3.7)*

Corollary 3.2.1. [5]
We have

$$\Gamma(s+1, x; c) = s\Gamma(s, x; c) + c\Gamma(s-1, x; c) + x^s e^{-x-cx^{-1}} \tag{3.12}$$

proof .9.
This follows from (3.7) when we take $\beta = 1$. *It is to be noted that the substitution* $c = 0$ *in (3.12) yields the recurrence relation*

$$\Gamma(s+1, x) = s\Gamma(s, x) + x^s e^{-x}$$

For the classical incomplete gamma function

Theorem 3.2.2. [11]
For $Re(c) \geq 0$

$$\gamma(s,x;c) + \Gamma(s,x;c) = \Gamma_c(s).$$

proof .10.
When we add lower and upper incomplete gamma functions, we get

$$\gamma(s,x;c) + \Gamma(s,x;c) = \int_0^x t^{s-1}e^{-t-ct^{-1}}dt + \int_x^\infty t^{s-1}e^{-t-ct^{-1}}dt$$

Hence

$$\int_0^\infty t^{s-1}e^{-t-ct^{-1}}dt = \Gamma_c(s)$$

Theorem 3.2.3. [5]
For $a > 0$

$$\int_x^\infty t^{s-1}e^{-at-t^{-1}}dt = a^{-s}\Gamma(s,ax;ac)$$

proof .11.
Substitution$t = \frac{\mu}{a}$ in $\int_x^\infty t^{s-1}e^{-at-t^{-1}}dt$ and use (3.1); we get

$$a^{-s}\int_{ax}^\infty \mu^{s-1}e^{-\mu-ac\mu^{-1}}d\mu = a^{-s}\Gamma(s,ax;ab)$$

Theorem 3.2.4. *(Parametric differentiation)*[5].

$$\frac{d}{dc}\left(\Gamma(s,x;c)\right) = -\Gamma(s-1,x;c)$$

We have

$$\Gamma(s+1,x;c) = s\Gamma(s,x;c) + c\Gamma(s-1,x;c) + x^s e^{-x-cx^{-1}}$$

proof .12.

$$\begin{aligned}\frac{d}{dc}\left(\Gamma(s,x;c)\right) &= \frac{d}{dc}\left(\int_x^\infty t^{s-1}e^{-t-\frac{c}{t}}dt\right)\\ &= -\int_x^\infty t^{s-2}e^{-t-\frac{c}{t}}dt\\ &= -\Gamma(s-1,x;c)\end{aligned}$$

Theorem 3.2.5. *(Decomposition formula)*[5]
The decomposition formula

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s).$$

For the classical incomplete gamma functions was proved to be a special case of the decomposition formula [?]

$$\gamma(s,x,c) + \Gamma(s,x,c) = 2b^{\frac{\alpha}{2}} K_\alpha(2\sqrt{b}).$$

For the generalized incomplete gamma functions. One could expect to have a similar formula for the extended incomplete gamma functions.

These extensions are useful and provide new connections with error and Whittaker functions, for $p = 0$ they will be reduced to the known gamma Instead of using the exponential function.
Chaudhry and Zubair [5] proposed a new generalized extensions of incomplete gamma functions in the following form

$$\gamma_\mu(\alpha, x; p) = \sqrt{\frac{2p}{\pi}} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right) dt, \tag{3.13}$$

$$\Gamma_\mu(\alpha, x; p) = \sqrt{\frac{2p}{\pi}} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right) dt. \tag{3.14}$$

Recently and Inspired by the work of Agarwal [2], abbas *et al.* in [1] introduce a new generalized incomplete gamma functions by replacing the Macdonald function $K_\alpha(z)$ by it's extended one developed by Boudjelkha , namely the function

$$R_K(z, \alpha, q, \lambda) = \frac{(z/2)^\alpha}{2} \int_0^\infty t^{-\alpha-1} \frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}} dt, \tag{3.15}$$

where $|\arg z^2| < \pi/2$, $0 < q \leq 1$ and $-1 \leq \lambda \leq 1$.

Definition 3.2.2. *The extended generalized incomplete gamma functions are given by [1]*

$$\gamma_\mu(\alpha, x; q; \lambda; p) = \sqrt{\frac{2p}{\pi}} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) dt \tag{3.16}$$

$$\Gamma_\mu(\alpha, x; q; \lambda; p) = \sqrt{\frac{2p}{\pi}} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) dt \tag{3.17}$$

where $\Re_e(x) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$ and $\Re_e(p) > 0$.

3.3 The Extended Beta Function

Definition 3.3.1. [4]
For $Re(c) > 0$; y and x arbitary complex number the Extended Beta Function is dened as

$$B_\mu(x, y; p; m) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{1}{2}} (1-t)^{y-\frac{1}{2}} (1-zt)^{-a} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) dt.$$

If $x, y \in Cm > 0$,and $R(\mu) \geq 0$.

Connection with other special functions [4]

As mentioned in the Introduction, this extension of the beta function is justified not only by the fact that most properties of the beta function are carried over simply, but also by the fact that this function is related to other special functions for particular values of the variables. In this section, we demonstrate this fact for the cases $y = -x$ and $y = x$, There may well be other such relations to be discovered for other special cases.

Theorem 3.3.1.
The extended beta function is related to the Whittaker function(Whittaker function[27]) by

$$B(\alpha, \alpha, b) = \sqrt{2\pi}^{-\alpha} b^{\frac{(\alpha-1)}{2}} e^{-2b} W_{\frac{-\alpha}{2}, \frac{\alpha}{2}}(4b); \qquad Re(b) > 0. \tag{3.18}$$

proof .13. *The substitution $y = x$ in*

$$B(x, y, b) = 2^{1-x-y} \int_{-1}^1 (1+t)^{x-1} (1-t)^{y-1} e^{\frac{-4b}{(1-t^2)}} dt$$

;yield

$$B(x, x, b) = 2^{1-2x} \int_{-1}^1 (1-t^2)^{x-1} e^{\frac{-4b}{(1-t^2)}} dt$$

Since the integrand on the right-hand side is even, it follows that:

$$B(x, x, b) = 2^{2-2x} \int_0^1 (1-t^2)^{x-1} e^{\frac{-4b}{(1-t^2)} dt.(3.19)}$$

The substitution $\xi = 1 - t^2$in (3.19) yields

$$B(x, x, b) = 2^{2-2x} \int_0^1 \xi^{x-1} (1-\xi)^{\frac{1}{2}-1} e^{\frac{-4b}{\xi}} d\xi. \tag{3.20}$$

The integral on the right-hand side of (3.20) is a special case of the result [27,p.384(3.471)(2)]

$$\int_0^u x^{(v-1)}(u-x)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \beta^{\frac{(v-1)}{2}}u^{\frac{(2\mu-v-1)}{2}}e^{\frac{-\beta}{2u}}\Gamma(\mu)W_{(\frac{(1-2\mu-v)}{2},\frac{v}{2})}(\frac{\beta}{u})$$

$(Re(\mu) > 0; Re(\beta) > 0; u > 0)$
With $\beta = 4b, u = 1, v = x$ *and* $\mu = \frac{1}{2}$, *this gives*

$$B(x,x,b) = \sqrt{2\pi}^{-x}b^{\frac{(x-1)}{2}}e^{-2b}W_{\frac{-x}{2},\frac{x}{2}}(4b); \qquad Re(b) > 0. \qquad (3.21)$$

Replacing x *by* α *in (3.21)completes the proof of (3.18).*

Properties 3.3.1. *[4]*

1. *For* $Re(c) \geq 0$;
$$B(x,y;c) = B(y,x;c).$$
2. *Functional relation*
$$B(x,y+1;c) + B(x+1,y;c) = B(x,y;c).$$
3. *For* $Re(c) > 0$
$$B(x,y;c) = \sum_{n=0}^{\infty} B(x+n,y+1;c).$$
4. *For* $Re(c) > 0$
$$B(x,1-y;c) = \sum_{n=0}^{\infty}\frac{(y)_n}{n!}B(x+n,1;c).$$

proof .14.

1.
$$B(x,y;c) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{c}{t(1-t)}}\,dt.$$

Replace $t = 1-\mu$, *we fined*

$$\begin{aligned} B(x,y;c) &= \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{c}{t(1-t)}}\,dt \\ &= \int_0^1 (1-\mu)^{x-1}\mu^{y-1}e^{-\frac{c}{t(1-t)}}\,d\mu \\ &= \int_0^1 \mu^{y-1}(1-\mu)^{x-1}e^{-\frac{c}{t(1-t)}}\,d\mu \\ &= B(y,x;c). \end{aligned}$$

2.

$$\begin{aligned} B(x,y+1;c)+B(x+1,y;c) &= \int_0^1 t^{x-1}(1-t)^y e^{-\frac{c}{t(1-t)}}\,dt + \int_0^1 t^x(1-t)^{y-1}e^{-\frac{c}{t(1-t)}}\,dt \\ &= \int_0^1 t^x(1-t)^y e^{-\frac{c}{t(1-t)}}[t^{-1}+(1-t)^{-1}]dt \\ &= \int_0^1 t^x(1-t)^y e^{-\frac{c}{t(1-t)}}[\frac{1}{t(1-t)}]dt \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{c}{t(1-t)}}\,dt \\ &= B(x,y;c). \end{aligned}$$

3. The factor $(1-t)^{y-1}$*has the series representations as the following*

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n.$$

So

$$(1-t)^{y-1} = (1-t)^y \sum_{n=0}^{\infty} t^n.$$

$$\begin{aligned} B(x,y;c) &= \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{c}{t(1-t)}}\,dt \\ &= \int_0^1 t^{x-1}(1-t)^y \sum_{n=0}^{\infty} t^n e^{-\frac{c}{t(1-t)}}\,dt \\ &= \sum_{n=0}^{\infty} \int_0^1 (1-t)^y t^{n+x-1} e^{\frac{c}{t(1-t)}}\,dt \\ &= \sum_{n=0}^{\infty} B(x+n,y+1;c). \end{aligned}$$

4. The factor $(1-t)^{-y}$*has the series representations as the following*

$$(1-t)^{-y} = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^n$$

Using the denition of extended beta function; we get

$$\begin{aligned}
B(x, 1-y; c) &= \int_0^1 t^{x-1}(1-t)^{-y} e^{-\frac{c}{t(1-t)}} dt \\
&= \int_0^1 t^{x-1} \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^n e^{-\frac{c}{t(1-t)}} dt \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^{n+x-1} e^{-\frac{c}{t(1-t)}} dt \\
&= \sum_{n=0}^{\infty} \frac{(y)_n}{n!} \int_0^1 t^{n+x-1} e^{-\frac{c}{t(1-t)}} dt \\
&= \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B(x+n, 1; c).
\end{aligned}$$

Recently and Inspired by the work of Agarwal [2], abbas *et al.* [1] introduce a new generalized Euler's beta functions by replacing the Macdonald function $K_\alpha(z)$ by it's extended one developed by Boudjelkha.

Definition 3.3.2. *The extended generalized beta function is given by*

$$B_\mu(x, y; q; \lambda; p; m) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}} R_K\left(\frac{p}{t^m(1-t)^m}, -\mu - \frac{1}{2}, q, \lambda\right) dt, \tag{3.22}$$

where $x, y \in \mathbb{C}$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$ *and* $\Re_e(p) > 0$.

Remark 3.3.1. *Taking* $\lambda = 0$ *and* $q = 1$, *the equation* (3.22) *reduces to the extended Euler beta function defined by Agarwal et al.*[2] .

Properties 3.3.2 (Functional relations)**.**

1. *The following formula holds:*

$$B_\mu(x, y; q; \lambda; p; m) = B_\mu(x+1, y; q; \lambda; p; m) + B_\mu(x, y+1; q; \lambda; p; m). \tag{3.23}$$

2. *Let* $n \in \mathbb{N}$. *Then the following summation formula holds:*

$$B_\mu(x, y; q; \lambda; p; m) = \sum_{k=0}^{n} B_\mu(x+k, y+n-k; q; \lambda; p; m). \tag{3.24}$$

proof .15.

1. *The right-hand side of* (3.23) *reads*

$$\sqrt{\frac{2p}{\pi}}\int_0^1\left\{t^{x-\frac{1}{2}}(1-t)^{y-\frac{3}{2}}+t^{x-\frac{3}{2}}(1-t)^{y-\frac{1}{2}}\right\}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,$$

which, after simplification, yields

$$\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,$$

which is equal to the left-hand side of (3.23).

2. *The case* $n=0$ *of* (3.24) *holds trivially. The case* $n=1$ *of* (3.24) *is just the relation* (3.23). *For the other cases we can easily proceed by induction on* n.

Properties 3.3.3. *The following formula holds*

$$B_\mu(x,1-y;q;\lambda;p;m)=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}B_\mu(x+n,1;q;\lambda;p;m). \tag{3.25}$$

proof .16. *We have*

$$B_\mu(x,1-y;q;\lambda;p;m)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{-y-\frac{1}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt. \tag{3.26}$$

By substituting the formula

$$(1-t)^{-y}=\sum_{n=0}^{\infty}(y)_n\frac{t^n}{n!},\quad (|t|<1,\quad y\in\mathbb{C}), \tag{3.27}$$

in the right-hand of (3.26) *and after interchanging the order of integral and summation we get* (3.25).

Properties 3.3.4. *The following formula holds*

$$B_\mu(x,y;q;\lambda;p;m)=\sum_{n=0}^{\infty}B_\mu(x+n,y+1;q;\lambda;p;m). \tag{3.28}$$

proof .17. *By substituting the following formula*

$$(1-t)^{y-1}=(1-t)^y\sum_{n=0}^{\infty}t^n,\quad (|t|<1), \tag{3.29}$$

in the right-hand of (3.22) *and similarly as in the proof of proposition 3.3.3 we get the desired result.*

3.4 The Extended Riemann-Liouville type fractional

We first recall that the classical Riemann-Liouville fractional derivative is defined by

$$D_z^v f(z) = \frac{1}{\Gamma(-v)} \int_0^z (z-t)^{-v-1} f(t) dt.$$

Where $R(v) < 0$ and the integration path is a line from 0 to z in the complex t-plane.
It coincides with the fractional integral of order $-v$. In case $m-1 < R(v) < m, m \in N$ it is customary to write:

$$D_z^v f(z) = \frac{d^m}{dz^m} D_z^{v-m} f(z) = \frac{d^m}{dz^m} \left\{ \frac{1}{\Gamma(m-v)} \int_0^z (z-t)^{m-v-1} f(t) dt \right\}.$$

We present the following new extended Riemann-Liouville-type fractional derivative operator

Definition 3.4.1. [2]
The extended Riemann-Liouville fractional derivative of $f(z)$ of order v is defined by:
For $Re(v) > 0,\ Re(p) > 0,\ Re(m) > 0,$ and $Re(\mu) \geq 0.$

$$D_z^{v,\mu;p;m} f(z) = \frac{1}{\Gamma(-v)} \sqrt{\frac{2p}{\pi}} \int_0^z (z-t)^{-v-1} f(t) K_{\mu+\frac{1}{2}} \left(\frac{pz^{2m}}{t^m (z-t)^m} \right) dt.$$

Remark 3.4.1.
If we take $m = 0,\ \mu = 0$ and $p \to 0$ Then the above extended Riemann-Liouville fractional derivative operator reduces to the classical Riemann-Liouville fractional derivative operator.

Lemma 3.4.1. [2]
If $R(v) < 0,$ then we have:

$$D_z^{v,\mu;p;m}\{z^\lambda\} = \frac{z^{\lambda-v}}{\Gamma(-v)} B_\mu(\lambda + \frac{3}{2}, -v\frac{1}{2}; p; m).$$

proof .18.
Using definition of The extended Riemann-Liouville fractional derivative; we

have:

$$D_z^{v,\mu;p;m}\{z^\lambda\} = \frac{1}{\Gamma(-v)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-v-1}f(t)K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right)dt$$
$$= \frac{z^{\lambda-v}}{\Gamma(-v)}\sqrt{\frac{2p}{\pi}}\int_0^1 (1-u)^{(-v+\frac{1}{2})-\frac{3}{2}}u^{(\lambda+\frac{3}{2})-\frac{3}{2}}K_{\mu+\frac{1}{2}}\left(\frac{p}{u^m(z-u)^m}\right)du$$
$$= \frac{z^{\lambda-v}}{\Gamma(-v)}B_\mu(\lambda+\frac{3}{2},-v\frac{1}{2};p;m).$$

Lemma 3.4.2. [2]
Let $R(v) < 0$, and suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $(z < \rho)$ for some $\rho \in R_+$. Then we have:

$$D_z^{v,\mu;p;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{v,\mu;p;m}\{z^n\}.$$

proof .19.

Using definition of The extended Riemann-Liouville fractional derivative To the function $f(z)$ with its series expansion, we have:

$$D_z^{v,\mu;p;m}\{f(z)\} = \frac{1}{\Gamma(-v)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-v-1}K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right)\sum_{n=0}^{\infty} a_n t^n dt.$$

Since the power series converges uniformly on any closed disk centered at the origin with its radius smaller than ρ, so does the series on the line segment from 0 to a fixed z for $z < \rho$. This fact guarantees term-by-term integration as follows:

$$D_z^{v,\mu;p;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n \left\{\frac{1}{\Gamma(-v)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-v-1}K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right)t^n dt\right\}$$
$$= \sum_{n=0}^{\infty} a_n D_z^{v,\mu;p;m}\{z^n\}.$$

As a consequence we have the following result.

Theorem 3.4.1.
Let $R(v) < 0$ and suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty}$ $(|z| < \rho)$ for some $\rho \in \mathbb{R}_+$.

Then we have:

$$\begin{aligned} D_z^{v,\mu;p;m}\{z^{\lambda-1}f(z)\} &= \sum_{n=0}^{\infty} a_n D_z^{v,\mu;p;m}\{z^{\lambda+n-1}\} \\ &= \frac{z^{\lambda-v-1}}{\Gamma(-v)}\sum_{n=0}^{\infty} a_n B_\mu\left(\lambda+n+\frac{1}{2}, -v+\frac{1}{2}; p; m\right) z^n \end{aligned}$$

r r rrr

Finally we'll see a new extended generalized Riemann-Liouville fractional derivative operator [1].

Definition 3.4.2. *The extended generalized Riemann-Liouville fractional derivative is defined as*

$$D_z^{\delta,\mu;p;q;\lambda;m} f(z) := \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1} f(t) R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right) dt, \tag{3.30}$$

where $\Re_e(\delta) < 0$, $\Re_e(p) > 0$, $\Re_e(m) > 0$, $\Re_e(\mu) > 0$ *and* $0 < q \leq 1$, $-1 \leq \lambda \leq 1$.
For $n-1 < \Re_e(\delta) < n$, $n \in \mathbb{N}$ *we write*

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m} f(z) := \frac{d^n}{dz^n} D_z^{\delta-n,\mu;p;q;\lambda;m} f(z) &= \frac{d^n}{dz^n}\left\{\frac{1}{\Gamma(n-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{n-\delta-1} f(t)\right. \\ &\times \left. R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right) dt\right\} \end{aligned} \tag{3.31}$$

Remark 3.4.2.
If we take $m = 0$, $\mu = 0$, $\lambda = 1$, *and* $p \to 0$, *then the above extended generalized Riemann-Liouville fractional derivative operator reduces to the classical Riemann-Liouville fractional derivative operator*

In order to calculate the extended generalized fractional derivatives for some functions, We begin by two results involving extended generalized Riemann-Liouville fractional derivative operator of some elementary functions which will be useful in the sequel.

Lemma 3.4.3. *Let* $-m-1 < \Re_e(\delta) < -m$ *for some positif integer* m *and* $\beta > -\frac{3}{2}$. *Then we have*

$$D_z^{\delta,\mu;p;q;\lambda;m}\{z^\beta\} = \frac{z^{\beta-\delta}}{\Gamma(-\delta)} B_\mu(\beta+\frac{3}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m). \tag{3.32}$$

proof .20. *Using definition 3.4.2, and a local setting $t = zu$ we obtain*

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{z^\beta\} &= \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1}t^\beta R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right)dt \\ &= \frac{z^{\beta-\delta}}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (1-u)^{(-\delta+\frac{1}{2})-\frac{3}{2}}u^{(\beta+\frac{3}{2})-\frac{3}{2}} \\ &\times\ R_K\left(\frac{p}{u^m(1-u)^m}, -\mu-\frac{1}{2}, q, \lambda\right)dt \\ &= \frac{z^{\beta-\delta}}{\Gamma(-\delta)}B_\mu(\beta+\frac{3}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m). \end{aligned}$$

More generally, we give the extended generalized Riemann-Liouville fractional derivative of an analytic function f at the origin.

Lemma 3.4.4. *Let $-m-1 < \Re_e(\delta) < -m$ for some positif integer m. If a function f is analytic at the origin then we have*

$$D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\}.$$

proof .21.
Since f is analytic at the origin, its Maclaurin expansion is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (for $|z| < \rho$ with $\rho \in \mathbb{R}^+$ is the convergence radius). Substitute entire power series in definition 3.4.2, we obtain

$$D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} = \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1}R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}; q; \lambda\right)\sum_{n=0}^{\infty} a_n t^n dt.$$

By virtue of the uniform continuity on the convergence disk, we can do integration term by term in the equation above, so we obtain yet:

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} &= \sum_{n=0}^{\infty} a_n\left(\frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1}\right) \\ &\times\ R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}; q; \lambda\right)t^n dt \\ &= \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\}. \end{aligned}$$

proof .22.

Using binomial theorem for $(1-z)^{-\alpha}$ and lemma 3.4.3 we obtain:

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\} &= D_z^{\delta,\mu;p;q;\lambda;m}\left\{\sum_{n=0}^{\infty}(\alpha)_n\frac{z^n}{n!}\right\} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\} \\ &= \frac{z^{-\delta}}{\Gamma(-\delta)}\sum_{n=0}^{\infty}(\alpha)_n B_\mu\left(n+\frac{3}{2},-\delta+\frac{1}{2};p,q;\lambda;m\right)\frac{z^n}{n!}. \end{aligned}$$

Whence the result.

Combining the previous lemmas we obtain again a generalized extended derivative of the product of analytic with power function.

Theorem 3.4.2. *Let $m-1 \le \Re_e(\beta) < m$ for some $m \in \mathbb{N}$. Suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $(|z| < \rho)$ for some $\rho \in \mathbb{R}^+$. Then we have*

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}f(z)\} &= \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta+n-1}\} \\ &= \frac{z^{\beta-\delta-1}}{\Gamma(-\delta)} \\ &\times \sum_{n=0}^{\infty} a_n B_\mu(\beta+n+\frac{1}{2},-\delta+\frac{1}{2};p;q;\lambda;m)z^n. \end{aligned}$$

proof .23.
Since the function $z^{\beta-1}f(z)$ can be rewritten as a serie expansion, by definition 3.4.2, we get

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}f(z)\} &= \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta+n-1}\} \\ &= \frac{z^{\beta-\delta-1}}{\Gamma(-\delta)}\sum_{n=0}^{\infty} a_n B_\mu\left(\beta+n+\frac{1}{2},-\delta+\frac{1}{2};p;q;\lambda;m\right)z^n \end{aligned} \tag{3.33}$$

3.5 Conclusion

This thesis introduced the concept of Fractional calculus. The branch of mathematics that explores fractional integrals and their derivatives. In the introduction, we gave a brief of the Fractional Calculus history and then we started with some Basic techniques and functions, such as the gamma function, the beta function, The Error function,The Mittag-Leffler function And the Function of Mellin-Ross, which was necessary to understand the rest of these papers.

Thereafter we proved the construction of the Riemann-Liouville method to define a differintegral and his properties; Then we checked in some examples for Rimann-liouville fractional integral and derivative.

Next we studied the Caputo fractional derivative and his properties and some examples.

We also explored Hadamard fractional integrals to left sided and to right sided then Hadamard Fractional Derivatives to left sided and to right sided with properties and some examples.

I added Caputo-Fabrizio fractional integrals and Fractional Derivative with properties.

At last we spoked about Generalized incomplete gamma and Beta functions with some theorems plus The Extended Riemann-Liouville type fractional .

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