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Essential of conformable fractional calculus

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Dedication

I dedicate this work

To my mom and my father who supported and encouraged me during these years of study. May she find here the testimony of my deep gratitude to my brothers, my grandparents and those who shared with me all the moments of emotion during the completion of this work. They have warmly supported and encouraged me throughout my journey.

To my family, loved ones and those who give me love and liveliness.

To all my teachers are in the Department of Mathematics. All of my classmates for the 2020/2021 class, each with his own name.

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Contents

Abstract

The concept of derivatives of non-integer order, known as fractional derivatives, first appeared in the letter between L'Hôpital and Leibniz in which the question of a half- order derivative was posed. Since then, many formulations of fractional derivatives have appeared. Recently, a new definition of fractional derivative, named "conformable fractional derivative", has been introduced. This new fractional derivative is compatible with the classical derivative and it has attracted attention in domains such as mechanics, electronics and anomalous diffusion.
Motivated by the considerable attention and the wide resonance in the scientific community that conformable fractional derivative have received it. This master thesis is devoted to the theory of conformable fractional calculus, it summarizes the most recent contributions in this area, and vastly expands on them to create a comprehensive theory conformable fractional calculus.

Key words: Fractional derivatives, Conformable fractional derivatives, Fractional calculus, Conformable fractional calculus.

List Of Notations And Symbols & Acronyms

List Of Notations And Symbols

- E_α : The Mittag-Leffer function.
- $E_{\alpha,\beta}$: The Generalized Mittag-Leffer function in two arguments α and β.
- $\Gamma(\alpha)$: The Gamma function.
- $B(x, y)$: The Beta function of x and y.
- $(\Omega, \mathcal{F}, \mathbb{P})$: The probability space.
- $\mathcal{F}_t$: The filtration on probability space.
- $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$: The filtered probability space.
- $\mathcal{V}(S, T)$: The class of real measurable functions $f(t, w)$.
- $\mathcal{L}\{f(.); s\}$: The Laplace transform of a function f.
- $\mathcal{L}^{-1}F(s)$: The transformation reversal of Laplace of a function f.
- (X, Σ, μ) : The measure space.
- $\mathbb{L}^\infty$: The associated Lebesgue $\infty-$space.
- ${}_aI_x^\alpha(f)$: The Riemann-Liouville left-sided fractional integral of order α.
- ${}_xI_b^\alpha(f)$: The Riemann-Liouville right-sided fractional integral of order α.
- ${}_aD_x^\alpha(f)$: The Riemann-Liouville left-sided fractional derivative of order α.
- ${}_xD_b^\alpha(f)$: The Riemann-Liouville right-sided fractional derivative of order α.
- ${}_a^CD_x^\alpha(f)$: The Caputo left-sided fractional derivative of order α.
- ${}_x^CD_b^\alpha(f)$: The Caputo right-sided fractional derivative of order α.
- $\mathbb{H}^2([0, T])$: The space of all the processes X which are measurable.

- $\mathbb{H}^2([0,T], \|\ .\ \|_{\mathbb{H}^2})$: Banach space.
- $(p)_{n,k}^{\alpha}$: The Pochhammer symbol.
- Γ_k^{α} : The $(\alpha,k)-$Gamma function.
- $B_k^{\alpha}(p,q)$: The $(\alpha,k)-$Beta function.
- $B_k(p,k)$: The $k-$Beta function.
- $T_\alpha(f)$: The conformable fractional derivative of f of order α.
- $I_\alpha^a(f)$: The conformable fractional integral of f of order α.
- $\mathcal{L}_k^{\alpha}\{f(.);s\}$: The $(\alpha,k)-$Laplace transform.

Acronyms

- M-L: Means "Mettag -Leffler".
- R-L: Means "Riemann-Liouville".
- G-L: Means "Grunwlad-Letnikov".
- H-ss: Means "Scaling exponent of self similar processes".
- H-sssi: Means "H-ss with stationary increments".
- FDE: Means "Fractional differential equation".
- FSDE: Means "Fractional stochastic differential equation".
- CFDE: Means "Conformable fractional derivative equation".
- CFSDE: Means "Conformable fractional stochastic derivative equation"

Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term fractional is a misnomer, but it is retained following the prevailing use.

The fractional calculus may be considered an old and yet novel topic. It is an old topic since, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed up to nowadays. In fact the idea of generalizing the notion of derivative to non integer order, in particular to the order 1/2, is contained in the correspondence of Leibniz with Bernoulli, L'Höpital and Wallis. Euler took the first step by observing that the result of the evaluation of the derivative of the power function has a a meaning for non-integer order thanks to his Gamma function.
In the last few decades, fractional differentiation has been used applied scientists for solving several fractional differential equations and they proved that the fractional calculus is very useful in several fields of applications with some restrictions such as: Physics (quantum mechanics and thermodynamics), chemistry, biology, economics, engineering, signal and image processing and control theory.
For Economics and Finance we mention the relation between fractional differencing and long memory processes. The Grunwald-Letnikov fractional difference Δ_T^α of order α with the step T is defined by the equation of ARFIMA model, where $\alpha = d$: .

$$\Delta_T^\alpha y(t) := (1 - L_T)^\alpha y(t) := \sum_{m=0}^{\infty} (-1)^m . \binom{d}{m} . y(t - m.T)$$

Where $\{y_t, t = 1, 2, ..., T\}$ is an $ARFIMA(0, d, 0)$ model if we have the following equation of discrete time stochastic process $(1 - L)^d y_t = \epsilon(t)$ is the fractional difference equation with the Grunwald-Letnikov fractional difference of order $\alpha = d$.
For Continuum Mechanics: we mention some applications of related techniques in mechanics dealing with fractional kinematics, where the symmetric fractional derivative had

been used in the definition of strain[6], namely

$$\mathbf{K}_\alpha(x,t) = 1/2({}_0D_x^\alpha -_x D_L^\alpha)u(x,t)$$

where K_α is a fractional strain, D^α is a fractional derivative, x is a spatial coordinate, t denotes time, and u is a displacement.

In Physics: for example the fractional differential equation for the RC circuit has the form

$$\frac{d^\gamma q}{dt^\gamma} + \frac{1}{\tau_\gamma} q(t) = \frac{C}{\tau_\gamma} v(t).$$

Where $\tau_\gamma = \dfrac{RC}{\sigma^{1-\gamma}}$ is the time constant measured in seconds, R (resistance), C (capacitance) and $v(t)$ is the voltage source, $q(t)$(charge), the parameter σ characterizes the fractional structures.

The main advantage of fractional derivatives lies in that it is more suitable for describing memory properties of various materials and processes in comparison with classical integer-order derivative. However, some objection has been revealed for the slightly burdensome mathematical formula of its definition and the resultant complexities in the solutions of the differential equations of fractional order.

At present, there exist a number of definitions of fractional derivatives in the literature, each depending on a given set of assumptions. But it is worth noting that these kinds of derivatives do not satisfy the classical chain rule. The discrepancies between known definitions can be solved in simple way by presenting a new fractional definition which is called the "Conformable Fractional Derivative". Khalil and al. [18] proposed this new fractional derivative that has some basic characteristics of the first-order derivative such as the product rule and the chain rule which seems more appropriate to describe many more models.

This new definition has attracted a great deal of attention from many researchers. see [18] and references therein. For the basic properties of the conformable fractional derivative, some results have been obtained [25, 33]. In [1], Abdeljawad proves chain rules, exponential functions, Gronwall's inequality, fractional integration by parts, Taylor power series expansions and Laplace transforms for the conformable fractional calculus. Furthermore, linear differential systems are discussed [33]. In [2], Batarfi and al. obtain the Green function for a conformable fractional linear problem and then the study of nonlinear con-

formable fractional differential equations, its several applications and generalizations were also discussed in [33, 7].
In this master thesis a new kind of fractional derivative is introduced the most important properties of the conformable fractional derivative and integral have been introduced, some interesting results of ordinary fractional calculus are extended to conformable one. Finally, using obtained results the conformable fractional stochastic differential equations are established.
After this introduction, this master thesis is organized as follows.
In Chapter 1: **"Preliminaries"** we recall the most essential definitions from the classical calculus and remind some techniques which are necessary for the understanding of our work. In particular, we introduce some basic concepts concerning continuous time stochastic processes.

Chapter 2: **"Fractional Calculus"** gives the basic approaches to define a fractional integral or derivative namely the Grünwald-Letnikov, Riemann-Liouville and the Caputo integral and derivatives, their most important properties, composition rules, as well as Laplace transforms. Then we start the study of differential equations containing fractional derivatives, firstly the so called ordinary fractional differential equations (FDEs). We restrict ourselves to linear FDEs because there is a more compact theory. We give conditions for existence and uniqueness of solution for linear initial-value problems. Secondly stochastic fractional differential equations (SFDEs), we study a result on the global existence and uniqueness of solution for Caputo fractional stochastic differential equations.

In chapter 3: **" Conformable Fractional Calculus"** we discuss the basic theory of the conformable fractional calculus. We introduce, for the first time, this new concept of derivatives, give some important properties and examples, introduce and prove some distinguishing features and basic theorems of these derivatives. Second we prove the existence and uniqueness result on the solution of differential equations containing conformable fractional derivatives. In the final section of this chapter existence and uniqueness results of solution of a class of stochastic differential equations of the considered fractional derivative are discussed.
Finally, we give a conclusion. In witch we summarize the main results of this work.

Chapter 1

Preliminaries.

In this first chapter we gather some preliminary and basic notions used throughout the course of this master thesis and has the object to be a library of basic results. In short, we give here a quick reminder of the fundamental useful results. In particular, we first introduce some basic concepts on stochastic processes theory, stochastic integration and stochastic differential equations, finally recall the notion laplace transformation and fixed point theory. For more details, see the following references [8, 9, 12, 16, 23].

1.1 Basics tools for stochastic calculus.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable X is a rule for assigning to every outcome ω of an experiment Ω a number $X(\omega)$. A stochastic process X_t is a rule for assigning to every $\omega \in \Omega$ a function $X_t(\omega)$. Thus, a stochastic process could be seen as a family of time functions depending on the parameter ω (a collection of paths or trajectories) or, equivalently, a family of random variables depending on a time parameter t, or a function of t and ω as well.

1.1.1 Stochastic processes.

Definition 1.1.1. *(Stochastic process). We define real valued* $(one - dimensional)$ ***stochastic process*** *a family of random variables* $\{X_t\}_{t\in I}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$*:*

$$X_t : \Omega \longrightarrow \mathbb{R}, t \in \mathrm{I} \subseteq \mathbb{R}_+.$$

We shall say that $\{X_t\}_{t\in I}$ is a discrete-state process if its values are countable. Otherwise, it is a continuous-state process. The set $\mathcal{S} \subseteq \mathbb{R}$, whose elements are the values of the process, is called state space. A stochastic process could be a discrete time or a continuous time process, according as the set I *is countable or continuous.*

Definition 1.1.2. *(Finite dimensional distributions). For any natural number $k \in \mathbb{N}$ and a "time" sequence $\{t_i\}_{i=1,\dots,k} \in \mathrm{I}$, the finite-dimensional distributions of the real valued stochastic process $X_t = \{X_t\}_{t\in I}$ are the measures $\mu_{t_1,\dots,t_k}$, defined on $\mathbb{R}^k$, such that*

$$\mu_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = \mathbb{P}(\{X_{t_1} \in A_1, \cdots, X_{t_k} \in A_k\}), \tag{1.1}$$

where $\{A_1, \dots, A_k\}$ are Borel sets on $\mathbb{R}$.

Theorem 1.1.1. *(Kolmogorov extension theorem [16]). For all $\{t_i\}_{i=1,\dots,k} \in \mathrm{I}$, $k \in \mathbb{N}$ let $\nu_{t_1,\dots,t_k}$ be probability measures on $\mathbb{R}^k$, such that :*

1. *for all permutations π on $\{1, 2, \dots, k\}$,*

$$\nu_{t_{\pi(1)},\dots,t_{\pi(k)}}(A_1 \times ... \times A_k) = \nu_{t_1,\dots,t_k}(A_{\pi^{-1}(1)} \times \cdots \times A_{\pi^{-1}(k)})$$

2. *for any $m \in \mathbb{N}$,*

$$\nu_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = \nu_{t_1,\dots,t_k,t_{k+1},\dots,t_{k+m}}(A_1 \times \cdots \times A_k \times \mathbb{R} \times \cdots \times \mathbb{R}),$$

where of course the set on the right side as a total of $k + m$ factors. Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a real valued stochastic process X defined on it, such that:

$$\nu_{t_1,\dots,t_k}(A_1 \times ... \times A_k) = \mathbb{P}(\{X_{t_1} \in A_1, ..., X_{t_k} \in A_k\}),$$

for any $t_i \in \mathrm{I}$, $k \in \mathbb{N}$ and $A_i \in \mathcal{B}$.

Definition 1.1.3. *(Filtration). An increasing family $\mathcal{F}_t = \{\mathcal{F}_t\}_{t\in I}$ of complete sub σ-fields of $\mathcal{F}$ is said a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$.*
Consider a stochastic process $X = \{X_t\}_{t\in I}$ and let:

$$\mathcal{F}_t^X = \sigma(\{X_s; 0 \leq s \leq t\}) = \sigma(\{\mathcal{N} \cup \{X_s^{-1}(H); 0 \leq s \leq t, H \in \mathcal{B}\}\}),$$

where $\mathcal{B}$ is the Borel σ-algebra and $\mathcal{N}$ indicates the class of null-sets. Clearly if $0 \leq s \leq t$ one has

$$\mathcal{F}_s^X \subseteq \mathcal{F}_t^X \subseteq \mathcal{F}.$$

Therefore, $\mathcal{F}^X = \{\mathcal{F}_t^X\}_{t\in I}$ *defines a filtration, termed natural filtration of* $\{X_t\}_{t\in I}$.

Definition 1.1.4. *(Adapted process). A stochastic process* $\{X_t\}_{t\in I}$ *is said adapted to the filtration* $\{\mathcal{F}_t\}_{t\in I}$ *if for each* $t \geq 0$*:*

$$\mathcal{F}_t^X \subseteq \mathcal{F}_t.$$

In other words, for each t, the r.v. $X(t)$ *is* $\mathcal{F}_t$*-measurable.*

Definition 1.1.5. *(Predictable). A stochastic process* $\{X(t) : t \in [0, T]\}$ *is predictable if there exists* $\mathcal{F}_t-$*adapted and left-continuous processes* $\{X_n(t) : t \in [0, T]\}$ *such that* $X_n(t) \longrightarrow X(t)$ *as* $n \longrightarrow \infty$ *for* $t \in [0, T]$.

Definition 1.1.6. *(Martingale). A stochastic process* $M = \{M_t\}_{t\geq 0}$ *is a martingale with respect to the filtration* $\mathcal{F}_t$ *and the measure* $\mathbb{P}$ *if, for any* $t \geq 0$*, one has:*

1. $M_t \in L^1(\Omega, \mathbb{P})$,
2. $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$, $0 \leq s \leq t$.

   This means that M_t *is* $\mathcal{F}_t$*-adapted.Moreover, the expected value of* M_t *does not depend on time. Indeed,*

   $$\mathbb{E}(M_t) = \mathbb{E}(\mathbb{E}(M_t|\mathcal{F}_0)) = \mathbb{E}(M_0).$$

Definition 1.1.7. *(Gaussian process). A real stochastic process* $\{X_t\}_{t\in I}$ *is Gaussian if and only if, for every finite sequence* $\{t_1, t_2, ..., t_k\} \in I$,

$$X_{t_1,...,t_k} = (X_{t_1}, \cdots, X_{t_k})$$

has a multivariate normal distribution.

Definition 1.1.8. *(Stationary process). A stochastic process* $\{X_t\}_{t\geq 0}$ *is said a stationary process if any collection* $\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$ *has the same distribution of* $\{X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_n+\tau}\}$ *for each* $\tau \geq 0$*. That is,*

$$\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\} \stackrel{d}{=} \{X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_n+\tau}\}.$$

Let X be a stationary process, then the following elementary properties hold:

- Varying t, all the random variables X_t have the same low; i.e. $X_{t_1} \stackrel{d}{=} X_{t_2}$ for any t_1, $t_2 \geq 0$.

- All the moments, if they exist, are constant in time.
- The distribution of X_{t_1} and X_{t_2} depends only on the difference $\tau = t_1 - t_2$ (time lag).

Therefore, the autocovariance function $\gamma(t_1, t_2) = \gamma(t_1 - t_2)$ depends only on $\tau = t_1 - t_2$. We write

$$\gamma(\tau) = \mathbb{E}(X_t - \mu)(X_{t-\tau} - \mu) = Cov(X_t, X_{t-\tau}), \tag{1.2}$$

where $\mu = \mathbb{E}(X(t)$ and $\gamma(\tau)$ indicates the autocovariance coefficient at the lag τ.

Definition 1.1.9. *(Stationary increment process). A stochastic process $\{X_t\}_{t\geq 0}$ is said a stationary increment process, shortly si, if for any $h \geq 0$:*

$$\{X_{t+h} - X_h\}_{t\geq 0} \overset{d}{=} \{X_t - X_0\}_{t\geq 0}. \tag{1.3}$$

Definition 1.1.10. *(Self-similar processes). A real valued stochastic process $X = \{X_t\}_{t\geq 0}$ is said self-similar with index $H \geq 0$, shortly H-ss, if for any $a \geq 0$:*

$$\{X_{at}\}_{t\geq 0} \overset{d}{=} \{a^H X_t\}_{t\geq 0}.$$

We observe that the transformation scales differently "space" and "time", for this reason one often prefers using the word self-affine process. The index H is said Hurst's exponent or scaling exponent of the process.

Remark 1.1.1. *Observe that, if X is an H-ss process, then all the finite-dimensional distributions of X in $[0, \infty]$ are completely determined by the distribution in any finite real interval.*

Theorem 1.1.1. *[23] For $p \in (0, \infty)$, let $\mathbb{L}^p = \mathbb{L}^p(\Omega; \mathbb{R}^d)$ be the family of $\mathbb{R}^d-$valeud random variables X with $\mathbb{E}|X|^p < \infty$. In $\mathbb{L}^1$, we have $|\mathbb{E}(X)| \leq \mathbb{E}|X|$. Moreover, the following three inequalities are very useful:*

1. ***Hölder's inequality:***

$$\mathbb{E}(|XY|) \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} (\mathbb{E}|Y|^q)^{\frac{1}{q}}$$

if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $X \in \mathbb{L}^p$, $Y \in \mathbb{L}^q$;

2. ***Minkovski's inequality:***

$$\left(\mathbb{E}|X+Y|^P\right)^{\frac{1}{p}} \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} + (\mathbb{E}|Y|^p)^{\frac{1}{p}}$$

if $p > 1$, $X, Y \in \mathbb{L}^p$;

3. ***Chebyshev's inequality:***

$$\mathbb{P}\{w : |X(w)| \geq c\} \leq c^{-p}\mathbb{E}(|X|^p)$$

if $c > 0$, $p > 0$, $X \in \mathbb{L}^p$.

4. *A simple application of Hölder's inequality implies*

$$(\mathbb{E}|X|^r)^{\frac{1}{r}} \leq (\mathbb{E}|X|^p)^{\frac{1}{p}}$$

if $0 < r < p < \infty$, $X \in \mathbb{L}^p$.

Theorem 1.1.2. *(Monotonic convergence theorem [23]). If* X_k *is an increasing sequence of nonnegative random variables, then*

$$\lim_{k\longrightarrow\infty} \mathbb{E}(X_k) = \mathbb{E}\left(\lim_{k\longrightarrow\infty} X_k\right).$$

Theorem 1.1.3. *(Dominated convergence theorem [23]). Let* $p \geq 1$, $X_k \subset \mathbb{L}^p(\Omega, \mathbb{R}^d)$ *and* $Y \subset \mathbb{L}^p(\Omega, \mathbb{R})$. *Assume that* $|X_k| \leq Y$ *a.s. and* X_k *converges to* X *in probability. Then* $X \in \mathbb{L}^p(\Omega, \mathbb{R}^d)$, X_k *converges to* X *in* $\mathbb{L}^p$, *and*

$$\lim_{k\longrightarrow\infty} \mathbb{E}(X_k) = \mathbb{E}(X).$$

Definition 1.1.11. *(Volterra integral equations [8]). For the first kind Volterra integral equations, the unknown function* $u(x)$ *occurs only under the integral sign in the form:*

$$f(x) = \int_0^x K(x,t)u(t)dt. \tag{1.4}$$

However, Volterra integral equations of the second kind, the unknown function $u(x)$ *occurs inside and outside the integral sign. The second kind is represented in the form:*

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt. \tag{1.5}$$

The kernel $K(x,t)$ *and the function* $f(x)$ *are given real-valued functions, and* λ *is a parameter.*

1.1.2 Brownian motion

Brownian motion is the name given to the irregular movement of pollen grains, suspended in water, observed by the Scottish botanist Robert Brown in 1828. The motion mathematically it is $W(t)$ Let us now give the mathematical definition of Brownian motion.

Definition 1.1.12. *(Brownian motion). A stochastic process* $W = \{W_t\}_{t\geq 0}$ *is an ordinary (standart) Brownian motion (Bm) if:*
(i) $W(0) = 0$ *a.s. and it is* $\mathcal{F}_t$*-adapted,*
(ii) it has independent increments, That is the random variables $W_{t_2} - W_{t_1}$ *and* $W_{t_4} - W_{t_3}$ *are independent for any* $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$*, it has stationary increments,*
(iii) for each $t > 0$*,* $W(t)$ *has a Gaussian distribution with mean zero and variance* t*, and covariance* $\mathbb{E}(W(t)W(s)) = \min(t, s)$*.*
(iv) its sample paths are continuous a.s. (The Bm trajectories starts in zero a.s. and are continuous).

Proposition 1.1.1. *[23] The Brownian motion* $W(t)$ *is an* $\mathcal{F}_t$*-martingale.*

Theorem 1.1.4. W *is a* H*-ss process with* $H = 1/2$

Proof: It is enough to show that for every $a > 0$, $\{a^{1/2}W(at)\}$ is also Brownia nmotion. Conditions (i), (ii) and (iv) follow from the same conditions for $\{W(t)\}$. As to (iii), Gaussianity and mean-zero property also follow from the properties of $\{W(t)\}$. As to the variance, $\mathbb{E}\left[\left(a^{1/2}W(at)^2\right)\right] = t$. Thus $\{a^{1/2}W(at)\}$ is a Brownian motion.

Proposition 1.1.1. *[10]*

1. *Self-similarity: The Brownian motion is* $\frac{1}{2}$*-SSSI.*

2. *Symmetry:* $\{-B(t), t \geq 0\}$ *is also a Brownian motion.*

3. *Markov Property: Brownian motion is a Markov process.*

4. *Hölder continuous: A Brownian motion has paths a.s. locally* $\gamma-$*Hölder continuous for* $\gamma \in [0, 1/2)$*.*

5. *Nondifferentiability of Paths: The Brownian motion's sample paths are a.s. nowhere differentiable.*

Theorem 1.1.2. *[16]. For a Gaussian sequence* $\{X_k, k \geq 0\}$ *to be Markovian each of the following condition is necessary and sufficient.*

- *For any* $k \leq n$

$$\mathbb{E}(X_k|X_1, ..., X_{k-1}) = \mathbb{E}(X_k|X_{k-1}). \tag{1.6}$$

- *For* $j \leq l \leq k \leq n$

$$\rho_{jk} = \rho_{jl}\rho_{lk}. \tag{1.7}$$

Definition 1.1.13. *(Markovian process). We say that* X *is Markovian if any finite collection* $\{X(t_1), ..., X(t_n)\}$*,* $t_i \in I$*, is Markovian.*

Let the process X be Gaussian and Markovian. In view of (1.6) we have the Markov property:

$$\mathbb{E}(X(t+h)|\{X(s), s \leq t\}) = \mathbb{E}(X(t+h)|X(t)), \tag{1.8}$$

for any $h > 0$. Moreover, by (1.7), one has:

$$\rho(s,t) = \rho(s,h)\sigma(h,t), s \leq h \leq t, \tag{1.9}$$

with $\rho(s,t) = \mathbb{E}(X(s)X(t))/\sigma(s)\sigma(t)$.

Definition 1.1.14. *(H-sssi processes). A stochastic process* $X = \{X_t\}_{t\in I}$*,* $\mathcal{F}$*-adapted, which is* H*-ss with stationary increments, is said* H*-sssi process with exponent* H*.*

1.2 Stochastic integration

let us consider the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of the Bm $W(t)$, $t \geq 0$. We introduce the following class of functions.

Definition 1.2.1. *Let* $\mathcal{V}(S,T)$ *be the class of real measurable functions* $f(t,\omega)$*, defined on* $[0,\infty) \times \Omega$*, such that:*

1. $f(t,\omega)$ *is* $\mathcal{F}_t$*-adapted.*
2. $\mathbb{E}\left(\int_S^T f(t,\cdot)^2 dt\right) < \infty$.

1.2.1 Itô integral

Let $f \in \mathcal{V}(S,T)$. We want to define the Itô integral of f in the interval $[S,T)$. Namely:

$$\mathcal{I}(f)(\omega) = \int_S^T f(t,\omega)dW_t(\omega), \tag{1.10}$$

where W_t is a standard ($\mathbb{E}(W(1)^2) = 1$) one dimensional Brownian motion. We begin defining the integral for a special class of functions:

Definition 1.2.2. *(Simple functions). A function $\phi \in \mathcal{V}(S,T)$ is called simple function (or elementary), if it can be expressed as a superposition of characteristic functions.*

$$\phi(t,\omega) = \sum_{k\geq 0} e_k(\omega) 1_{[t_k,t_{k+1})}(t), \tag{1.11}$$

Definition 1.2.3. *Let $\phi \in \mathcal{V}(S,T)$ be a simple function of the form of (1.11), then we define the stochastic integral:*

$$\int_S^T \phi(t,\omega)dW_t = \sum_{k\geq 0} e_k(\omega)(W_{t_{k+1}} - W_{t_k})(\omega). \tag{1.12}$$

Lemma 1.2.1. *(Itô isometry [23]). Let $\phi \in \mathcal{V}(S,T)$ be a simple function, then:*

$$\mathbb{E}\left((\int_S^T \phi(t,\cdot)dW_t)^2 \right) = \mathbb{E}\left(\int_S^T \phi(t,\cdot)^2 dt \right). \tag{1.13}$$

Remark 1.2.1. *Observe that (1.13) is indeed an isometry. In fact, it can been written as equality of norms in $\mathbb{L}^2$ spaces:*

$$\left\| \int_S^T \phi(t,\cdot)dW_t \right\|_{\mathbb{L}^2(\Omega,\mathbb{P})} = \|\phi\|_{\mathbb{L}^2([S,T]\times\Omega)}.$$

We have the following important proposition.

Proposition 1.2.1. *[10] Let $f \in \mathcal{V}$, then there exists a sequence of simple functions $\phi_n \in \mathcal{V}, n \in \mathbb{N}$, which converges to f in the $\mathbb{L}^2$-norm. Namely,*

$$\lim_{n\longrightarrow\infty} \int_S^T \mathbb{E}\left((f(t,\cdot) - \phi_n(t,\cdot))^2 \right) dt = \lim_{n\longrightarrow\infty} ||f - \phi_n||^2_{\mathbb{L}^2([S,T]\times\Omega)} = 0. \tag{1.14}$$

Given $f \in \mathcal{V}(S,T)$, the proposition above, together with Ito isometry, implies that the sequence $\left\{ \int_S^T \phi_n(t,\omega)dW_t(\omega), n \in \mathbb{N} \right\}$ is Cauchy on $\mathbb{L}^2(\Omega,\mathbb{P})$. So that, it converges to a limit in $\mathbb{L}^2(\Omega,\mathbb{P})$. We call this limit the Itô integral of f .

Definition 1.2.4. *(Itô integral). Let $f \in \mathcal{V}(S,T)$ The Itô integral from S to T of f is defined as the $\mathbb{L}^2(\Omega,\mathbb{P})$ limit:*

$$\mathcal{I}(f) = \int_S^T f(t,\omega)dW_t(\omega) = \lim_{n\longrightarrow\infty} \phi_n(t,\omega)dW_t(\omega), \tag{1.15}$$

where $\phi_n \in \mathcal{V}$, $n \in \mathbb{N}$, is a sequence of simple functions which converges to $f \in \mathbb{L}^2([S,T]\times\Omega)$.

Remark 1.2.2. *Observe, in view of (1.14), that the definition above does not depend on the actual choice of $\{\phi_n, n \in \mathbb{N}\}$.*
By definition, we have that Itô isometry holds for Itô integrals:

Corollary 1.2.1. *(Itô isometry for Itô integrals [10]). Let $f \in \mathcal{V}(S,T)$, then:*

$$\mathbb{E}\left((\int_S^T f(t,\cdot)dW_t)^2 \right) = \mathbb{E}\left(\int_S^T f^2(t,\cdot)dt \right). \tag{1.16}$$

Moreover,

Corollary 1.2.2. *[10] If $f_n(t,\omega) \in \mathcal{V}(S,T)$ converges to $f(t,\omega) \in \mathcal{V}(S,T)$ as $n \longrightarrow \infty$ in the $\mathbb{L}^2([S,T]\times\Omega)$-norm, then:*

$$\int_S^T f_n(t,\cdot)dW_t \longrightarrow \int_S^T f(t,\cdot)dW_t, \tag{1.17}$$

in the $\mathbb{L}^2(\Omega,\mathbb{P})$-norm.

Proposition 1.2.2. *[16] Let $f, g \in \mathcal{V}(0,T)$ and let $0 \le S < U < T$. Then:*

1. $\int_S^T f dW_t = \int_S^U f dW_t + \int_U^T f dW_t$.

2. *For some constant $a \in \mathbb{R}$,* $\int_S^T (af+g)dW_t = a\int_S^T f dW_t + \int_S^T g dW_t$.

3. $\mathbb{E}\left[\int_S^T f dW_t\right] = 0$.

4. $\int_S^T f dW_t$ *is $\mathcal{F}_T$-measurable.*

5. *The process $M_t(\omega) = \int_T^0 f(t,\omega)dW_s(\omega)$ where $f \in \mathcal{V}(0,T)$ for any $t > 0$, is a martingale with respect to $\mathcal{F}_t$.*

The construction of the Itô Integral can be extended to a class of function $f(t, \omega)$ which satisfies a weak integration condition. This generalization is indeed necessary because it is not difficult to find functions which do not belong to $\mathcal{V}$. For instance, take a function of Bm which increase rapidly $f(t, \omega) = exp(W_t(\omega)^2)$. Therefore, we introduce the following class of functions:

Definition 1.2.5. *Let $\mathcal{W}(S, T)$ be the class of real measurable functions $f(t, \omega)$, defined on $[0, \infty) \times \Omega$, such that*

1. *$f(t, \omega)$ is $\mathcal{F}_t$-adapted.*
2. *$\mathbb{P}(\int_S^T f(t, \cdot)^2 dt < \infty) = 1$.*

Remark 1.2.3. *Clearly, $\mathcal{V} \subset \mathcal{W}$.*
In the construction of stochastic integrals for the class of functions belonging to Ω we can no longer use the $\mathbb{L}^2$ notion of convergence, but rather we have to use convergence in probability . In fact, for any $f \in \mathcal{W}$, one can show that there exists a sequence of simple functions $\phi_n \in \mathcal{W}$ such that

$$\int_S^T |\phi_n(t, \cdot) - f(t, \cdot)|^2 dt \longrightarrow 0 \tag{1.18}$$

in probability. For such a sequence one has that the sequence $\{\int_S^T |\phi_n(t, \Delta) dW_t(\omega), n \in \mathbb{N}\}$ converges in probability to some random variable. Moreover, the limit does not depends on the approximating sequence ϕ_n. Thus, we may define:

Definition 1.2.6. *(Itô integral II). Let $f \in \mathcal{W}(S, T)$. The Itô integral from S to T of f is defined as the limit in probability:*

$$\int_S^T f(t, \omega) dW_t(\omega) = \lim_{n \longrightarrow \infty} \int_S^T \phi_n(t, \omega) dW_t(\omega), \tag{1.19}$$

where $\phi_n \in \mathcal{W}, n \in \mathbb{N}$, is a sequence of simple functions which converges to f in probability.

Remark 1.2.4. *Note that this integral is not in general a martingale. However, it is a local martingale.*

1.2.2 One dimentionel Ito formula

Definition 1.2.7. *(Itô processes). Let X_t be a stochastic process, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that for any $t \geq 0$:*

$$X_t = X(0) + \int_0^t u_s ds + \int_0^t v_s dW_s, \tag{1.20}$$

where $u, v \in \mathcal{W}$. Then, X_t is called (one-dimensional) Itô process.

Theorem 1.2.1. *(Itô formula [10]). Let $g(t,x) \in C^2(\mathbb{R}_+ \times \mathbb{R})$ and let X_t be an Itô process of the form:*

$$dX_t = u_t dt + v_t dW_t.$$

Then, the process

$$Y_t = g(t, X_t), t \geq 0,$$

is again an Itô process, and the following Itô formula holds:

$$dY_t = dg(t,X_t) = \left(\partial_t g(t,X_t) + u_t \partial_x f(t,X_t) + \frac{1}{2} v_t^2 \partial_{xx} f(t,X_t)\right) dt + v_t \partial_x f(t,X_t) dW_t, \tag{1.21}$$

or equivalently:

$$dg(t,X_t) = \partial_t g(t,X_t)dt + \partial_x g(t,X_t)dX_t + \frac{1}{2}\partial_{xx} g(t,X_t) d\langle X\rangle_t, \tag{1.22}$$

where $\langle X\rangle_t = \int_0^t v_s^2 ds$ is the quadratic variation of the Itô diffusion.

Theorem 1.2.2. *(Gronwall's inequality [16]). Let $T > 0$, $c > 0$ and $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0,T]$, let $v(\cdot)$ be a nonnegative integrable function on $[0,T]$. If*

$$u(t) \leq c + \int_0^t v(s)u(s)ds, 0 \leq t \leq T, \tag{1.23}$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds\right), 0 \leq t \leq T. \tag{1.24}$$

1.2.3 Stochastic differential equations

The equation has to be interpreted as

$$X_t = X_{t_0} + \int_{t_0}^t a(s,X_s)ds + \int_{t_0}^t \sigma(s,X_s)dW_s, t_0 \leq t \leq T, \tag{1.25}$$

where the first integral is a Lebesgue (or Riemann) integral for each sample path and the second is an Itô integral.

As with deterministic ordinary and partial differential equations, it is important to know whether a given SDE has a solution, and whether or not it is unique.

Definition 1.2.8. *(Strong and weak solutions). If the version W_t of Brownian motion defined in the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is given in advance and the solution X_t constructed from it is $\mathcal{F}_t$-adapted, the solution is called a strong solution. If we are only given the functions $a(t,x)$ and $\sigma(t,x)$ and ask for a pair of processes (X_t, W_t), then the solution X_t (or more precisely (X_t, W_t)) is called a weak solution.*

The hypothesis of an existence and uniqueness theorem are usually sufficient but not necessary, conditions. Some are quite strong, but can be weakened in several ways. Most of the assumptions concern the coefficients $a, \sigma : [t_0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$.

Existence and uniqueness conditions

- A1. Measurability: $a(t,x)$ and $\sigma(t,x)$ are L^2-measurable in $[t_0, T] \times \mathbb{R}$.
- A2. Lipschitz condition: there exists a constant $K > 0$ such that for any $t \in [t_0, T]$ and $x, y \in \mathbb{R}$:
$$|a(t,x) - a(t,y)| \leq K|x-y|, \tag{1.26}$$
and
$$|\sigma(t,x) - \sigma(t,y)| \leq K|x-y|. \tag{1.27}$$
- A3. Linear growth bound: there exists a constant $K > 0$ such that for any $t \in [t_0, T]$ and $x, y \in \mathbb{R}$:
$$|a(t,x)|^2 \leq K^2(1+|x|^2), \tag{1.28}$$
and
$$|\sigma(t,x)|^2 \leq K^2(1+|x|^2). \tag{1.29}$$
- A4. Initial value: X_{t_0} is $\mathcal{F}_{t_0}$-measurable with $\mathbb{E}(|X_{t_0}|^2) < \infty$.

Theorem 1.2.3. *(Existence and uniqueness theorem for stochastic differential equations [16]).*
Let $T > 0$ and $a(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$ be measurable functions satisfying

$$| a(t,x) | + | \sigma(t,x) | \leq C(1+|x|), x \in \mathbb{R}^n, t \in [0,T]. \tag{1.30}$$

for some constant C, (where $|\sigma| = \sum | \sigma_{ij} |^2$) and such that

$$| a(t,x) - a(t,y) | + | \sigma(t,x) - \sigma(t,y) | \leq K | x - y |; x, y \in \mathbb{R}^n, t \in [0,T]. \tag{1.31}$$

for some constant K. Let Z be a random variable which is independent of the $\sigma-$algebra $\mathcal{F}_\infty^{(m)}$ generated by $W_s(\cdot)$, $s \geq 0$ and such that

$$\mathbb{E}[|Z|^2] < \infty.$$

Then the stochastic differential equation

$$\begin{cases} dX_t &= a(t,X_t)dt + \sigma(t,X_t)dW_t, \quad 0 \leq t \leq T, \\ X_0 &= Z. \end{cases} \tag{1.32}$$

has a unique $t-$continuous solution $X_t(w)$ with the property that $X_t(w)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $W_s(\cdot)$; $s \leq t$ and

$$\mathbb{E}\left[\int_0^T | X_t |^2 \, dt\right] < \infty. \tag{1.33}$$

Remark 1.2.5. *Conditions (1.30) and (1.31) are natural in view of the following two simple examples from deterministic differential equations (i.e.$\sigma = 0$): The equation*

$$\frac{dX_t}{dt} = X_t^2, \quad X_0 = 1 \tag{1.34}$$

corresponding to $a(x) = x^2$ (which does not satisfy (1.30)) has the (unique) solution

$$X_t = \frac{1}{1-t}; \quad 0 \leq t < 1.$$

Thus it is impossible to find a global solution (defined for all t) in this case. More generally, condition (1.30) ensures that the solution $X_t(w)$ of (1.32) does not explode, i.e. that $| X_t(w) |$ does not tend to ∞ in a finite time. Thus condition (1.31) guarantees that equation (1.32) has a unique solution. Here uniqueness means that if $X_1(t,w)$ and $X_2(t,w)$ are two $t-$continuous processes satisfying (1.32) and (1.33) then

$$X_1(t,w) = X_2(t,w), t \leq T, a.s. \tag{1.35}$$

1.3 Laplace Transform

The Laplace Transform is a function transformation commonly used in the solution of complicated differential equations. With the Laplace transform it is frequently possible to avoid working with equations of different differential order directly by translating the problem into a domain where the solution presents itself algebraically.

Definition 1.3.1. *The Laplace transform of a function f of the real variable $t \in \mathbb{R}_+$ is defined by:*

$$\mathcal{L}\{f(s);s\} = \int_0^{\infty} e^{-st} f(t)dt, s \in \mathbb{R} \tag{1.36}$$

$f(t)$ is called the original of $\mathcal{L}f(s)$.

Definition 1.3.2. *The transformation reversal of Laplace is carried out by means of an integral in the complex plan, pure and positive:*

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-\infty}^{\gamma+\infty} e^{st}F(s)ds$$

where γ is chosen to ensure that the integral is convergent.which implies that γ is greater than the actual singularity part of $F(s)$.

Proposition 1.3.1. *Suppose that $f(t)$ and $g(t)$ are two functions, which are equal to zero for $t < 0$ and for which the Laplace transforms $F(t)$ and $G(t)$ exist. The following statements hold (see [13]):*

(*a*) *The Laplace transform and its inverse are linear operators, suppose that $\lambda \in \mathbb{R}$, then :*

$$\mathcal{L}\{\lambda f(t) + g(t); s\} = \lambda\mathcal{L}\{f(t);s\} + \mathcal{L}\{g(t);s\} = \lambda F(s) + G(s)$$

$$\mathcal{L}^{-1}\{\lambda F(t) + G(t); s\} = \lambda\mathcal{L}^{-1}\{F(t);s\} + \mathcal{L}^{-1}\{G(t);s\} = \lambda f(s) + g(s)$$

(*b*) *For the Laplace transform of the convolution of $f(t)$ and $g(t)$ is follows:*

$$\mathcal{L}\{f(t) * g(t); s\} = F(s)G(s)$$

where the convolution is defined by:

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

(c) *The limit of the function $sF(s)$ for $s \to \infty$ is given by*

$$\lim_{s \longrightarrow \infty} sF(s) = f(0)$$

(d) *The Laplace transform of the $n-th$ derivative $(n \in \mathbb{N})$ of $f(t)$ is given by:*

$$\mathcal{L}\{f^{(n)}(t);s\} = s^n F(s) - \sum_{k=0}^{n-1} s^{s-k-1} f^k(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{s-k-1}(0)$$

.

Definition 1.3.3. *The derivative of a function f is defined as*

$$D^1 f(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

Iterating this operation yields an expression for the $n - th$ derivative of a function. As can be easily seen for any natural number n:

$$D^n f(x) = \lim_{h \to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x + (n-m)h)$$

or equivalently,

$$D^n f(x) = \lim_{h \to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x - mh) \tag{1.37}$$

1.4 Some Results from Nonlinear Analysis

Definition 1.4.1. *(**Banach space**) A normed space X is called a Banach space if it is complete, i.e., if every Cauchy sequence is convergent. That is*

$$\{f_n\}_{n\in\mathbb{N}} \text{ is cauchy in } X \Rightarrow \exists f \in X \text{ such that } f_n \to f$$

Definition 1.4.2. *[12](**Contractive function**) Let (X, d) be a complete metric space. A function $f : X \longrightarrow X$ is called a contractive function if there exists $k < 1$ such that for any $x, y \in X$,*

$$d(f(x), f(y)) \leq kd(x, y).$$

Definition 1.4.3. *[12](**Fixed point**) A fixed point of a mapping $T : X \longrightarrow X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is*

$$Tx = x.$$

Definition 1.4.4. *[12](**Banach's fixed point theorem**) Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a contraction on X. Then T has a unique fixed point $x \in X$ (such that $T(x) = x$).*

Chapter 2

Fractional Calculus

This Chapter mainly introduces definitions and basic properties of fractional derivatives, including Riemann-Liouville fractional derivative, Caputo fractional derivative and some basics properties of these derivatives are discussed. The difference between Caputo and Riemann-Liouville formulas for the fractional derivatives also is mentioned. Some basic tools of fractional differential equations are introduced, such as existence results of fractional ordinary equations are obtained and those of stochastic fractional equations are given at the end of the chapter. See [3, 17, 20, 21, 26] and their references for details on the fractional calculus.

2.1 Fractional Calculus

In this section, we shall give some basic formulas and techniques which are necessary to better understand the rest of this work. We start off with the Gamma function.

2.1.1 Special Functions

The Gamma Function

The most basic interpretation of the Gamma function is simply the generalization of the factorial for all real numbers.

Definition 2.1.1. *Its definition is given by*

$$\Gamma(x) = \int_0^\infty e^{(-t)} t^{x-1} dt, x \in \mathbb{R}^+ \tag{2.1}$$

The Gamma function has some unique properties. By using its recursion relations we can obtain formulas:

$$\Gamma(x+1) = x\Gamma(x), x \in \mathbb{R}^+$$

$$\Gamma(x) = (x-1)!, x \in \mathbb{R}^+$$

Example 2.1.1.

$$\Gamma(1) = \Gamma(2) = 1$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(n+1/2) = \frac{\sqrt{\pi}}{2^n}(2n-1), n \in \mathbb{N}$$

The Beta Function

Like the Gamma function, the Beta function is defined by a definite integral.

Definition 2.1.2. *It's given by :*

$$\boldsymbol{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, x, y \in \mathbb{R}^+ \tag{2.2}$$

The Beta function can also be defined in terms of the Gamma function:

$$\boldsymbol{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, x, y \in \mathbb{R}^+ \tag{2.3}$$

The Mittag-Leffler Function

The Mittag-Leffler function is named after a Swedish mathematician who defined and studied it in (1903, [21]). The function is a direct generalization of the exponential function, $\exp(x)$, and it plays a major role in fractional calculus.

Definition 2.1.3. *The standard definition of the Mittag-Leffler is given by :*

$$\mathrm{E}_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \tag{2.4}$$

It is also common to represent the Mittag-Leffler function in two arguments α and β. Such that

$$\mathrm{E}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \beta > 0, \alpha > 0. \tag{2.5}$$

The exponential series defined by (2.5) generalization of (2.4).

As a result of the definition given in (2.5), the following relations hold:

$$\mathrm{E}_{\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)} + x\mathrm{E}_{\alpha,\alpha+\beta}(x) \tag{2.6}$$

and

$$\mathrm{E}_{\alpha,\beta}(x) = \beta\mathrm{E}_{\alpha,\beta+1}(x) + \alpha x\frac{d}{dx}\mathrm{E}_{\alpha,\beta+1}(x) \tag{2.7}$$

Example 2.1.2.

$$\mathrm{E}_{\alpha,\beta}(0) = 1$$

$$\mathrm{E}_{1,1}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{x^k}{k!} = \mathbf{e}^x$$

$$\mathrm{E}_{1,2}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+2)} = \frac{1}{x}\sum_{k=0}^{\infty}\frac{x^{k+1}}{(k+1)!} = \frac{\mathbf{e}^x - 1}{x}$$

2.2 Basic fractional approche

2.2.1 Grunwald-Letnikov derivative

Grunwald-Letnikov derivative or also named Grunwald-Letnikov differintegral, is a direct generalization of the classical derivative. The idea behind is that h should approach 0 as n approaches infinity,

$$\begin{aligned} f^1(x) &= \lim_{h\to 0}\frac{f(x) - f(x-h)}{h} \\ f^2(x) &= \lim_{h\to 0}\frac{f^1(x) - f^1(x-h)}{h} \\ &= \lim_{h_1\to 0}\frac{\lim\limits_{h_2\to 0}\dfrac{f(x+h_2) - f(x)}{h_2} - \lim\limits_{h_2\to 0}\dfrac{f(x-h_1-h_2) - f(x-h_1)}{h_2}}{h_1} \end{aligned}$$

when $h_1 = h_2 = h$

$$f^2(x) = \lim_{h\to 0}\frac{f(x-2h) - 2f(x-h) + f(x)}{h^2}$$

continuing for n times we have

$$f^n(x) = D^n f(x) = \lim_{h\to 0}\frac{1}{h^n}\sum_{m=0}^{n}(-1)^m\binom{m}{n}f(x-mh).$$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

This can be replaced by Gamma functions as $\frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)}$ for non-integer n, α. Therefore, differentiation in fractional order is

$$ {}_aD^{\alpha}f(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{m=0}^{[\frac{x-a}{h}]}(-1)^m\frac{(\alpha-1)!}{m!(\alpha-m+1)!}f(x-mh). $$

For negative α, the process will be integration. Therefore, for integration we write

$$ {}_aD^{-\alpha}f(x) = \lim_{h\to 0}h^{\alpha}\sum_{m=0}^{[\frac{x-a}{h}]}\frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)}f(x-mh) \tag{2.8} $$

or equivalently,

$$ {}_aD^{-\alpha}f(x) = \lim_{n\to\infty}\left(\frac{n}{x-a}\right)^{\alpha}\sum_{m=0}^{n}\frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)}f\left(x-m\left(\frac{x-a}{n}\right)\right) \tag{2.9} $$

2.2.2 Riemann-Liouville approche

2.1.3.1 Riemann-Liouville fractional integrals

We begin by introducing a fractional integral of integer order n in the form of Cauchy formula.

$$ {}_aD_x^{-n}f(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1}f(t)dt \tag{2.10} $$

It will be shown that the above integral can be expressed in terms of n-multiple integral, that is

$$ {}_aD_x^{-n}f(x) = \int_0^x dx_1\int_a^{x_1}dx_2\int_a^{x_2}dx_3...\int_a^{x_{n-1}}f(t)dt \tag{2.11} $$

When $n=2$, by using the well-known Dirichlet formula, namely

$$ \int_a^b dx\int_a^x f(x,y)dy = \int_a^b dy\int_y^b f(x,y)dx \tag{2.12} $$

(2.11) becomes

$$ \begin{aligned} \int_a^x dx_1\int_a^{x_1}f(t)dt &= \int_a^x dt f(t)\int_t^x dx_1 \\ &= \int_a^x (x-t)f(t)dt. \end{aligned} $$

This shows that the two-fold integral can be reduced to a single integral with the help of Dirichlet formula. For $n=3$, the integral in (2.11) gives

$$ \begin{aligned} {}_aD_x^{-3}f(x) &= \int_a^x dx_1\int_a^{x_1}dx_2\int_a^{x_2}f(t)dt, \\ &= \int_a^x dx_1\left[\int_a^{x_1}dx_2\int_a^{x_2}f(t)dt\right]. \end{aligned} \tag{2.13} $$

By using the result in (2.13) the integrals within big brackets simplify to yield

$$ {}_aD_x^{-3}f(x) = \int_a^x dx_1 \left[\int_a^{x_1} (x_1 - t) f(t) dt \right]. \tag{2.14} $$

If we use (2.12), then the above expression reduces to

$$ {}_aD_x^{-3}f(x) = \int_a^x dt f(t) \int_x^t (x_1 - t) dx_1 = \int_a^x \frac{(x-t)^2}{2!} f(t) dt. \tag{2.15} $$

Continuing this process, we finally obtain

$$ {}_aD_x^{-n}f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt. \tag{2.16} $$

It is evident that the integral in (2.16) is meaningful for any number n provided its real part is greater than zero.

Definition 2.2.1. *Let $f(x) \in \mathbb{L}(a,b)$, $\alpha > 0$, then*

$$ {}_aI_x^{\alpha} f(x) =_a D_x^{-\alpha} f(x) = I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt. \tag{2.17} $$

and

$$ {}_xI_b^{\alpha} f(x) =_x D_b^{-\alpha} f(x) = I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt. \tag{2.18} $$

for $x > a$ is called Riemann-Liouville left-sided and right-sided fractional integral of order α, respectively.

Theorem 2.2.1. *[5] Let $f \in \mathbb{L}_1[a,b]$ and $\alpha > 0$. Then, the integral $I_a^{\alpha} f(x)$ exists for almost every $x \in [a,b]$. Moreover, the function $I_a^{\alpha} f$ itself is also an element of $\mathbb{L}_1[a,b]$.*
Proof: *We write the integral in question as*

$$ \int_a^x (x-t)^{\alpha-1} f(t) dt = \int_{-\infty}^{+\infty} \phi_1(x-t) \phi_2(t) dt, $$

where

$$ \phi_1(u) = \begin{cases} u^{\alpha-1} & for \quad 0 < u \leq b-a \\ 0 & else \end{cases} $$

and

$$ \phi_2(u) = \begin{cases} f(u) & for \quad a < u \leq b \\ 0 & else \end{cases} $$

By construction, $\phi_j \in \mathbb{L}(\mathbb{R})$ for $j \in \{1,2\}$ and thus by a classical result on Lebesgue integration.

Example 2.2.1. *If* $f(x) = (x-a)^{\beta-1}$*, then find the value of* ${}_aI_x^{\alpha} f(x)$*.*

Soultion: We have

$$ {}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1} dt. $$

If we substitute $t = a + y(x-a)$ in the above integral, it reduces to

$$ \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1} $$

where $\beta > 0$. Thus

$$ {}_aI_x^{\alpha} f(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1} $$

Proposition 2.2.1. *Fractional integrals obey the following properties:*

$$ {}_aI_x^{\alpha}\, {}_aI_x^{\beta} \phi = {}_aI_x^{\alpha+\beta} \phi = {}_aI_x^{\beta}\, {}_aI_x^{\alpha} \phi $$

$$ {}_xI_b^{\alpha}\, {}_xI_b^{\beta} \phi = {}_xI_b^{\alpha+\beta} \phi = {}_xI_b^{\beta}\, {}_xI_b^{\alpha} \phi \tag{2.19} $$

Proof: *By virtue of the definition (2.17), it follows that*

$$ \begin{aligned} {}_aI_x^{\alpha}\, {}_aI_x^{\beta} \phi &= \tfrac{1}{\Gamma(\alpha)} \int_a^x \frac{dt}{(x-t)^{1-\alpha}} \frac{1}{\Gamma(\beta)} \int_a^t \frac{\phi(u)du}{(t-u)^{1-\beta}} \\ &= \tfrac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x du \phi(u) \int_u^x \frac{dt}{(x-t)^{1-\alpha}(t-u)^{\beta}}. \end{aligned} \tag{2.20} $$

If we use the substitution $y = \frac{t-u}{x-u}$*, the value of the second integral is*

$$ \frac{1}{\Gamma(\alpha)\Gamma(\beta)(x-u)^{1-\alpha-\beta}} \int_0^1 y^{\beta-1}(1-y)^{\alpha-1} dy = \frac{(x-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}. $$

which, when substituted in (2.20) yields the first part of (2.19). The second part can be similarly established. In particular,

$$ {}_aI_x^{n+\alpha} f = {}_aI_x^{n}\, {}_aI_x^{\alpha} f, n \in \mathbb{N}, \alpha > 0 \tag{2.21} $$

which shows that the n-fold differentiation

$$ \frac{d^n}{dx^n}\, {}_aI_x^{n+\alpha} f(x) = {}_a I_x^{\alpha} f, n \in \mathbb{N}, \alpha > 0 \tag{2.22} $$

for all x*. When* $\alpha = 0$*, we obtain*

$$ {}_aI_x^{0} f(x) = f(x); \quad {}_aI_x^{-n} f(x) = \frac{d^n}{dx^n} f(x) = f^{(n)}(x) \tag{2.23} $$

The property given in (2.19) is called semigroup property of fractional integration.

2.1.3.2 Riemann-Liouville fractional derivative

Having established these fundamental properties of Riemann-Liouville integral operators, we now come to the corresponding differential operators.

Definition 2.2.2. *Let $(n-1) \leq \alpha < n$. The operator ${}_aD_x^\alpha$, defined by*

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, $$

and

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}} dt. $$

is called the Riemann-Liouville left-sided and right-sided fractional differential operator of order α, respectively.
For $\alpha = 0$, we set $D^0 := I$, the identity operator.

2.2.3 Caputo fractional derivative

The Caputo fractional derivative is considered to be an alternative definition for Riemann-Liouville definition, it is introduced by the Italian Mathematician Caputo in 1967.

Definition 2.2.3. *Let $\alpha > 0$, the Caputo left-sided and right-sided fractional differential operator of order α is given by:*

$$ {}_a^CD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, $$

and

$$ {}_x^CD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt. $$

2.2.4 Main properties of fractional operator

Lemma 2.2.1. *(Representation [14])*

- *The Riemann Liouville fractional derivative is equivalent to the composition of the same operator (($n-\alpha$)-fold integration and $n-th$ ordre differentiation) but in reverse ordre i.e*

$$ {}_aD_x^\alpha f(x) = D^n I_a^{n-\alpha} f(x) $$

- *Let $n-1<\alpha<n$, $n\in\mathbb{N}$, $\alpha\in\mathbb{R}$ and $f(x)$ be such that ${}^{C}D_{a}^{\alpha}f(x)$ exists. Then,*

$$ {}_{a}^{C}D_{x}^{\alpha}f(x)=I_{a}^{n-\alpha}D^{n}f(x). $$

Proposition 2.2.1. *In general the two operators, Riemann-Liouville and Caputo, do not coincide, i.e,*

$$ {}_{a}D_{x}^{\alpha}f(x)\neq {}_{a}^{C}D_{x}^{\alpha}f(x) $$

Proof: The well-known Taylor series expansion about the point 0 is

$$ \begin{aligned} f(x) &= f(0)+xf^{(1)}(0)+\tfrac{x^2}{2!}f^{(2)}(0)+\tfrac{x^3}{3!}f^{(3)}(0)+...+\tfrac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_{n-1} \\ &= \sum_{k=0}^{n-1}\frac{x^k}{\Gamma(k+1)}f^{(k)}(0)+R_{n-1} \\ R_{n-1} &= \int_0^x \frac{f^{(n)}(s)(x-s)^{n-1}}{(n-1)!}ds=\frac{1}{\Gamma(n)}\int_0^x f^{(n)}(s)(x-s)^{n-1}ds \\ &= I^n f^{(n)}(x). \end{aligned} $$

Using the linearity property of R-L and representation property of Caputo

$$ {}_{a}^{C}D_{x}^{\alpha}f(x)=I^{n-\alpha}D^{n}f(x). $$

and

$$ \begin{aligned} {}_{a}D_{x}^{\alpha}f(x) &= {}_{a}D_{x}^{\alpha}\left(\sum_{k=0}^{n-1}\frac{x^k}{\Gamma(k+1)}f^{(k)}(0)+R_{n-1}\right) \\ &= \sum_{k=0}^{n-1}\frac{{}_{a}D_{x}^{\alpha}x^k}{\Gamma(k+1)}f^{(k)}(0)+{}_{a}D_{x}^{\alpha}R_{n-1} \\ &= \sum_{k=0}^{n-1}\frac{x^{k-\alpha}}{\Gamma(k+1)}f^{(k)}(0)+{}_{a}D_{x}^{\alpha}I^n f^{(n)}(x) \\ &= \sum_{k=0}^{n-1}\frac{x^{k-\alpha}}{\Gamma(k+1)}f^{(k)}(0)+I^{n-\alpha}f^{(n)}(x) \\ &= \sum_{k=0}^{n-1}\frac{x^{k-\alpha}}{\Gamma(k+1)}f^{(k)}(0)+{}_{a}^{C}D_{x}^{\alpha}f(x). \end{aligned} $$

This means that

$$ {}_{a}D_{x}^{\alpha}f(x)\neq {}_{a}^{C}D_{x}^{\alpha}f(x) $$

Proposition 2.2.2. *The relation between the Riemann-liouville and Caputo fractional derivatives is given by:*

$$ {}_{a}^{C}D_{x}^{\alpha}f(x)={}_{a}D_{x}^{\alpha}\left(f(x)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right). $$

Proof: The proof result of Proposition 2.2.1 is

$$ {}_aD_x^\alpha f(x) = \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} f^{(k)}(0) +_a^C D_x^\alpha f(x) $$

This means that

$$ {}_a^C D_x^\alpha f(x) =_a D^\alpha \left(f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f^{(k)}(0) \right). $$

Lemma 2.2.2. *(Interpolation)*

- *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(t)$ be such that $D^\alpha f(t)$ exists. Then the following properties for the R-L operator hold*

$$ \begin{aligned} \lim_{\alpha \longrightarrow n} D^\alpha f(t) &= f^{(n)}(t), \\ \lim_{\alpha \longrightarrow n-1} D^\alpha f(t) &= f^{(n-1)}(t). \end{aligned} \tag{2.24} $$

- *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(t)$ be such that ${}^C D^\alpha f(t)$ exists. Then the following properties for the Caputo operator hold*

$$ \begin{aligned} \lim_{\alpha \longrightarrow n} {}^C D^\alpha f(t) &= f^{(n)}(t), \\ \lim_{\alpha \longrightarrow n-1} {}^C D^\alpha f(t) &= f^{(n-1)}(t) - f^{(n-1)}(0). \end{aligned} \tag{2.25} $$

Proof: *The proof uses integration by parts.*

$$ \begin{aligned} {}^c D^\alpha f(t) &= \tfrac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds \\ &= \tfrac{1}{\Gamma(n-\alpha)} \left(-f^{(n)}(s) \tfrac{(t-s)^{n-\alpha}}{n-\alpha} |_{s=0}^t - \int_0^t -f^{(n-1)}(s) \frac{(t-s)^{n-\alpha}}{n-\alpha} ds \right) \\ &= \tfrac{1}{\Gamma(n-\alpha+1)} \left(f^{(n)}(0) t^{n-\alpha} + \int_0^t f^{(n+1)}(s)(t-s)^{n-\alpha} ds \right). \end{aligned} $$

Now, by taking the limit for $\alpha \longrightarrow n$ and $\alpha \longrightarrow n-1$, respectively, it follows

$$ \lim_{\alpha \longrightarrow n} {}^C D^\alpha f(t) = \left(f^{(n)}(0) + f^{(n)}(s) \right) |_{s=0}^t = f^{(n)}(t) $$

and

$$ \begin{aligned} \lim_{\alpha \longrightarrow n-1} {}^C D^\alpha f(t) &= \left(f^{(n)}(0) + f^{(n)}(s)(t-s) \right) |_{s=0}^t - \int_0^t -f^{(n)}(s) ds \\ &= f^{(n-1)}(s)|_{s=0}^t \\ &= f^{(n-1)}(t) - f^{(n-1)}(0). \end{aligned} $$

For the Riemann-Liouville fractional differential operator the corresponding interpolation property reads

$$ \lim_{\alpha \longrightarrow n} D^\alpha f(t) = f^{(n)}(t), $$

$$ \lim_{\alpha \longrightarrow n-1} D^\alpha f(t) = f^{(n-1)}(t). $$

Lemma 2.2.3. *(Linearity)*

- *Let $n-1<\alpha<n$, $n\in\mathbb{N}$, $\alpha,\lambda\in\mathbb{R}$ and the function $f(x)$ and $g(x)$ be such that both ${}_aD_x^\alpha f(x)$ and ${}_aD_x^\alpha g(x)$ exist. The Riemann-Liouville fractional derivative is a linear operator i.e,*

$$D^\alpha(\lambda f(x)+g(x))=\lambda D^\alpha f(x)+D^\alpha g(x)$$

- *Let $n-1<\alpha<n$, $n\in\mathbb{N}$, α, $\lambda\in\mathbb{R}$ and the function $f(x)$ and $g(x)$ be such that both ${}_a^CD_x^\alpha f(x)$ and ${}_a^CD_x^\alpha g(x)$ exist. The Caputo fractional derivative is a linear operator i.e,*

$${}_a^CD_x^\alpha(\lambda f(x)+g(x))=\lambda{}_a^CD_x^\alpha f(x)+{}_a^CD_x^\alpha g(x) \tag{2.26}$$

Proof: *The proof follows straight forwardly from the definition of fractional integration and the fact that the integral and the classical integer-ordre derivative are linear operator.*

Lemma 2.2.4. *(Non-commutation)*

- *Let $n-1<\alpha<n$, $m,n\in\mathbb{N}$, $\alpha\in\mathbb{R}$ and the function $f(x)$ is such that ${}_aD_x^\alpha f(x)$ exists. Then in general Riemann Liouville operator is also non-commutative and satisfies*

$$D^m\left({}_aD_x^\alpha f(x)\right)={}_aD_x^{\alpha+m}f(x)\neq{}_aD_x^\alpha\left(D^m f(x)\right) \tag{2.27}$$

- *Let $n-1<\alpha<n$, $m,n\in\mathbb{N}$, $\alpha\in\mathbb{R}$ and the function $f(x)$ is such that ${}_a^CD_x^\alpha f(x)$ exists. Then in general*

$${}_a^CD_x^\alpha\left(D^m f(x)\right)={}_a^CD_x^{\alpha+m}f(x)\neq D^m\left({}_a^CD_x^\alpha f(x)\right) \tag{2.28}$$

Proof: Let $\alpha=\frac{1}{2}$, $f(x)=1$, $m=1$ using the definition of D_x^α,

$$D_x^{\frac{1}{2}}D^1(1)=D^{\frac{1}{2}}(0)=0,$$

$$D_x^{\frac{3}{2}}(1)=-\frac{1}{2\sqrt{(\pi)}}x^{-\frac{3}{2}},$$

$$D_x^{\frac{1}{2}}D^1(1)=0\neq D_x^{-\frac{3}{2}}.$$

That means

$$D^{\frac{1}{2}}D^1(1)\neq D^1D^{\frac{1}{2}}(1)$$

The same proof of Caputo.

Corollary 2.2.1. *(Leibniz Rule [5])*

- *Let* $t > 0$, $\alpha \in \mathbb{R}$, $n-1 < \alpha < n$, $n \in \mathbb{N}$. *If* $f(\tau)$ *and* $g(\tau)$ *are* $\mathcal{C}^{\infty}([0,x])$. *The Riemann-Liouville fractional derivative of Leibniz rule is given by*

$$ {}_aD_x^{\alpha}(f(x)g(x)) = \sum_{k=0}^{\infty} \binom{k}{\alpha} \left({}_aD_x^{\alpha-k} f(x)\right) g^{(k)}(x) \tag{2.29}$$

- *Let* $t > 0$, $\alpha \in \mathbb{R}$, $n-1 < \alpha < n$, $n \in \mathbb{N}$. *If* $f(\tau)$ *and* $g(\tau)$ *are* $\mathcal{C}^{\infty}([0,x])$. *The Caputo fractional derivative of Leibniz rule is given by*

$$ {}_a^CD_x^{\alpha}(f(x)g(x)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left({}_aD_x^{\alpha-k} f(x)\right) g^{(k)}(x) - \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)} \left((f(x)g(x))^{(k)}(0) \right). \tag{2.30}$$

Lemma 2.2.5. *(Laplace transforms for the basic fractional operators) Suppose that* $p > 0$ *and* $F(s)$ *is the Laplace transform of* $f(t)$. *Then the following statements hold(see Podlubny [27]):*

(*a*) *The Laplace transform of the fractional integral of order* α *is given by:*

$$ \mathcal{L}\{I^{\alpha} f(t); s\} = s^{-\alpha} F(s). \tag{2.31}$$

(*b*) *The Laplace transform of R-L of order* α *is given by*

$$\begin{aligned} \mathcal{L}\{D^{\alpha} f(t); s\} &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{k} [D^{\alpha-k-1} f(t)]_{t=0} \\ &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} [D^{k} I^{n-\alpha} f(t)]_{t=0}, \quad n-1 < \alpha < n \end{aligned}$$

(*c*) *The Laplace transform of the Caputo fractional derivative of order* α *is given by*

$$ \mathcal{L}\{{}^CD_x^{\alpha} f(t); s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha < n \tag{2.32}$$

Proof: To show the validity of (2.32). Using representation formula of Caputo,

$$ {}^CD_x^{\alpha} f(x) = I_a^{n-\alpha} D^n f(x). $$

Let $g(x) = D^n f(x)$. Then (2.32) becomes

$$ {}^CD_x^{\alpha} f(x) = I_a^{n-\alpha} g(x). $$

By the Laplace transform of the fractional integral and the representation formula of Caputo

$$\mathcal{L}\{{}^{C}D_x^{\alpha} f(x); s\} = \mathcal{L}\{I_a^{n-\alpha} g(x); s\} = s^{-(n-\alpha)} G(s). \tag{2.33}$$

where $G(s) = \mathcal{L}\{g(x); s\}$ and

$$G(s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0). \tag{2.34}$$

Finally, substituting (2.34) in (2.33), we have

$$\mathcal{L}\{{}^{C}D_x^{\alpha} f(x); s\} = s^{-(n-\alpha)} \left(s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \right) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0).$$

is proved.

To summarize all this results. A comparison between the Caputo and Riemann-Liouville fractional derivatives is given in the following table (see [12]).

Property	Riemann-Liouville	Caputo
Representation	$D^{\alpha} f(t) = D^n I^{n-\alpha} f(t)$	${}^{C}D^{\alpha} f(t) = I^{n-\alpha} D^n f(t)$
Interpolation	$\lim_{\alpha\to n} D^{\alpha} f(t) = f^{(n)}(t)$ $\lim_{\alpha\to n-1} D^{\alpha} f(t) = f^{(n-1)}(t)$	$\lim_{\alpha\to n} {}^{C}D^{\alpha} f(t) = f^{(n)}(t)$ $\lim_{\alpha\to n-1} {}^{C}D^{\alpha} f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$
Linearity	$D^{\alpha}(\lambda f(t) + g(t)) = \lambda D^{\alpha} f(t) + D^{\alpha} g(t)$	${}^{C}D^{\alpha}(\lambda f(t) + g(t))$ $= \lambda {}^{C}D^{\alpha} f(t) + {}^{C}D^{\alpha} g(t)$
Non-commutation	$D^m D^{\alpha} f(t) = D^{\alpha+m} f(t) \neq D^{\alpha} D^m f(t)$	${}^{C}D^{\alpha}\left(D^m f(t)\right) = {}^{C}D^{\alpha+m} f(t)$ $\neq D^m \left({}^{C}D^{\alpha} f(t)\right)$
Laplace transform	$s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(t)]_{t=0}$	$s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$
Leibniz rule	$D^{\alpha}(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{k}{\alpha} (D^{\alpha-k} f(t)) g^{(k)}(t)$	${}^{C}D^{\alpha}(f(t)g(t))$ $= \sum_{k=0}^{\infty} \binom{k}{\alpha} (D^{\alpha-k} f(t)) g^{(k)}(t)$ $- \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left((f(t)g(t))^{(k)}(0)\right)$
$f(t) = r = const$	$D^{\alpha} r = \frac{r}{\Gamma(1-\alpha)} t^{\alpha} \neq 0, \quad r = const$	${}^{C}D^{\alpha} r = 0, \quad r = const$

Table1: The comparison between the Caputo and Riemann-Liouville derivative fractional

2.3 Ordinary fractional differential equation

Fractional order differential equations have become an important tool in mathematical modeling. Although there are many possible generalizations of $\frac{d^n}{dt^n}f(t)$, the most commonly used definitions are Riemann−Liouville and Caputo fractional derivatives.
We use a transformation in the equivalent fractional Volterra integral equation of given fractional differential equation (FDE) and obtain its exact solution in terms of the solution of an integer order differential equation.

2.3.1 The main results

Consider the condition initial with Caputo type FDE given by

$$\begin{cases} {}^{C}D^{\alpha}X(t) &= f(t, X(t)) \\ X(0) &= X_0, \end{cases} \tag{2.35}$$

where $f \in \mathcal{C}([0,T] \times \mathbb{R}, \mathbb{R})$, $0 < \alpha < 1$.
Since f is assumed to be a continuous function, every solution of (2.35) is also a solution of the following Volterra fractional integral equation.

$$X(t) = X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, X(\tau))d\tau, \quad t \in [0,T]. \tag{2.36}$$

Furthermore, every solution of(2.36) is a solution of (2.35).
We note that (2.35) is equivalent to the following system

$$\begin{cases} D^{\alpha}(X(t) - X_0) &= f(t, X(t)) \\ X(0) = X_0 & \end{cases}$$

The following existence theorem is given for (2.35).

Theorem 2.3.1. *[4]Assume that $f \in \mathcal{C}([\mathbb{R}_0, \mathbb{R}])$.*
Where $\mathbb{R}_0 = [(t, X) : 0 \leq t \leq a$ and $|X - X_0| \leq b]$ and let $|f(t, X)| \leq M$ on $\mathbb{R}_0$. Then there exists at least one solution for FDE (2.35) on $0 \leq t \leq \gamma$ where $\gamma = \min\left(a, [\frac{b}{M}\Gamma(\alpha+1)]^{\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$.

Theorem 2.3.2. *Consider the FDE given by (2.35). Let*

$$g(v, X_*(v)) = f\left(t - (t^{\alpha} - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}, X(t - (t^{\alpha} - v\Gamma(\alpha+1))^{\frac{1}{\alpha}})\right)$$

and assume that the conditions of Theorem 2.3.1 hold. Then, a solution of (2.35), $X(t)$, is given by

$$X(t) = X_*(t^\alpha/\Gamma(\alpha+1))$$

where $X_*(v)$ is a solution of the integer order differential equation

$$\begin{cases} \frac{d(X_*(v))}{dv} &= g(v, X_*(v)) \\ X_*(0) &= X_0 \end{cases} \tag{2.37}$$

Proof: *The existence of the solution of (2.35) follows from Theorem 2.3.1. If $X(t)$ is a solution of (2.35) then, it is also a solution of (2.36).*
Let $\tau = t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}$. So, Volterra fractional integral equation (2.36) can be written as

$$X(t) = X_0 + \int_0^{t^\alpha/\Gamma(\alpha+1)} f(t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}, X(t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}))dv$$

$$X(t) = X_0 + \int_0^{t^\alpha/\Gamma(\alpha+1)} g(v, X_*(v))dv. \tag{2.38}$$

On the other hand, consider the system of FDE given by (2.37). Every solution of (2.37) is also a solution of the Volterra integral equation given below and vice versa.

$$X_*(v) = X_0 + \int_0^v g(s, X_*(s))ds, \quad 0 \le v \le a^\alpha/\Gamma(\alpha+1). \tag{2.39}$$

Since $0 \le t^\alpha/\Gamma(\alpha+1)$, the right-hand side of equation (2.38) is equal to $X_*(t^\alpha/\Gamma(\alpha+1))$. The theorems given below are simple generalizations of Theorems 2.3.1 and 2.3.2, respectively.

Theorem 2.3.3. *[4] Let $\|\cdot\|$denote any convenient norm on $\mathbb{R}^n$. Assume that $f \in \mathcal{C}([\mathbb{R}_1, \mathbb{R}^n])$, where $\mathbb{R}_1 = [(t, X) : 0 \le t \le a$ and $\| X - X_0 \| \le b]$, $f = (f_1, f_2, ..., f_n)^T$, $X = (x_1, x_2, ..., x_n)^T$ and let $\| f(t, X) \| \le M$ on R_1. Then, there exists at least one solution for system of FDE's given by*

$$\begin{cases} {}^C D^\alpha X(t) &= f(t, X(t)) \\ X(0) &= X_0 \end{cases} \tag{2.40}$$

on $0 \le t \le \beta$ where $\beta = \min\left(a, \left[\frac{b}{M}\Gamma(\alpha+1)\right]^{\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$.

Theorem 2.3.4. *[4] Consider the system of FDE given by (2.40) of order α, $0 < \alpha < 1$. Let*

$$g(v, X_*(v)) = f\left(t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}, X(t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}})\right)$$

and assume that the conditions of Theorem 2.3.3 hold. Then, a solution of (2.35), $X(t)$, can be given by

$$X(t) = X_*(t^\alpha/\Gamma(\alpha+1))$$

where $X_*(v)$ is a solution of the system of integer order differential equations

$$\begin{cases} \frac{d(X_*(v))}{dv} &= g(v, X_*(v)) \\ X_*(0) &= X_0 \end{cases}$$

Remark 2.3.1. *Although the Caputo derivative is more commonly used in applied problems, also exist the models with Riemann Liouville type derivative. Theorem 2.3.2 also holds if*

$$\begin{cases} D^\alpha(x(t) - x_0) &= f(t, x(t)) \\ x(0) &= x_0 \end{cases}$$

Riemann Liouville for system of FDE's is considered. But, generally the system of FDE are given in the form

$$\begin{cases} D^\alpha x(t) &= f(t, x(t)) \\ x(0) &= x_0 \end{cases}$$

To apply the given solution technique to these kind of problems,one should set

$$h(t, x(t)) = f(t, x(t)) - \frac{x_0 t^{-\alpha}}{\Gamma(1-\alpha)}$$

and solve the problem

$$D^\alpha x(t) = h(t, x(t)).$$

Most of the fractional differential equations of order α, $0 < \alpha < 1$, are given in the following form

$$D^\alpha(x(t)) = f(t, x(t)) \tag{2.41}$$

In order to use Theorem 2.3.2 to solve (2.41) with the initial condition $x(0) = x_0$, set $h(t, x(t)) = f(t, x(t)) - \frac{x_0 t^{-\alpha}}{\Gamma(1-\alpha)}$ and solve

$$D^\alpha(x(t) - x_0) = h(t, x(t))$$

2.3.2 Examples

In this subsection two examples are chosen such that the exact solutions can be evaluated analytically to show that the technique given a works properly.

Example 2.3.1. *Consider the system of fractional order given by*

$$\begin{cases} {}^{C}D^{\frac{1}{2}}x(t) = t \\ x(0) = x_0 \end{cases} \tag{2.42}$$

For this example,

$$g(v) = 2\sqrt{t}\Gamma\left(\frac{3}{2}\right)v - v^2\Gamma^2\left(\frac{3}{2}\right).$$

The solution of the corresponding integer order of FDE given in Theorem 2.3.2 is

$$x_1(v) = \sqrt{t}\Gamma\left(\frac{3}{2}\right)v^2 - \frac{\Gamma^2(\frac{3}{2}v^3)}{3} + x_0.$$

So, the solution of the given fractional order of FDE is

$$x(t) = x_1\left(\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\right) = \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}} + x_0. \tag{2.43}$$

Indeed, it can be shown that (2.43) is a solution of (2.42), by using the fractional derivative.

Example 2.3.2. *Consider the system of linear fractional differential equation given by*

$$\begin{cases} {}^{C}D^{\frac{1}{2}}x(t) &= t + x(t) \\ x(0) &= x_0 \end{cases} \tag{2.44}$$

The corresponding differential equation of this FDE is

$$\begin{cases} \frac{dx_1(v)}{dv} &= f_1(v) = x_1(v) + 2\sqrt{t}\Gamma\left(\frac{3}{2}\right)v - v^2\Gamma^2\left(\frac{3}{2}\right) \\ x(0) &= x_0 \end{cases}$$

The solution of this system of (2.44) is

$$x_1(v) = -2\sqrt{t}\Gamma\left(\frac{3}{2}\right)(v+1) + \Gamma^2\left(\frac{3}{2}\right)(v^2+2v+2) + e^v\left(x_0 + 2\sqrt{t}\Gamma\left(\frac{3}{2}\right) - 2\Gamma^2\left(\frac{3}{2}\right)\right).$$

Consequently, the solution of the system of (2.44) is

$$x(t) = x_1\left(\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\right) = -t + \frac{\pi}{2} + e^{2\sqrt{t}/\sqrt{\pi}}\left(x_0 + \sqrt{t\pi} - \frac{\pi}{2}\right). \tag{2.45}$$

2.4 Stochastic fractional differential equation

Several forms of fractional stochastic differential equations have been proposed in standard models and there has been significant interest in studying their solution.
In this section we shall discuss the the global existence and uniqueness of solution of a class of a Caputo fractional stochastic differential equations. Using a temporally weighted norm and whose coefficients satisfy a standard Lipschitz condition.

2.4.1 Preliminary

Consider a Caputo fractional stochastic differential equation (for short Caputo FSDE) of order $\alpha \in (\frac{1}{2}, 1)$ of the following form

$$^{C}D^{\alpha}_{0+}X(t) = b(t, X(t)) + \sigma(t, X(t))\frac{dW_t}{dt}, \tag{2.46}$$

where b,$\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$, are measurable and $(W_t)_{t\in[0,\infty)}$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$. For each $t \in [0, \infty)$, let $\mathfrak{X}_t := \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ denote the space of all $\mathcal{F}_t-$ measurable, mean square integrable functions $f = (f_1, , ..., f_d)^T : \Omega \to \mathbb{R}^d$ with

$$\|f\|_{ms} := \sqrt{\sum_{i=1}^{d} \mathbb{E}(|f_i|^2)} = \sqrt{\mathbb{E}\|f\|^2},$$

where $\mathbb{R}^d$ is endowed with the standard Euclidean norm.
A process X : $[0, \infty) \to \mathbb{L}(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathbb{F}-$ adapted if $X(t) \in \mathfrak{X}_t$ for all $t \geq 0$. For each $\eta \in \mathfrak{X}_0$ a $\mathbb{F}-$adapted process X is called a solution of (2.46) with the initial condition $X(0) = \eta$ if the following equality holds for $t \in [0, \infty)$

$$X(t) = \eta + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-\tau)^{\alpha-1} b(\tau - X(\tau))d\tau + \int_0^t (t-\tau)^{\alpha-1}\sigma(\tau - X(\tau))dW_\tau\right), \tag{2.47}$$

where $\Gamma(\alpha)$ is the Gamma function.

2.4.2 The main results

In the remaining of this section, we assume that the coefficients b and σ satisfy the following standard conditions:

- (H1) There exists L > 0 such that for all $x, y \in \mathbb{R}^d$, $t \in [0, \infty)$

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq L\|x - y\|.$$

- (H2) $\sigma(., 0)$ is essentially bounded, i.e.

$$\|\sigma(., 0)\|_\infty := esssup_{\tau\in[0,\infty)}\|\sigma(\tau, 0)\| < \infty$$

and $b(.,0)$ is $\mathbb{L}^2$ integrable, i.e.

$$\int_0^\infty \|b(\tau,0)\|^2 d\tau < \infty$$

Theorem 2.4.1. *Suppose that (H1) and (H2) hold. Then*

- *(i) for any $\eta \in \mathfrak{X}_0$, the initial value problem (2.46) with the initial condition $X(0) = \eta$ has a unique global solution on the whole interval $[0,\infty)$ denoted by $\varphi(.,\eta)$;*
- *(ii) on any bounded time interval $[0,T]$, where $T > 0$, the solution $\varphi(.,\eta)$ depends continuously on η, i.e.*

$$\lim_{\zeta\to\eta} \sup_{t\in[0,T]} \| \varphi(t,\zeta) - \varphi(t,\eta) \|_{ms} = 0$$

2.4.3 Proof of the main result

In order to prove the theorem 2.4.1 it is equivalent to show the existence and uniqueness solutions on an arbitrary interval $[0,T]$, where $T > 0$ is arbitrary. In what follows we choose and fix a $T > 0$ arbitrarily.

Let $\mathbb{H}^2([0,T])$ be the space of all the processes X which are measurable, $\mathbb{F}_T$−adapted, where $\mathbb{F}_T := \{\mathcal{F}_t\}_{t\in[0,T]}$, and satisfies that

$$\| X \|_{\mathbb{H}^2} =: \sup_{0\le t\le T} \| X(t) \|_{ms} < \infty$$

Obviously, $\mathbb{H}^2([0,T], \| \,.\, \|_{\mathbb{H}^2})$, is a Banach space. For any $\eta \in \mathfrak{X}_0$, we define an operator $\tau_\eta : \mathbb{H}^2([0,T]) \to \mathbb{H}^2([0,T])$ by

$$\tau_\eta\xi(t) = \eta + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-\tau)^{\alpha-1} b(\tau-\xi(\tau))d\tau + \int_0^t (t-\tau)^{\alpha-1}\sigma(\tau-\xi(\tau))dW_\tau\right) \tag{2.48}$$

The following lemma is devoted to showing that this operator is well-defined.

Lemma 2.4.1. *For any $\eta \in \mathfrak{X}_0$, the operator τ_η is well-defined.*

Proof: Let $\xi \in \mathbb{H}^2([0,T])$ be arbitrary. From the definition of $\tau_\eta\xi$ as in (2.48) and the inequality $\|x+y+z\|^2 \le 3(\|x\|^2 + \|y\|^2 + \|z\|^2)$ for all $x,y,z \in \mathbb{R}^d$, we have for all $t \in [0,T]$

$$\begin{aligned} \|\tau_\eta\xi(t)\|^2_{ms} \ \le\ & \|3\eta\|^2_{ms} + \frac{3}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau\right\|^2\right) \\ & + \ \frac{3}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\xi(\tau))dW_\tau\right\|^2\right) \end{aligned} \tag{2.49}$$

By the Hölder inequality, we obtain

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau\right\|^2\right) &\leq \int_0^t (t-\tau)^{2\alpha-2}d\tau\mathbb{E}\left(\int_0^t \|b(\tau,\xi(\tau))\|^2 d\tau\right)\\ &= \frac{t^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^t \|b(\tau,\xi(\tau))\|^2 d\tau\right) \qquad (2.50)\end{aligned}$$

From (H1), we derive

$$\begin{aligned}\|b(\tau,\xi(\tau))\|^2 &\leq 2(\|b(\tau,\xi(\tau))-b(\tau,0)\|^2+\|b(\tau,0)\|^2)\\ &\leq 2L^2\|\xi(\tau)\|^2+2\|b(\tau,0)\|^2\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left(\int_0^t \|b(\tau,\xi(\tau))\|^2 d\tau\right) &\leq 2L^2\mathbb{E}\left(\int_0^t \|\xi(\tau)\|^2 d\tau\right)+2\int_0^t \|b(\tau,0)\|^2 d\tau\\ &\leq 2L^2 T\sup_{t\in[0,T]}\mathbb{E}(\|\xi(t)\|^2)+2\int_0^T \|b(\tau,0)\|^2 d\tau\end{aligned}$$

which together with (2.50) implies that

$$\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau\right\|^2\right) \leq \frac{2L^2T^{2\alpha}}{2\alpha-1}\|\xi\|^2_{\mathbb{H}^2}+\frac{2T^{2\alpha-1}}{2\alpha-1}\int_0^T \|b(\tau,0)\|^2 d\tau \qquad (2.51)$$

Now, using Itô's isometry, we obtain

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\xi(\tau))dW_\tau\right\|^2\right) &= \sum_{1\leq i\leq d}\mathbb{E}\left(\int_0^t (t-\tau)^{\alpha-1}\sigma_i(\tau,\xi(\tau))dW_\tau\right)^2\\ &= \mathbb{E}\left(\int_0^t (t-\tau)^{2\alpha-2}\left|\sum_i \sigma_i(\tau,\xi(\tau))\right|^2 d\tau\right)\\ &= \mathbb{E}\left(\int_0^t (t-\tau)^{2\alpha-2}\|\sigma(\tau,\xi(\tau))\|^2 d\tau\right)\end{aligned}$$

From (H1), we also have

$$\|\sigma(\tau,\xi(\tau))\|^2 \leq 2L^2\|\xi(\tau)\|^2+2\|\sigma(\tau,0)\|^2 \leq 2L^2\|\xi(\tau)\|^2+2\|\sigma(.,0)\|^2_\infty.$$

Therefore, for all $t\in[0,T]$ we have

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\xi(\tau))dW_\tau\right\|^2\right) &\leq 2L^2\mathbb{E}\left(\int_0^t (t-\tau)^{2\alpha-2}\|\xi(\tau)\|^2 d\tau\right)\\ &+ 2\|\sigma(.,0)\|^2_\infty\mathbb{E}\left(\int_0^t (t-\tau)^{2\alpha-2}d\tau\right)\\ &\leq 2L^2\frac{T^{2\alpha-1}}{2\alpha-1}\|\xi\|^2_{\mathbb{H}^2}+\frac{2T^{2\alpha-1}}{2\alpha-1}\|\sigma(.,0)\|^2_\infty\end{aligned}$$

This together with (2.49) and (2.51) implies that $\|\tau_\eta \xi\|_{\mathbb{H}^2} < \infty$. Hence, the map τ_η is well-defined.

To prove existence and uniqueness of solutions,we will show that the operator τ_η defined as above is contractive under a suitable temporally weighted norm (for the same method to prove the existence and uniqueness of solutions of stochastic differential equations). Here, the weight function is the Mittag-Leffler function $E_{2\alpha-1}(.)$ defined as:

$$E_{2\alpha-1}(.) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma((2\alpha-1)k+1)} \quad for\ all \quad t \in \mathbb{R}$$

Lemma 2.4.2. *For any $\alpha > \frac{1}{2}$ and $\gamma > 0$, the following inequality holds:*

$$\frac{\gamma}{\Gamma(2\alpha-1)} \int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}(\gamma \tau^{2\alpha-1}) d\tau \leq E_{2\alpha-1}(\gamma t^{2\alpha-1}).$$

Proof: Let $\gamma > 0$ be arbitrary. Consider the corresponding linear Caputo fractional differential equation of the following form

$$^{C}D_{0+}^{2\alpha-1} x(t) = \gamma x(t). \tag{2.52}$$

The Mittag-Leffler function $E_{2\alpha-1}(\gamma t^{2\alpha-1})$ is a solution of (2.52). Hence, the following equality holds:

$$E_{2\alpha-1}(\gamma t^{2\alpha-1}) = 1 + \frac{\gamma}{\Gamma(2\alpha-1)} \int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}(\gamma \tau^{2\alpha-1}) d\tau,$$

which completes the proof.

Proof of The theorem: Let $T > 0$ be arbitrary. Choose and fix a positive constant such that

$$\gamma > \frac{3L^2(T+1)\Gamma(2\alpha-1)}{\Gamma(\alpha)^2} \tag{2.53}$$

On the space $\mathbb{H}^2([0,T])$, we define a weighted norm $\|.\|_\gamma$ as below

$$\|X\|_\gamma := \sup_{t\in[0,T]} \sqrt{\frac{\mathbb{E}(\|X(t)\|^2)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}} \quad for\ all \quad X \in \mathbb{H}^2([0,T]). \tag{2.54}$$

Obviously, two norms $\|.\|_{\mathbb{H}^2}$ and $\|.\|_\gamma$ are equivalent. Thus, $(\mathbb{H}^2(0,T), \|.\|_\gamma)$ is also a Banach space.

- Choose and fix $\eta \in \mathfrak{X}_0$. By virtue of Lemma 2.4.1, the operator τ_η is well defined. We will prove that the map τ_η is contractive with respect to the norm $\|.\|_\gamma$.

For this purpose, let $\xi, \hat{\xi} \in \mathbb{H}^2([0,T])$ be arbitrary. From (2.48) and the inequality $\|x+y\|^2 \le 2(\| x \|^2 + \| y \|^2)$ for all $x, y \in \mathbb{R}^d$, we derive the following inequalities for all $t \in [0,T]$:

$$
\begin{aligned}
\mathbb{E}\left(\left\|\tau_\eta\xi(t) - \tau_\eta\hat{\xi}(t)\right\|^2\right) &\le \frac{2}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\xi(t)) - b(\tau,\hat{\xi}(t)))d\tau\right\|^2\right)\\
&+ \frac{2}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\xi(t)) - \sigma(\tau,\hat{\xi}(t)))dW_\tau\right\|^2\right)
\end{aligned}
$$

Using the Hölder inequality and (H1), we obtain

$$
\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\xi(\tau)) - b(\tau,\hat{\xi}(\tau)))d\tau\right\|^2\right) \le L^2 t\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\xi(\tau) - \hat{\xi}(\tau)\|^2)d\tau
$$

On the other hand, by Itô's isometry and (H1), we have

$$
\begin{aligned}
\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\xi(\tau)) - \sigma(\tau,\hat{\xi}(\tau)))dW_\tau\right\|^2\right) &= \mathbb{E}\int_0^t (t-\tau)^{2\alpha-2}\|\sigma(\tau,\xi(\tau))\\
&- \sigma(\tau,\hat{\xi}(\tau))\|^2 d\tau\\
&\le L^2\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\xi(\tau) - \hat{\xi}(\tau))\|^2)d\tau
\end{aligned}
$$

Thus, for all $t \in [0,T]$ we have

$$
\mathbb{E}\left(\left\|T_\eta\xi(t) - T_\eta\hat{\xi}(t)\right\|^2\right) \le \frac{2L^2(t+1)}{\Gamma(\alpha)^2}\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\xi(\tau) - \hat{\xi}(\tau))\|^2)d\tau,
$$

which together with the definition of $\|.\|_\gamma$ as in (2.54) implies that

$$
\frac{\mathbb{E}\left(\left\|T_\eta\xi(t) - T_\eta\hat{\xi}(t)\right\|^2\right)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \le \frac{2L^2(t+1)}{\Gamma(\alpha)^2}\frac{\int_0^t (t-\tau)^{2\alpha-2}E_{2\alpha-1}(\gamma t^{2\alpha-1})d\tau}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\|\xi - \hat{\xi}\|^2{}_\gamma.
$$

In light of Lemma 2.4.2, we have for all $t \in [0,T]$

$$
\frac{\mathbb{E}\left(\left\|T_\eta\xi(t) - T_\eta\hat{\xi}(t)\right\|^2\right)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \le \frac{2\Gamma(2\alpha-1)L^2(T+1)}{\Gamma(\alpha)^2\gamma}\|\xi - \hat{\xi}\|^2{}_\gamma.
$$

Consequently,

$$
\|T_\eta\xi - T_\eta\hat{\xi}\|_\gamma \le \kappa\|\xi - \hat{\xi}\|_\gamma, \qquad \textit{where}\, \kappa := \sqrt{\frac{2\Gamma(2\alpha-1)L^2(T+1)}{\Gamma(\alpha)^2\gamma}}
$$

By (2.53), we have $\kappa < 1$ and therefore the operator τ_η is a contractive map on $(\mathbb{H}^2([0,T]), \|.\|_\gamma)$. Using the Banach fixed point theorem, there exists a unique fixed point of this map in $(\mathbb{H}^2([0,T]))$. This fixed point is also the unique solution of (2.46) with the initial condition $X(0) = \eta$. The proof of this part is complete.

- Choose and fix $T > 0$ and $\eta, \zeta \in \mathbb{X}_0$. Since $\varphi(., \eta)$ and $\varphi(., \zeta)$ are solutions of (2.46) it follows that

$$\begin{aligned}\varphi(t,\eta) - \varphi(t,\zeta) = \eta - \zeta &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}(b(\tau,\varphi(\tau,\eta)) - b(\tau,\varphi(\tau,\zeta)))d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\varphi(\tau,\eta)) - \sigma(\tau,\varphi(\tau,\zeta)))dW_\tau\end{aligned}$$

Hence, using the inequality $\|x+y+z\|^2 \leq 3(\|x\|^2 + \|y\|^2 + \|z\|^2)$ for all $x, y, z \in \mathbb{R}^d$, (H1), the Hölder inequality and Itô's isometry, we obtain

$$\begin{aligned}\mathbb{E}(\|\varphi(t,\eta) - \varphi(t,\zeta)\|^2) &\leq \frac{3L^2(t+1)}{\Gamma(\alpha)^2} \int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\varphi(t,\eta) - \varphi(t,\zeta)\|^2)d\tau \\ &+ 3\mathbb{E}(\|\eta - \zeta\|^2).\end{aligned}$$

By definition of $\|.\|_\gamma$, we have

$$\begin{aligned}\frac{\mathbb{E}(\|\varphi(t,\eta) - \varphi(t,\zeta)\|^2)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \frac{\mathbb{E}(\|\varphi(t,\eta) - \varphi(t,\zeta)\|^2)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} &\leq \frac{3L^2(t+1)}{\Gamma(\alpha)^2} \frac{\int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}(\gamma\tau^{2\alpha-1})d\tau}{E_{2\alpha-1}(\gamma\tau^{2\alpha-1})} \\ &\quad \times \|\varphi(.,\eta) - \varphi(.,\zeta)\|^2{}_\gamma + 3\mathbb{E}(\|\eta - \zeta\|^2)\end{aligned}$$

By virtue of Lemma 2.4.2, we have

$$\|\varphi(.,\eta) - \varphi(.,\zeta)\|^2{}_\gamma \leq \frac{3L^2(T+1)\Gamma(2\alpha-1)}{\gamma\Gamma(\alpha)^2} \|\varphi(.,\eta) - \varphi(.,\zeta)\|^2{}_\gamma + 3\|\eta - \zeta\|^2{}_{ms}.$$

Thus, by (2.53) we have

$$\left(1 - \frac{3L^2(T+1)\Gamma(2\alpha-1)}{\gamma\Gamma(\alpha)^2}\right) \|\varphi(.,\eta) - \varphi(.,\zeta)\|^2{}_\gamma \leq 3\|\eta - \zeta\|^2{}_{ms}.$$

Hence,

$$\lim_{\eta\to\zeta} \sup_{t\in[0,T]} \|\varphi(t,\eta) - \varphi(t,\zeta)\|_{ms} = 0.$$

The proof is complete.

Chapter 3

Conformable Fractional Calculus

This chapter is devoted to conformable fractional calculus theory. A new definition of the fractional derivative was proposed and found wide resonance in the scientific community interested in fractional calculus. it was laid out by Khalil and al(2014, [18]). and called the conformable fractional derivative. Then developed in Abdeljawad (2015, [1]), and is currently under intensive investigations. More information about that theory can be found in [18, 24, 1, 25, 15].

3.1 Conformable Fractional Calculus

In this section, we present some necessary definitions and essentials results from the conformable fractional calculus theory, see [18], [1] and their references for more details on conformable fractional derivatives.

3.1.1 Special Functions

Definition 3.1.1. *Let* $p \in (0, \infty)$, $k > 0$, $\alpha \in (0, 1]$, *and* $n \in \mathbb{N}^+$ *Pochhammer symbol* $(p)^{\alpha}_{n,k}$ *is given by*

$$(p)^{\alpha}_{n,k} = (p + \alpha - 1)(p + \alpha - 1 + k\alpha)(p + \alpha - 1 + 2k\alpha)...(p + \alpha - 1 + (n - 1)k\alpha).$$

Proposition 3.1.1. *[29] Let* $\alpha \in (0, 1]$ *and* $\Gamma^{\alpha}_{k} : (0, \infty) \longrightarrow \mathbb{R}$. *For* $0 < p < \infty$. *Conformable gamma function* Γ^{α}_{k} *is given by*

$$\Gamma^{\alpha}_{k}(p) = \int_{0}^{\infty} t^{p-1} e^{-\frac{t^{k\alpha}}{k\alpha}} d_{\alpha}t = \lim_{n\to\infty} \frac{n!\alpha^n k^n (nk\alpha)^{\frac{p+\alpha-1}{\alpha}-1}}{(p)^{\alpha}_{n,k}}.$$

Proposition 3.1.2. *[29] The $(\alpha, k)-$Gamma function $\Gamma_k^\alpha(p)$ satisfies the following identities*

1. $\Gamma_k^\alpha(p+k) = (p+\alpha-1)\Gamma_k^\alpha(p)$
2. $\Gamma_k^\alpha(p+nk\alpha) = (p)_{n,k}^\alpha \Gamma_k^\alpha(p)$
3. $\Gamma_k^\alpha(p) = (k\alpha)^{\frac{p+\alpha-1}{k\alpha}-1}\Gamma\left(\frac{p+\alpha-1}{k\alpha}\right)$
4. $\Gamma_k^\alpha(p) = (\alpha)^{\frac{p+\alpha-1}{k\alpha}-1}\Gamma_k\left(\frac{p+\alpha-1}{\alpha}\right)$
5. $\Gamma_k^\alpha(k\alpha+1-\alpha) = 1$

Definition 3.1.2. *Let $\alpha \in (0,1]$. The $(\alpha, k)-$Beta function $B_k^\alpha(p,q)$ is given by the formula*

$$B_k^\alpha(p,q) = \frac{1}{k\alpha}\int_0^1 t^{\frac{p}{k\alpha}-1}(1-t)^{\frac{q}{k\alpha}-1} d_\alpha t, \quad p,q,k>0.$$

Proposition 3.1.3. *The $(\alpha, k)-$Beta function satisfies the following identities*

1. $B_k^\alpha(p, k\alpha) = \frac{1}{p+k\alpha(\alpha-1)}$,
2. $B_k^\alpha(k\alpha(2-\alpha), q) = \frac{1}{q}$.

Proof: From the definition of the $(\alpha, k)-$Beta function $B_k^\alpha(p,q)$, we have

$$B_k^\alpha(p,k\alpha) = \frac{1}{k\alpha}\int_0^1 t^{\frac{p}{k\alpha}-1} d_\alpha t = \frac{1}{p+k\alpha(\alpha-1)}$$

and similarly,

$$B_k^\alpha(k\alpha(2-\alpha),q) = \frac{1}{k\alpha}\int_0^1 t^{1-\alpha}(1-t)^{\frac{q}{k\alpha}-1} d_\alpha t = \frac{1}{q}.$$

This completes the proof.

Remark 3.1.1. *From the Proposition 3.1.3, we have*

$$B_k^\alpha(k\alpha, k\alpha) = \frac{1}{k\alpha^2}.$$

Remark 3.1.2. *By the Proposition 3.1.3 with $\alpha = 1$, we have the following properties for $k-$Beta function*

$$B_k(p,k) = \frac{1}{p}, \quad B_k(k,p) = \frac{1}{q}.$$

Proposition 3.1.4. *[29] The following property holds for $(\alpha, k)-$Beta function $B_k^\alpha(p,q)$*

$$B_k^\alpha(p,q) = \frac{p+k\alpha(\alpha-2)}{p+q+k\alpha(\alpha-2)} B_k^\alpha(p-k\alpha, q).$$

Proposition 3.1.5. *[29] The following identity holds*

$$B_k^\alpha(p,q) = B_k(p + k\alpha(\alpha-1), q) = \frac{1}{k\alpha} B(\frac{p}{k\alpha} + \alpha - 1, \frac{q}{k\alpha})$$

where $B_k(x,y)$ is $k-$Beta function and $B(x,y)$ is classical Beta function.

Proposition 3.1.6. *[29] The following property holds for $(\alpha,k)-$Beta function in terms of $(\alpha,k)-$gamma function*

$$B_k^\alpha(p + k\alpha(1-\alpha), q) = \frac{\Gamma_k^\alpha(p)\Gamma_k^\alpha(q)}{\Gamma_k^\alpha(p+q+1-\alpha)}.$$

Remark 3.1.3. *By the Proposition 3.1.6 with $\alpha = 1$, we have the following property*

$$B_k(p,q) = \frac{\Gamma_k(p)\Gamma_k(q)}{\Gamma_k(p+q)}.$$

3.1.2 Conformable Fractional Derivative

The definition of fractional derivative don't have a standard form. But the basic definitions are Riemann-Liouville definition and Caputo definition. The fractional derivative can also be seen as an approximation of the classical derivative. This is not the case in general. This is due to the setbacks of these definitions and from them:

- When α is not a natural number. The derivative of constant is difficult (Riemann-Liouville derivative is: ${}_aD_t^\alpha \neq 0$ but ${}_a^CD_t^\alpha = 0$ for Caputo derivative).
- All precedent definitions do not satisfy the known formula of the derivative of the product of two functions:

$$D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f).$$

- The same problem of the derivative of the quotient of two functions:

$$D^\alpha\left(\frac{f}{g}\right) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}.$$

- The same problem of the chain rule:

$$D^\alpha\left(f \circ g\right)(t) = f^{(\alpha)}(g(t))g^{(\alpha)}(t).$$

- The same problem of commutation:

$$D^\alpha D^\beta f = D^{\alpha+\beta} f.$$

Therefore. (Khalil and al [18]) was presented a new definition called conformable fractional derivative with $\alpha \in (0,1)$ and it satisfies classical properties mentioned above.

Definition 3.1.3. *Let $\alpha \in (0,1)$, $t > 0$. Given a function $f : [0,\infty) \to \mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{3.1}$$

If f is $\alpha-$differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define: $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$.
We can write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. Moreover, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

Definition 3.1.4. *The conformable fractional derivative starting from a of a function $f : [a,\infty) \longrightarrow \mathbb{R}$ of order α is defined by*

$$T^a_\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}.$$

If $T^a_\alpha f(t)$ exists on (a,b), then $T^a_\alpha f(a) = \lim_{t\to a} T^a_\alpha f(t)$.

Theorem 3.1.1. *Let $\alpha \in (0,1]$ and $f : [0,\infty) \to \mathbb{R}$ is differentiable function at $t > 0$ then f is $\alpha-$differentiable function at $t > 0$, then*

$$f^{(\alpha)}(t) = t^{1-\alpha}\frac{d}{dt}f(t) \tag{3.2}$$

Proof: Let $h = \varepsilon t^{1-\alpha}$ in (3.1), and then we have $\varepsilon = t^{\alpha-1}h$. Therefore,

$$\begin{aligned} T_\alpha(f)(t) &= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \\ &= \lim_{h\to 0}\frac{f(t+h) - f(t)}{ht^{\alpha-1}} \\ &= t^{1-\alpha}\lim_{h\to 0}\frac{f(t+h) - f(t)}{h} \\ &= t^{1-\alpha}\tfrac{d}{dt}f(t). \end{aligned}$$

Theorem 3.1.2. *If a function $f : [0,\infty) \longrightarrow \mathbb{R}$ is $\alpha-$differentiable at $t_0 > 0$, $\alpha \in (0,1]$, then f is continuous at t_0.*

Proof: Since $f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0) = \frac{f(t_0+\varepsilon t_0^{1-\alpha}) - f(t_0)}{\varepsilon}\varepsilon$. Then,

$$\lim_{\varepsilon\to 0}[f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0)] = \lim_{\varepsilon\to 0}\frac{f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0)}{\varepsilon}\lim_{\varepsilon\to 0}\varepsilon$$

Let $h = \varepsilon t_0^{1-\alpha}$. Then,

$$\lim_{h\to 0}[f(t_0+h) - f(t_0)] = f^{(\alpha)}(t_0).0$$

which implies that

$$\lim_{h\to 0} f(t_0+h) = f(t_0).$$

Hence, f is continuous at t_0.

Properties 3.1.1. *Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then*

1. $T_\alpha(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$.

2. $T_\alpha(c) = 0$, *for all constant functions* $f(t) = c$.

3. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.

4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

5. $T_\alpha\left(\frac{f}{g}\right) = \dfrac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

6. $T_\alpha(f \circ g)(t) = f^{'}(g(t))T_\alpha(g)(t)$.

Proof: *Using (3.2), all properties will be proven consecutively.*

Now, for $\alpha \in (0,1]$,

$$\begin{aligned} T_\alpha(t^p) &= t^{1-\alpha}\tfrac{d}{dt}t^p \\ &= pt^{1-\alpha}t^{p-1} \\ &= pt^{p-\alpha}. \end{aligned}$$

This is prove of property 1. Secondly, for property number two,

$$T_\alpha(c) = t^{1-\alpha}\frac{d}{dt}c = 0.$$

Then, for property 3,

$$\begin{aligned} T_\alpha(af+bg)(t) &= t^{1-\alpha}\tfrac{d}{dt}(af+bg)(t) \\ &= t^{1-\alpha}(af^{'}(t) + bg^{'}(t)) \\ &= at^{1-\alpha}\tfrac{d}{dt}f(t) + bt^{1-\alpha}\tfrac{d}{dt}g(t) \\ &= aT_\alpha(f)(t) + bT_\alpha(g)(t). \end{aligned}$$

And from it the linear property is verified for this definition. Then the property 4 is prove by

$$\begin{aligned} T_\alpha(fg)(t) &= t^{1-\alpha}\tfrac{d}{dt}(fg)(t) \\ &= t^{1-\alpha}(g\tfrac{d}{dt}f + f\tfrac{d}{dt}g)(t) \\ &= t^{1-\alpha}f'(t)g(t) + t^{1-\alpha}g'(t)f(t) \\ &= g(t)T_\alpha(f)(t) + f(t)T_\alpha(g)(t). \end{aligned}$$

Then, for 5

$$\begin{aligned} T_\alpha\left(\tfrac{f}{g}\right)(t) &= t^{1-\alpha}\left(\tfrac{f}{g}\right)'(t) \\ &= t^{1-\alpha}\frac{(g(t)f'(t) - f(t)g'(t))}{(g(t))^2} \\ &= \frac{g(t)t^{1-\alpha}f'(t) - f(t)t^{1-\alpha}g'(t)}{(g(t))^2} \\ &= \frac{g(t)T_\alpha(f)(t) - f(t)T_\alpha(g)(t)}{(g(t))^2}. \end{aligned}$$

Finally, for property 6 is prove by

$$\begin{aligned} T_\alpha(f\circ g)(t) &= t^{1-\alpha}(f\circ g)'(t) \\ &= t^{1-\alpha}f'(g(t))g'(t) \\ &= f'(g(t))t^{1-\alpha}g'(t) \\ &= f'(g(t))T_\alpha(g)(t). \end{aligned}$$

The proof is complete.

Corollary 3.1.1. *[24] Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then*

- *Quotient Property:* $T_\alpha\left(\frac{1}{f(t)}\right) = -\frac{T_\alpha(f)(t)}{(f(t))^2}$.

- *Product Property:* $T_\alpha(f(t))^2 = 2(f(t)T_\alpha(f)(t))$

Theorem 3.1.3. *(Rolle's Theorem for Conformable Fractional Differentiable Functions) Let $a > 0$ and $f : [a,b] \longrightarrow \mathbb{R}$ be a given function that satisfies*

1. *f is continuous on $[a,b]$,*

2. *f is $\alpha-$differentiable for some $\alpha \in (0,1)$,*

3. $f(a) = f(b)$.

Then, there exist $c \in (a,b)$, *such that* $f^{(\alpha)}(c) = 0$

Proof: *Since* f *is continuous on* $[a,b]$ *and* $f(a) = f(b)$, *there exists* $c \in (a,b)$ *at which is a point of local extrema. With no loss of generality, assume* c *is a point of local minimum. So*

$$f^{(\alpha)}(c) = \lim_{\varepsilon \to 0^+} \frac{f(c + \varepsilon c^{1-\alpha}) - f(c)}{\varepsilon} = \lim_{\varepsilon \to 0^-} \frac{f(c + \varepsilon c^{1-\alpha}) - f(c)}{\varepsilon}.$$

But, the two limits have opposite sign, so $f^{(\alpha)}(c) = 0$.

Theorem 3.1.4. *(Mean Value Theorem for Conformable Fractional Differentiable Functions) Let* $a > 0$ *and* $f : [a,b] \longrightarrow \mathbb{R}$ *be a given function that satisfies*

1. f *is continuous on* $[a,b]$,

2. f *is* $\alpha-$*differentiable for some* $\alpha \in (0,1)$.

Then, there exists $c \in (a,b)$, *such that* $f^{(\alpha)}(c) = \dfrac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$

Proof: *Consider the function*

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right).$$

Then, the function g *satisfies the conditions of the fractional Rolle's theorem. Hence there exists* $c \in (a,b)$, *such that* $g^{(\alpha)}(c) = 0$. *Using the fact that* $T_\alpha(\frac{1}{\alpha}x^\alpha) = 1$, *the result follows.*

Definition 3.1.5. *Let* $\alpha \in (n, n+1]$, *and* f *be an* $n-$*differentiable at* t, *where* $t > 0$. *Then the conformable fractional derivative of* f *of order* α *is defined as*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{([\alpha]-1)}(t + \varepsilon t^{([\alpha]-\alpha)}) - f^{([\alpha]-1)}(t)}{\varepsilon}$$

Where $[\alpha]$ *is the smallest integer greater than or equal to* α.

Remark 3.1.4. *As a consequence of definition 3.1.5, one can easily show that*

$$T_\alpha(f)(t) = t^{([\alpha]-\alpha)} f^{[\alpha]}(t)$$

Where $\alpha \in (n, n+1]$, *and* f *is* $(n+1)-$*differentiable at* $t > 0$.

Theorem 3.1.5. *(Conformable fractional derivative of Known functions)*

1. $T_\alpha(e^{ct}) = ct^{1-\alpha}e^{ct}$

2. $T_\alpha(\sin(at)) = at^{1-\alpha}\cos(at), \quad a \in \mathbb{R}$

3. $T_\alpha(\cos(at)) = -at^{1-\alpha}\sin(at), \quad a \in \mathbb{R}$

4. $T_\alpha(\tan(at)) = at^{1-\alpha}\sec^2(at), \quad a \in \mathbb{R}$

5. $T_\alpha(\frac{1}{\alpha}t^\alpha) = 1$

Proof:

1).

$$\begin{aligned} T_\alpha(e^{ct}) &= \lim_{\varepsilon\to 0}\frac{e^{c(t+\varepsilon t^{1-\alpha})}-e^{ct}}{\varepsilon} = e^{ct}\lim_{\varepsilon\to 0}\frac{e^{c\varepsilon t^{1-\alpha}}-1}{\varepsilon} \\ &= e^{ct}\lim_{\varepsilon\to 0}\frac{t^{1-\alpha}e^{c\varepsilon t^{1-\alpha}}-t^{1-\alpha}}{\varepsilon t^{1-\alpha}} = e^{ct}t^{1-\alpha}\lim_{\varepsilon\to 0}\frac{e^{c\varepsilon t^{1-\alpha}}-1}{\varepsilon t^{1-\alpha}} \end{aligned}$$

Let $h = \varepsilon t^{1-\alpha}$. Then by using L'Hôpital rule, we get

$$\begin{aligned} &= t^{1-\alpha}e^{ct}\lim_{h\to 0}\frac{e^{ch}-1}{h} = ct^{1-\alpha}e^{ct}\lim_{h\to 0}\frac{e^{ch}}{1} \\ &= ct^{1-\alpha}e^{ct} \end{aligned}$$

2).

$$\begin{aligned} T_\alpha(\sin(at)) &= \lim_{\varepsilon\to 0}\frac{\sin(a(t+\varepsilon t^{1-\alpha}))-\sin(at)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{\sin(at)\cos(a\varepsilon t^{1-\alpha})+\cos(at)\sin(a\varepsilon t^{1-\alpha})-\sin(at)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\sin(at)\left[\frac{\cos(a\varepsilon t^{1-\alpha})-1}{\varepsilon}\right] + \lim_{\varepsilon\to 0}\frac{\cos(at)\sin(a\varepsilon t^{1-\alpha})}{\varepsilon} \\ &= t^{1-\alpha}\sin(at)\lim_{\varepsilon\to 0}\left[\frac{\cos(a\varepsilon t^{1-\alpha})-1}{\varepsilon t^{1-\alpha}}\right] + t^{1-\alpha}\cos(at)\lim_{\varepsilon\to 0}\frac{\sin(a\varepsilon t^{1-\alpha})}{\varepsilon t^{1-\alpha}} \end{aligned}$$

Let $h = \varepsilon t^{1-\alpha}$ then we get

$$= t^{1-\alpha}\sin(at)\lim_{h\to 0}\left[\frac{\cos(ah)-1}{h}\right] + t^{1-\alpha}\cos(at)\lim_{h\to 0}\frac{\sin(ah)}{h}$$

By using L'Hôpital Rule, we get

$$\begin{aligned} &= t^{1-\alpha}\sin(at)\lim_{h\to 0}\frac{-a\sin(ah)}{1} + t^{1-\alpha}\cos(at).a \\ &= at^{1-\alpha}\cos(at). \end{aligned}$$

3).Similar to (2)

4).

$$\begin{aligned}
T_\alpha(\tan(at)) &= T_\alpha\left(\frac{\sin(at)}{\cos(at)}\right)\\
&= \frac{\cos(at)T_\alpha(\sin(at))-\sin(at)T_\alpha(\cos(at))}{\cos^2(at)}\\
&= \frac{\cos(at)(at^{1-\alpha}\cos(at))-\sin(at)(-at^{1-\alpha}\sin(at))}{\cos^2(at)}\\
&= \frac{at^{1-\alpha}\cos^2(at)+at^{1-\alpha}\sin^2(at)}{\cos^2(at)}\\
&= at^{1-\alpha}(1+tan^2(at))\\
&= at^{1-\alpha}\sec^2(at).
\end{aligned}$$

5).

$$\begin{aligned}
T_\alpha(\tfrac{1}{\alpha}t^\alpha) &= t^{1-\alpha}\tfrac{d}{dt}\tfrac{1}{\alpha}t^\alpha\\
&= t^{1-\alpha}\tfrac{1}{\alpha}\alpha t^{\alpha-1}\\
&= t^{1-\alpha}t^{\alpha-1}\\
&= 1.
\end{aligned}$$

3.1.3 Conformable fractional integral

When it comes to integration, the most important class of functions to define the integral is the space of continuous functions. The conformable fractional integral is discussed as follows.

Definition 3.1.6. *Let $a \geq 0$ and $\alpha \in (0,1)$. Also, let f be a continuous function such that $I^\alpha f$ exists. Then:*

$$I_\alpha^a(f(t)) = I_1^a(t^{1-\alpha}f(t)) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx \tag{3.3}$$

If the Riemann improper integral exists.

Definition 3.1.7. *The fractional integral starting from a of a function $f : [a,\infty) \longrightarrow \mathbb{R}$ of order α is defined by:*

$$I_\alpha^a f(t) = \int_a^t (s-a)^{\alpha-1} f(s)ds.$$

Theorem 3.1.6. *(Inverse Property) $T_\alpha I_\alpha^a(f)(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of I_α.*

Proof: *Since f is continuous, then $I_\alpha^a(f)(t)$ is clearly differentiable. Hence,*

$$\begin{aligned} T_\alpha\left(I_\alpha^a(f)\right)(t) &= t^{1-\alpha}\tfrac{d}{dt}I_\alpha^a(f)(t) \\ &= t^{1-\alpha}\tfrac{d}{dt}\int_a^t \frac{f(x)}{x^{1-\alpha}}dx \\ &= t^{1-\alpha}\tfrac{f(t)}{t^{1-\alpha}} \\ &= f(t). \end{aligned}$$

Theorem 3.1.7. *(Conformable Fractional Integral of Conformable Fractional Derivative) Let $f:(a,b)\longrightarrow\mathbb{R}$ be $\alpha-$differentiable and $\alpha\in(0,1]$. For all $x>a$ then*

$$I_\alpha^a\left[T_\alpha(f)(t)\right] = f(t) - f(a).$$

Proof: *Using (3.3), it is easily seen that*

$$\begin{aligned} I_\alpha^a\left[T_\alpha(f)(t)\right] &= \int_a^t x^{\alpha-1}T_\alpha(f)(x)dx \\ &= \int_a^t x^{\alpha-1}x^{1-\alpha}\frac{d}{dx}f(x)dx \\ &= \int_a^t \frac{d}{dx}f(x)dx \\ &= f(t)-f(a). \end{aligned}$$

Definition 3.1.8. *(Conformable Fractional Integral as a Limit of a Sum) If f is a function defined for $a<x\leq b$. Then the definite fractional integral of f from a to be b is*

$$\int_a^b \frac{f(x)}{x^{1-\alpha}}dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{f(x_i)}{x_i^{1-\alpha}}\Delta x.$$

Where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$.

Theorem 3.1.8. *(Mean Value Theorem for Conformable Fractional Integral) If $f:[a,b]\longrightarrow\mathbb{R}$ is a continuous function on $[a,b]$. Then, there exists c in $[a,b]$ such that,*

$$\int_a^b \frac{f(x)}{x^{1-\alpha}}dx = f(c)\left(\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha\right).$$

Proof: *Using (3.3). Since $f(t)$ is continuous and recall that from theorem 3.1.6 $I_\alpha^a(f(t))$ is continuous on $[a,b]$, $\alpha-$differentiable on (a,b) and $T_\alpha\left(I_\alpha^a f(t)\right) = f(t)$. Now, from the theorem it can be stated that there is a number c such that $c\in(a,b)$ and*

$$I_\alpha^a f(b) - I_\alpha^a f(a) = T_\alpha\left[I_\alpha^a f(t)\right]\left(\frac{1}{\alpha}b^\alpha - \frac{a}{\alpha}a^\alpha\right).$$

However, it is known that $T_\alpha\left[I_\alpha a f(c)\right] = f(c)$ *and*

$$I_\alpha^a f(b) = \int_a^b \frac{f(x)}{x^{1-\alpha}} dx = \int_a^b \frac{f(t)}{t^{1-\alpha}} dt, \quad I_\alpha^a f(a) = \int_a^a \frac{f(x)}{x^{1-\alpha}} dx = 0$$

Thus

$$\int_a^b \frac{f(x)}{x^{1-\alpha}} dx = f(c)\left(\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha\right).$$

Theorem 3.1.9. *(Second Mean Value Theorem for Conformable Fractional Integral) Let* f *and* g *be functions satisfying the following continuous on* $[a,b]$*. Bounded and integrable on* $[a,b]$*,* $m = \inf\{f(x), x \in [a,b]\}$ *and* $M = \sup\{f(x), x \in [a,b]\}$*. Then, there exists a number* $c \in (a,b)$ *such that*

$$\int_a^b \frac{f(x)g(x)}{x^{1-\alpha}} dx \leq c \int_a^b \frac{g(x)}{x^{a-\alpha}} dx.$$

Proof: *If* $m = \inf f$*,* $M = \sup f$ *and* $g(x) \geq 0$ *in* $[a,b]$*, then*

$$mg(x) < f(x)g(x) < Mg(x) \tag{3.4}$$

Divide (3.4) by $x^{1-\alpha}$ *and integrate (3.4) with respect to* x *over* (a,b)*, resulting*

$$m\int_a^b \frac{g(x)}{x^{1-\alpha}} dx \leq \int_a^b \frac{f(x)G(x)}{x^{1-\alpha}} dx \leq M\int_a^b \frac{g(x)}{x^{1-\alpha}} dx.$$

Then there exists a number $c \in [m, M]$ *such that*

$$\int_a^b \frac{f(x)g(x)}{x^{1-\alpha}} dx \leq c \int_a^b \frac{g(x)}{x^{1-\alpha}} dx.$$

Theorem 3.1.10. *(Extended Mean Value Theorem for Conformable Fractional Differentiable Functions) Let* $a > 0$ *and* $f, g : [a,b] \longrightarrow \mathbb{R}$ *be functions that satisfy*

- f, g *is continuous on* $[a,b]$*,*
- f, g *is* $\alpha-$*differentiable for some* $\alpha \in (0,1)$*.*

Then, there exist $c \in (a,b)$*, such that*

$$\frac{f^{(\alpha)}(c)}{g^{(\alpha)}(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Remark 3.1.5. *Observe that Theorem 3.1.4 is a special case of this Theorem 3.1.10 for* $g(x) = \frac{x^\alpha}{\alpha}$

Proof: Consider the function

$$F(x) = f(x) - f(a) + \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)(g(x) - g(a))$$

Since F is continuous on $[a, b]$, $\alpha-$differentiable on (a, b), and $F(a) = 0 = F(b)$, then by Theorem 3.1.3, there exist a $c \in (a, b)$ such that $F^{(\alpha)}(c) = 0$ for some $\alpha \in (0, 1)$. Using the linearity of T_α and the fact that the $\alpha-$derivative of a constant is zero, our result follows.

Theorem 3.1.11. *Let $a > 0$ and $f : [a, b] \longrightarrow \mathbb{R}$ be a given function that satisfies*

- *f is continuous on $[a, b]$,*
- *f is $\alpha-$differentiable for some $\alpha \in (0, 1)$*

If $f^{(\alpha)}(x) = 0$ for all $x \in (a, b)$, then f is a constant on $[a, b]$.

Proof: *Suppose $f^{(\alpha)}(x) = 0$ for all $x \in (a, b)$. Let x_1, x_2 be in $[a, b]$ with $x_1 < x_2$. So, the closed interval $[x_1, x_2]$ is contained in $[a, b]$, and the open interval (x_1, x_2) is contained in (a, b).*

Hence, f is continuous on $[x_1, x_2]$ and $\alpha-$differentiable on (x_1, x_2). So, by Theorem 3.1.4, there exist c between x_1 and x_2 with

$$\frac{f(x_2) - f(x_1)}{\frac{x_2^\alpha}{\alpha} - \frac{x_1^\alpha}{\alpha}} = f^{(\alpha)}(c) = 0.$$

Therefore, $f(x_2) - f(x_1) = 0$ and $f(x_2) = f(x_1)$.

Since x_1 and x_2 are arbitrary numbers in $[a, b]$ with $x_1 < x_2$, then f is a constant on $[a, b]$.

Theorem 3.1.12. *Let $a > 0$ and $f : [a, b] \longrightarrow \mathbb{R}$ be a given function that satisfies*

- *f is continuous on $[a, b]$,*
- *f is $\alpha-$differentiable for some $\alpha \in (0, 1)$.*

Then we have the following:

1. *If $f^{(\alpha)}(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.*
2. *If $f^{(\alpha)}(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Proof: *Following similar line of argument as given in the proof of Theorem 3.1.11, there exist c between x_1 and x_2 with*

$$\frac{f(x_2) - f(x_1)}{\frac{x_2^\alpha}{\alpha} - \frac{x_1^\alpha}{\alpha}} = f^{(\alpha)}(c).$$

1. *If $f^{(\alpha)}(c) > 0$, then $f(x_2) > f(x_1)$ for $x_1 < x_2$.*
   Therefore, f is increasing on $[a, b]$ since x_1 and x_2 are arbitrary numbers of $[a, b]$.

2. *If $f^{(\alpha)}(c) < 0$, then $f(x_2) < f(x_1)$ for $x_1 < x_2$.*
   Therefore, f is decreasing on $[a, b]$ since x_1 and x_2 are arbitrary numbers of $[a, b]$.

Now we give an example to illustrate Theorem 3.1.12.

Example 3.1.1. *Let $f : [0.5, 3] \longrightarrow \mathbb{R}$ be defined by $f(x) = x^3 - 3x + 2$. Find where f is increasing and decreasing.*

Solution: *We first compute $f^{(\alpha)}(x)$ for any $\alpha \in (0, 1)$. By definition, we have*

$$f^{(\alpha)}(x) = 3x^{1-\alpha}(x^2 - 1).$$

So, $f^{(\alpha)}(x) = 0$ if and only if $x = -1, 0$ or 1.
All numbers less than 0 will not be considered since they do not lie in the domain under consideration.
To this end, we will consider all positive numbers less than one (in particular,$x \in [0.5, 1)$) and all numbers greater or equal to one (in particular,$x \in [1, 3]$).

- *For $x \in [0.5, 1)$, $x - 1 < 0$ and $x + 1 > 0$. This implies that for all $\alpha \in (0, 1)$, $f^{(\alpha)}(x) < 0$ for all $x \in [0.5, 1)$. So, f is decreasing on $[0.5, 1)$.*

- *For $x \in [1, 3]$, $x - 1 \geq 0$ and $x + 1 > 0$. This implies that for all $\alpha \in (0, 1)$, $f^{(\alpha)}(x) > 0$ for all $x \in [1, 3]$. So, f is increasing on $[1, 3]$.*

Theorem 3.1.13. *Let $0 < a < b$ and $f : [a, b] \longrightarrow \mathbb{R}$ be continuous function. Then for $\alpha \in (0, 1)$*

$$| I_\alpha^a(f)(t) | \leq I_\alpha^a(| f |)(t).$$

Proof: *The result follows directly since*

$$
\begin{aligned}
\mid I_\alpha^a(f)(t) \mid &= \left| \int_a^t \frac{f(x)}{x^{1-\alpha}} dx \right| \\
&\leq \int_a^t \left| \frac{f(x)}{x^{1-\alpha}} \right| dx \\
&= \int_a^t \frac{\mid f(x) \mid}{x^{1-\alpha}} dx \\
&= I_\alpha^a(\mid f \mid)(t).
\end{aligned}
$$

Corollary 3.1.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous function such that*

$$M = \sup_{[a,b]} \mid f \mid .$$

Then for any $t \in [a, b]$, $\alpha \in (0, 1)$,

$$|I_\alpha^a(f)(t)| \leq M \left(\frac{t^\alpha}{\alpha} - \frac{a^\alpha}{\alpha} \right).$$

Proof: *From Theorem 3.1.13, we have that for any $t \in [a, b]$, $\alpha \in (0, 1)$,*

$$
\begin{aligned}
\mid I_\alpha^a(f)(t) \mid &\leq I_\alpha^a(\mid f \mid)(t) \\
&= \int_a^t \frac{\mid f(x) \mid}{x^{1-\alpha}} dx \\
&\leq M \int_a^t x^{\alpha-1} dx \\
&= M \left(\tfrac{t^\alpha}{\alpha} - \tfrac{a^\alpha}{\alpha} \right).
\end{aligned}
$$

Definition 3.1.9. *($(\alpha, k)-$Laplace transform)Let $\alpha \in (0, 1]$, $k > 0$, and $f : [0, \infty) \longrightarrow \mathbb{R}$ be a function. Then the fractional Laplace transform of order α of f defined by*

$$\mathcal{L}_k^\alpha\{f(t); s\} = F_k^\alpha(s) = \int_0^\infty e^{-s\frac{t^{k\alpha}}{k\alpha}} f(t) d_\alpha t \tag{3.5}$$

Which is called $(\alpha, k)-$Laplace transform. Some properties of the $(\alpha, k)-$Laplace Transform

1. $\mathcal{L}_k^\alpha\{0; s\} = 0$
2. $\mathcal{L}_k^\alpha\{f(t) + g(t); s\} = \mathcal{L}_k^\alpha\{f(t); s\} + \mathcal{L}_k^\alpha\{g(t); s\}$
3. $\mathcal{L}_k^\alpha\{cf(t); s\} = c\mathcal{L}_k^\alpha\{f(t); s\}, \quad c$ *is a constant.*

Properties 2) and 3) together means that the Laplace transform is linear.

Theorem 3.1.14. *[29](Laplace transform for conformable fractional derivative) Let $\alpha \in (0,1]$, $k > 0$, and $f : (0,\infty) \longrightarrow \mathbb{R}$ be differentiable function. Then*

$$\mathcal{L}_k^{\alpha}\{T_{\alpha}(f)(t); s\} = s\mathcal{L}_k^{\alpha}\{t^{\alpha(k-1)}f(t); s\} - f(0). \tag{3.6}$$

Theorem 3.1.15. *[29]Let $\alpha \in (0,1]$, $c \in \mathbb{R}$ and $k > 0$. Then we have the following results*

1. $\mathcal{L}_k^{\alpha}\{1; s\} = s^{-\frac{1}{k}}\Gamma_k^{\alpha}(1)$,
2. $\mathcal{L}_k^{\alpha}\{t; s\} = s^{-\frac{1+\alpha}{k\alpha}}\Gamma_k^{\alpha}(2)$,
3. $\mathcal{L}_k^{\alpha}\{t^p; s\} = s^{-\frac{p+\alpha}{k\alpha}}\Gamma_k^{\alpha}(p+1)$,
4. $\mathcal{L}_k^{\alpha}\left\{e^{c\frac{t^{k\alpha}}{k\alpha}}; s\right\} = (s-c)^{-\frac{1}{k}}\Gamma_k^{\alpha}(1)$,
5. $\mathcal{L}_k^{\alpha}\{f(t)e^{c\frac{t^{k\alpha}}{k\alpha}}; s\} = F_k^{\alpha}(s-c)$,
6. $\mathcal{L}_k^{\alpha}\{f(ct); s\} = \frac{1}{c^{\alpha}}F_k^{\alpha}(\frac{s}{c^{k\alpha}})$.

Example 3.1.2. *Let us consider the function $f(t) = \sin v\frac{t^{\alpha}}{\alpha}$, then by using the property $T_{\alpha}(\cos v\frac{t^{\alpha}}{\alpha}) = -v\sin v\frac{t^{\alpha}}{\alpha}$, we can write*

$$\mathcal{L}_k^{\alpha}\left\{\sin v\frac{t^{\alpha}}{\alpha}; s\right\} = \int_0^{\infty} e^{-s\frac{t^{k\alpha}}{k\alpha}} \sin v\frac{t^{\alpha}}{\alpha} d_{\alpha}t$$

Therefore, using integration by part for conformable integral, we have

$$\begin{aligned}
-\tfrac{1}{v}\int_0^{\infty} e^{-s\frac{t^{k\alpha}}{k\alpha}} T_{\alpha}\left(\cos v\frac{t^{\alpha}}{\alpha}\right) d_{\alpha}t &= -\tfrac{1}{v}\left\{e^{-s\frac{t^{k\alpha}}{k\alpha}} \cos v\tfrac{t^{\alpha}}{\alpha} \,|_0^{\infty} - \int_0^{\infty} \cos v\frac{t^{\alpha}}{\alpha} T_{\alpha}\left(e^{-s\frac{t^{k\alpha}}{k\alpha}}\right) d_{\alpha}t\right\} \\
&= \tfrac{1}{v} - \tfrac{s}{v}\int_0^{\infty} t^{k\alpha-\alpha}e^{-s\frac{t^{k\alpha}}{k\alpha}} \cos v\frac{t^{\alpha}}{\alpha} d_{\alpha}t \\
&= \tfrac{1}{v} - \tfrac{s}{v^2}\int_0^{\infty} t^{k\alpha-\alpha}e^{-s\frac{t^{k\alpha}}{k\alpha}} T_{\alpha}\left(\sin v\frac{t^{\alpha}}{\alpha}\right) d_{\alpha}t.
\end{aligned}$$

Similarly, we get

$$\mathcal{L}_k^{\alpha}\left\{\sin v\frac{t^{\alpha}}{\alpha}; s\right\} = \frac{1}{v} + \frac{s(k-\alpha)}{v^2}\mathcal{L}_k^{\alpha}\left\{t^{k-2\alpha}\sin v\frac{t^{\alpha}}{\alpha}; s\right\} - \frac{s^2}{v^2}\mathcal{L}_k^{\alpha}\left\{t^{k-\alpha}\sin v\frac{t^{\alpha}}{\alpha}; s\right\} \tag{3.7}$$

If we take $k = \alpha$ in (3.7), we have

$$\mathcal{L}_{\alpha}^{\alpha}\left\{\sin v\frac{t^{\alpha}}{\alpha}; s\right\} = \frac{v}{1+s^2}$$

which is proved by Abdeljawad in [1]

3.2 Ordinary Conformable Fractional Differential equations

In this section, we establish some criteria for the global existence of solutions to the local initial value problem by means of some fixed point theorems and by the use of the conformable fractional calculus. More details can be found, e.g. in [15].

3.2.1 Preliminary

$$T_\alpha^a x(t) = f(t, x(t)), \quad t \in [a, \infty), \quad 0 < \alpha < 1, \tag{3.8}$$

subject to the initial conditions

$$x(a) = x_a \tag{3.9}$$

where $T_\alpha^a x(t)$ denotes the conformable fractional derivative starting from a of a function x of order α, $f : [a, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. The condition (3.9) are often called local initial condition.

Lemma 3.2.1. *[33] If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on the subinterval $[c, d]$ of $[a, b]$ and if $T_\alpha^a f(t)$ exists on (c, d). Then there exists a point ξ in (c, d) such that*

$$f(d) - f(c) = \frac{1}{\alpha} T_\alpha^a f(\xi) \left[(d-a)^\alpha - (c-a)^\alpha\right].$$

Lemma 3.2.2. *[33] Let f and g be continuous, nonnegative functions on $[a, b]$ and λ a nonnegative constant such that*

$$f(t) \leq \lambda + I_\alpha^a (fg)(t) \qquad for \quad t \in [a, b],$$

then

$$f(t) \leq \lambda e^{I_\alpha^a g(t)} \qquad for \quad t \in [a, b].$$

We first make the following hypotheses, which will be adopted in the following discussion.

Let $\mathcal{D} = [a, \infty) \times \mathbb{R}$.

(H1) The function $f : \mathcal{D} \longrightarrow \mathbb{R}$ is continuous.

(H2) There exists a positive constant L such that, for any $(t, u), (t, v)$ in $\mathcal{D}$,

$$|f(t, u) - f(t, v)| \leq L|u - v|.$$

3.2.2 The main results

In this subsection, we establish some criteria for the global existence, extension, boundedness, and stabilities of solutions to the local initial value problem. By Theorems 3.1.6 and 3.1.7, the initial value problem (3.8) − (3.9) is easily transformed into an equivalent integral equation.

Lemma 3.2.3. *[33] If (H1) holds, then a function x in $\mathcal{C}([a,b])$ is a solution of the initial value problem (3.8)-(3.9) if and only if it is a continuous solution of the following integral equation:*

$$x(t) = x_a + I^a_\alpha f(t, x(t))$$

Now, we are in a position to present a result of existence and uniqueness of the solution to the initial value problem (3.8)-(3.9).

3.2.3 Proof of the main results

Theorem 3.2.1. *If (H1)-(H2) hold, then the initial value problem (3.8)-(3.9) has exactly one solution defined on $[a,b]$.*

Proof: *Write $I = [a,b]$. The assertion will be proven by Banach's contraction principle on $\mathcal{C}(I)$ equipped with an appropriate weighted maximum norm. To this end, given a positive number β in (L,∞), define a function $e(t)$ by*

$$e(t) = e^{-\beta\frac{(t-a)^\alpha}{\alpha}},$$

and then, for x in $\mathcal{C}(I)$, define

$$\| x \|_\beta = \| e(.)x(.) \|,$$

where $\| \, . \, \|$ denotes the maximum norm on $\mathcal{C}(I)$. It is easy to verify that $\| \, . \, \|_\beta$ is also a norm on $\mathcal{C}(I)$, which is equivalent to the maximum norm $\| \, . \, \|$ since

$$e(b) \, \| \, . \, \| \leq \| \, . \, \|_\beta \leq \| \, . \, \| \, .$$

Hence $(\mathcal{C}(I), \| \, . \, \|_\beta)$ is a Banach space. Define next an operator

$$\mathcal{T} : (\mathcal{C}(I), \| \, . \, \|_\beta) \longrightarrow (\mathcal{C}(I), \| \, . \, \|_\beta)$$

by

$$\mathcal{T}x(t) = x_0 + \int_a^t f(s, x(s))(s-a)^{\alpha-1} ds,$$

and then Lemma 3.2.3 ensures that the fixed points of the operator $\mathcal{T}$ are the solutions of the problem (3.8)-(3.9). We now show that $\mathcal{T}$ is a contraction on $(\mathcal{C}(I), \| \, . \, \|_\beta)$. Indeed, let $x, y \in \mathcal{C}(I)$ and observe

$$\mathcal{T}x(t) - \mathcal{T}y(t) = \int_a^t \left[f(s, x(s)) - f(s, y(s)) \right] (s-a)^{\alpha-1} ds.$$

Thus, by (H2), a direct calculation gives, for every t in I,

$$\begin{aligned} \|\mathcal{T}x(t) - \mathcal{T}y(t)\| &\leq Le(t) \int_a^t e^{-1}(s) e(s) \|x(s) - y(s)\| (s-a)^{\alpha-1} ds \\ &\leq Le(t) \int_a^t e^{-1}(s)(s-a)^{\alpha-1} ds \parallel x - y \parallel \\ &\leq Le(t) I_\alpha^a e^{-1}(t) \parallel x - y \parallel \\ &\leq \tfrac{L}{\beta} e(t) \left(e^{-1}(t) - 1 \right) \parallel x - y \parallel \\ &\leq \tfrac{L}{\beta} \parallel x - y \parallel . \end{aligned}$$

Hence

$$\parallel \mathcal{T}x - \mathcal{T}y \parallel \leq \frac{L}{\beta} \parallel x - y \parallel .$$

Since $0 < \frac{L}{\beta} < 1$, the Banach contraction principle ensures that there is a unique x in $\mathcal{C}(I)$ with $x = \mathcal{C}x$, and equivalently the problem $(3.8) - (3.9)$ has a unique solution x in $\mathcal{C}(I)$. The proof is complete.

3.3 Stochastic Conformable Fractional Differential equations

In this section, we prove the existence and uniqueness result on the solution of a class of conformable fractional stochastic equation.

$$T_{\alpha,t}^a u(x,t) = \sigma(u(x,t)) \dot{W}_t, \quad x \in \mathbb{R}, 0 < a < t \leq T < \infty, 0 < \alpha < 1. \tag{3.10}$$

with an initial condition $u(x, 0) = u_0(x)$; where $T_{\alpha,t}^a$ is a conformable fractional derivative, $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz continuous function and $\dot{W}_t$ is a generalized derivative of Wiener process (Gaussian white noise).

Definition 3.3.1. *Given that $g(t)$ is any smooth and compactly supported function, then we define the generalized derivative $\dot{w}(t)$ of $w(t)$ (not necessarily differentiable) as*

$$\int_0^\infty g(t) \dot{w}(t) dt = - \int_0^\infty \dot{g}(t) w(t) dt.$$

Similarly, the generalized derivative $\dot{W}_t$ *of Wiener process with a smooth function* $g(t)$ *as follows*

$$\int_0^t g(s)\dot{W}_s ds = g(t)W_t - \int_0^t \dot{g}(s)W_s ds.$$

Theorem 3.3.1. *[25]. The following inequalities*

$$\exp\left(\frac{-ax}{a+1}\right) \leq \frac{a}{x^a}\gamma(a,x) \leq_1 F_1(a; a+1; -x) \leq \frac{1}{a+1}\left(1 + ae^{-x}\right)$$

hold, where ${}_1F_1(a; a+1; -x)$ *is a confluent hypergeometric (Kummer) function.*
Also, for $0 < a \leq 1$*,*

$$\frac{1-e^{-x}}{x} \leq \frac{a}{x^a}\gamma(a,x)$$

where $\gamma(z,x)$ is an incomplete gamma function given by

$$\gamma(z,x) = \int_0^x e^{-t}t^{z-1}dt, \quad x > 0$$

3.3.1 Main Results

Assume the following condition on σ; that is, σ is globally Lipschitz:

Condition 1. There exist a finite positive constant, Lip_σ such that for all $x, y \in \mathbb{R}$, we have

$$\mid \sigma(x) - \sigma(y) \mid \leq Lip_\sigma \mid x - y \mid,$$

with $\sigma(0) = 0$ for convenience.

Also, the assumption on u:

Condition 2. The random solution $u : \mathcal{D} \longrightarrow \mathbb{R}$ is $\mathbb{L}^2-$continuous (or continuous in probability).

Define the following $\mathbb{L}^2(\mathbb{P})$ norm

$$\parallel u \parallel_{2,\alpha,\beta} := \left\{ sup_{a \leq t \leq T} sup_{x \in \mathbb{R}} e^{-\frac{\beta}{\alpha}(t-a)^\alpha} \mathbb{E} \mid u(x,t) \mid^2 \right\}^{\frac{1}{2}}.$$

Following similar idea in [33], we give the following results:

Lemma 3.3.1. *[7] Given that Condition 2 holds, then a function u in* $\mathbb{L}^2(\mathbb{P})$ *is a solution of Equation (3.10) if and only if it is a solution of the integral equation*

$$u(x,t) = u_0(x) + I^a_{\alpha,t}[\sigma(u(x,t))\dot{W}_t].$$

Thus, the solution to Equation (3.10) is given as follows

$$\begin{aligned} u(x,t) &= u_0(x) + \int_a^t (s-a)^{\alpha-1}\sigma(u(x,s))\dot{W}_s ds \\ &= u_0(x) + \int_a^t (s-a)^{\alpha-1}\sigma(u(x,s))dW_s. \end{aligned}$$

3.3.2 Proof of the main results

Theorem 3.3.2. *Suppose* $\mathcal{C}_{\alpha,\beta,T} < \frac{1}{Lip_\sigma^2}$ *for positive constant* Lip_σ *together with both Conditions 1 and 2. Then there exists solution u that is unique up to modification, with*

$$\mathcal{C}_{\alpha,\beta,T} := \frac{(T-a)^{2\alpha-1}}{2\alpha(2\alpha-1)}\left(1+(2\alpha-1)e^{\frac{\beta}{\alpha}(T-a)}\right).$$

We start by defining the operator

$$\mathcal{A}u(x,t) = u_0(x) + \int_a^t (s-a)^{\alpha-1}\sigma(u(x,s))dw_s,$$

and the fixed point of the operator gives the solution of Equation (3.10).

The proof of the theorem is based on the following lemmas:

Lemma 3.3.2. *Suppose u is a predictable random solution such that* $\| u \|_{2,\alpha,\beta} < \infty$ *and Conditions 1 and 2 hold. Then there exists a positive constant* $\mathcal{C}_{\alpha,\beta,T}$ *such that*

$$\| \mathcal{A}u \|_{2,\alpha,\beta}^2 \leq c_1 + \mathcal{C}_{\alpha,\beta,T} Lip_\sigma^2 \| u \|_{2,\alpha,\beta}^2 .$$

Proof: *By the assumption that* u_0 *is bounded, we obtain*

$$\mathbb{E} \mid \mathcal{A}u(x,t) \mid^2 \leq c_1 + Lip_\sigma^2 \int_a^t (s-a)^{2(\alpha-1)} \mathbb{E} \mid u(x,s) \mid^2 ds.$$

Multiply through by $e^{-\frac{\beta}{\alpha}(t-a)^\alpha}$ *to obtain*

$$\begin{aligned} e^{-\frac{\beta}{\alpha}(t-a)^\alpha}\mathbb{E} \mid \mathcal{A}u(x,t) \mid^2 &\leq c_1 e^{-\frac{\beta}{\alpha}(t-a)^\alpha} \\ &+ Lip_\sigma^2 e^{-\frac{\beta}{\alpha}(t-a)^\alpha} \int_a^t (s-a)^{2(\alpha-1)} e^{\frac{\beta}{\alpha}(s-a)^\alpha} e^{-\frac{\beta}{\alpha}(s-a)^\alpha} \mathbb{E} \mid u(x,s) \mid^2 ds \\ &\leq c_1 e^{-\frac{\beta}{\alpha}(t-a)^\alpha} + Lip_\sigma^2 e^{-\frac{\beta}{\alpha}(t-a)^\alpha} \| u \|_{2,\alpha,\beta}^2 \int_a^t (s-a)^{2(\alpha-1)} e^{\frac{\beta}{\alpha}(s-a)^\alpha} ds \\ &\leq c_1 + Lip_\sigma^2 \| u \|_{2,\alpha,\beta}^2 \int_a^t (s-a)^{2(\alpha-1)} e^{\frac{\beta}{\alpha}(s-a)^\alpha} ds \end{aligned}$$

since $e^{-\frac{\beta}{\alpha}(t-a)^\alpha} \leq 1$, $a \leq t \leq T$, *that is,* $0 \leq t-a \leq T-a \Longrightarrow -\frac{\beta}{\alpha}(t-a) \leq 0$.

Thus taking sup over $t \in [a,T]$ and $x \in \mathbb{R}$ and evaluating the integral we have

$$\begin{aligned}
\| \mathcal{A}u \|^2_{2,\alpha,\beta} &\leq c_1 + Lip^2_\sigma \| u \|^2_{2,\alpha,\beta} \int_a^t (s-a)^{2(\alpha-1)} e^{\frac{\beta}{\alpha}(s-a)^\alpha} ds \\
&\leq c_1 + Lip^2_\sigma \| u \|^2_{2,\alpha,\beta} \times -\tfrac{\beta}{\alpha}(t-a)^{2\alpha} \left(\tfrac{\beta}{\alpha}\right)^{-2\alpha} (a-t)^{-2\alpha} \\
&\times \left[\Gamma(2\alpha-1) - \Gamma\left(2\alpha-1, \tfrac{\beta}{\alpha}(a-t)\right)\right] \\
&\leq c_1 + Lip^2_\sigma \| u \|^2_{2,\alpha,\beta} \times -(-1)^{-2\alpha} \left(\tfrac{\beta}{\alpha}\right)^{1-2\alpha} \left[\Gamma(2\alpha-1) - \Gamma\left(2\alpha-1, \tfrac{\beta}{\alpha}(a-t)\right)\right] \\
&\leq c_1 + Lip^2_\sigma \left(\tfrac{\beta}{\alpha}\right)^{1-2\alpha} \Gamma\left(2\alpha-1, \tfrac{\beta}{\alpha}(a-t)\right) \| u \|^2_{2,\alpha,\beta} .
\end{aligned}$$

By the estimate on the incomplete gamma function in Theorem 3.3.1, we obtain

$$\begin{aligned}
\Gamma\left(2\alpha-1, \tfrac{\beta}{\alpha}(a-t)\right) &\leq \frac{\left(\frac{\beta}{\alpha}\right)^{2\alpha-1}}{2\alpha(2\alpha-1)}(a-t)^{2\alpha-1}\left(1+(2\alpha-1)e^{\frac{\beta}{\alpha}(t-a)}\right) \\
&\leq \frac{\left(\frac{\beta}{\alpha}\right)^{2\alpha-1}}{2\alpha(2\alpha-1)}(t-a)^{2\alpha-1}\left(1+(2\alpha-1)e^{\frac{\beta}{\alpha}(t-a)}\right)
\end{aligned}$$

and therefore, since $0 < t-a < T-a$, we have

$$\begin{aligned}
\| \mathcal{A}u \|^2_{2,\alpha,\beta} &\leq c_1 + \tfrac{Lip^2_\sigma}{2\alpha(2\alpha-1)}(t-a)^{2\alpha-1}\left(1+(2\alpha-1)e^{\frac{\beta}{\alpha}(t-a)}\right) \| u \|^2_{2,\alpha,\beta} \\
&\leq c_1 + \tfrac{Lip^2_\sigma}{2\alpha(2\alpha-1)}(T-a)^{2\alpha-1}\left(1+(2\alpha-1)e^{\frac{\beta}{\alpha}(T-a)}\right) \| u \|^2_{2,\alpha,\beta}
\end{aligned}$$

Lemma 3.3.3. *[7] Suppose u and v are predictable random solutions such that $\| u \|_{2,\alpha,\beta} + \| v \|_{2,\alpha,\beta} < \infty$ and Conditions 1 and 2 hold. Then there exists a positive constant $\mathcal{C}_{\alpha,\beta,T}$ such that*

$$\| \mathcal{A}u - \mathcal{A}v \|^2_{2,\alpha,\beta} \leq \mathcal{C}_{\alpha,\beta,T} Lip^2_\sigma \| u - v \|^2_{2,\alpha,\beta} .$$

Remark 3.3.1. *By Fixed point theorem we have $u(x,t) = \mathcal{A}u(x,t)$ and*

$$\| u \|^2_{2,\alpha,\beta} = \| \mathcal{A}u \|^2_{2,\alpha,\beta} \leq c_1 + \mathcal{C}_{\alpha,\beta,T} Lip^2_\sigma \| u \|^2_{2,\alpha,\beta}$$

which follows that

$$\| u \|^2_{2,\alpha,\beta} \left[1 - \mathcal{C}_{\alpha,\beta,T} Lip^2_\sigma\right] \leq \infty \Longrightarrow \| u \|_{2,\alpha,\beta} \leq \infty \Longleftrightarrow \mathcal{C}_{\alpha,\beta,T} < \frac{1}{Lip^2_\sigma}.$$

Similarly,

$$\| u - v \|_{2,\alpha,\beta}^2 = \| \mathcal{A}u - \mathcal{A}v \|_{2,\alpha,\beta}^2 \leq \mathcal{C}_{\alpha,\beta,T} Lip_\sigma^2 \| u - v \|_{2,\alpha,\beta}^2,$$

thus $\| u - v \|_{2,\alpha,\beta}^2 [1 - \mathcal{C}_{\alpha,\beta,T} Lip_\sigma^2] \leq 0$ and therefore $\| u - v \|_{2,\alpha,\beta}^2 < 0$ if and only if $\mathcal{C}_{\alpha,\beta,T} < \frac{1}{Lip_\sigma^2}$.

The existence and uniqueness result follows by Banach's contraction principle.

Conclusion

In this master thesis a new kind of fractional derivative is introduced, the most important properties of the conformable fractional derivative and integral were given and proved, some interesting results of ordinary fractional calculus are extended to conformable one.

We first review the basic definitions and properties of fractional integral and derivative for the purpose of acquainting with sufficient fractional calculus theory. Many definitions and studies of fractional calculus have been proposed in the past two centuries. These definitions include Grünwald-Letnikov and the two most commonly used definitions Riemann-Liouville and Caputo fractional operators and with the help of them solution of differential equations are discussed.

Secondly the new derivative is introduced, important properties and examples are given, distinguishing features and basic theorems of these derivative and integral are introduced and proved.

Finally, using obtained results the conformable fractional ordinary differential equations were established. The existence and uniqueness result were obtained under some precise conditions for class of conformable time-fractional stochastic equation.

In the end, we hope and predict that researches in this subject will be active and promising since there are different questions which is still without any accurate answer. For example, it is possible to extend this new definition for many class of stochastic fractional differential equations which will be considered by others as a future work.

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