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Theme:

Controllability of Stochastic Differential Equations Driven by Mixed Fractional Brownian Motion

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⁂ *Dedication* ⁂

In the name of Allah, the Most Gracious, the Most Merciful

This thesis is dedicated to

*My **father**, who sincerely raised me with care and actively supported me in my determination to find and realize my potential, has been a constant source of strength and encouragement.*

*My beloved **mother**, who raised me with care and gave me all the love, she is the main reason for who I am today, and she is a source of strength to me.*

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Abstract

The purpose of this thesis is to present some theoretical results on stochastic differential equations driven by mixed fractional Brownian motion. We first recall some basic notions of Brownian motion and fractional Brownian motion. We then introduce the mixed fractional Brownian motion and study some of its properties. We develop the stochastic calculus needed to handle such equations and prove the existence and uniqueness of the associated solutions. Finally, we study the controllability problem for the related system in the mild formulation by using the control-to-state operator, and we establish a main result on controllability under appropriate assumptions.

Résumé

Ce travail se concentre sur l'analyse théorique des équations différentielles stochastiques pilotées par un mouvement brownien mixte fractionnaire. Nous commençons par présenter les concepts essentiels relatifs au mouvement brownien et au mouvement brownien fractionnaire, avant d'introduire le mouvement brownien fractionnaire mixte et ses principales caractéristiques. Nous construisons ensuite le calcul stochastique adapté à ce processus et démontrons des théorèmes d'existence et d'unicité pour les solutions des équations stochastiques étudiées. Par la suite, nous examinons la contrôlabilité du système associé en exploitant la formulation faible et le concept d'opérateur entrée-sortie, et nous établissons un résultat fondamental concernant la contrôlabilité sous des conditions convenables.

Contents

Introduction

Brownian motion has a long history in probability theory and remains one of the most important stochastic processes in mathematics and its applications. Since its first observation in 1827 by the Scottish botanist Robert Brown [9], Brownian motion has played a fundamental role in the development of modern probability theory, stochastic analysis, and many other fields. Brown observed pollen grains exhibiting erratic motion while suspended in water. Initially, this phenomenon was not understood, but it was later shown to result from the random collisions of fluid molecules with the particles. The motion now known as Brownian motion is a fundamental physical phenomenon. A rich mathematical theory has since been developed to describe it, and it has become a cornerstone of stochastic process theory and random dynamical systems.

A succession of mathematicians and physicists founded the theory of Brownian motion. Their work, however, was, in many cases, a contribution to the general theory of random walks and probabilistic models. The first reference to such models was due to Thorvald Thiele in 1880 [35]. In finance, the first stochastic model of stock price fluctuations was established by Louis Bachelier in his Ph.D. thesis, The Theory of Speculation (1900) [4]. It was Albert Einstein in 1905 [13] who provided the fundamental physical explanation of Brownian motion in terms of the kinetic molecular theory of gases. The mathematical model of Brownian motion was then formulated by Norbert Wiener in 1923 [36] within the framework of stochastic process theory. The Wiener process is one of the most fundamental stochastic processes and has many applications in finance, biology, and physics.

Brownian motion is a continuous-time stochastic process with continuous sample paths, independent increments, and Gaussian distributions. These characteristics make it a primary framework for modeling random variations across diverse fields and form the essential basis for Itô calculus and a significant portion of modern stochastic differential equation theory. However, the standard Brownian motion model is insufficient to address all random processes, particularly those that exhibit memory or long-range dependence.

To overcome this limitation, Mandelbrot and Van Ness introduced fractional Brownian motion [21]. This process generalizes standard Brownian motion by incorporating a Hurst parameter $H \in (0,1)$, which governs the roughness of sample paths and the dependence structure of increments. When $H = \frac{1}{2}$, fractional Brownian motion reduces to classical Brownian motion. For $H > \frac{1}{2}$, the process exhibits long-range dependence, while for $H < \frac{1}{2}$, it displays an anti-persistent behavior. Because of these features, fractional Brownian motion has become a powerful tool for modeling phenomena with memory effects.

More recently, Brownian motion and fractional Brownian motion have been combined to create mixed fractional Brownian motion, a flexible model that captures both short- and long-term dependence. This hybrid process combines the local randomness of Brownian motion with the persistence of fractional Brownian motion, making it a natural framework for studying stochastic differential equations driven by mixed noise.

This thesis is devoted to the theoretical study of mixed fractional Brownian motion and the controllability of stochastic differential equations driven by this process. In Chapter 1, we recall the basic notions and properties of Brownian motion and fractional Brownian motion. In Chapter 2, we introduce mixed fractional Brownian motion and study its main characteristics. In Chapter 3, we develop the stochastic calculus needed to treat equations driven by mixed fractional Brownian motion and establish existence and uniqueness results for their solutions. In Chapter 4, we investigate the controllability of the corresponding system using the mild formulation and the control-to-state operator, and we present the main controllability result under suitable assumptions.

This organization allows us to move gradually from classical stochastic processes to more advanced hybrid models, and then to the analysis of controllability for stochastic systems driven by mixed fractional Brownian motion.

Chapter 1

Generalities on Fractional Brownian Motion

1.1 Brownian Motion

Brownian motion is the continuous random motion of microscopic particles when suspended in a fluid medium. Brownian motion was first observed (1827) by the Scottish botanist Robert Brown[9] (1773-1858) when studying pollen grains in water. The effect was finally explained in 1905 by Albert Einstein [13], who realized it was caused by water molecules colliding randomly with the particles. Over a century later, Brownian motion can still cause problems for scientists studying small biological particles in solution because they move around too much.

1.1.1 Definition of Brownian Motion

To formally define Brownian motion, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space on which we define the process $(B_t)_{t\geq 0}$.

Definition 1.1.1. *A stochastic process* $(B_t)_{t\geq 0}$ *is called a* **standard Brownian motion** *if it satisfies the following conditions:*

1. $B_0 = 0 \ \mathbb{P} - a.s.$
2. *For all* $n \geq 1$, *for all times* $0 = t_0 \leq t_1 \leq ... \leq t_n$, *The increments* $B_{t_n} - B_{t_{n-1}}, ..., B_{t_1} - B_{t_0}$ *are independent random variables ("independent increments").*
3. *For any given times* $0 \leq s \leq t$, $B_t - B_s$ *has the Gaussian distribution* $\mathcal{N}(0, t-s)$ *with mean zero and variance* $t - s$.
4. *Almost surely, the function* $t \to B_t$ *is continuous.*

Remark 1.1.1.

1. We can rewrite the second condition as follows: for $s \leq t$, the random variable $B_t - B_s$ is independent of $\sigma(B_r, r \leq s)$.

2. The natural filtration of the Brownian motion is $\mathcal{F}_t^B = \sigma(B_s, s \leq t)$.

3. We can define the Brownian motion without the last condition of continuous paths, because with a stochastic process satisfying the second and the third conditions, by applying the Kolmogorov's continuity theorem, there exists a modification of $(B_t)_{t\geq 0}$ which has continuous paths almost surely.

Proposition 1.1.2. *The Brownian motion $B = (B_t, t \geq 0)$ is a centered Gaussian process with covariance :*

$$Cov(B_t, B_s) = \mathbb{E}(B_s B_t) = min(s,t) = s \wedge t, s \geq 0, t \geq 0.$$

Proof. We have that $B_t = B_t - B_0$. Thus $B_t \sim \mathcal{N}(0, t)$ by definition. Moreover, without loss of generality, we assume $s < t$. Hence, we have

$$Cov(B_t, B_s) = \mathbb{E}(B_s B_t) = E(B_s(B_t - B_s) + B_s^2) = 0 + Var(B_s) = 0 + s = s, s < t,$$

hence

$$Cov(B_t, B_s) = \mathbb{E}(B_s B_t) = min(s,t) = s \wedge t, s \geq 0, t \geq 0.$$

Note that since the Brownian motion is a continuous Gaussian process, the proposition 1.1.2 characterizes uniquely the Brownian motion. ■

1.1.2 Properties of Brownian Motion

In this section, we will present some properties of Brownian motion.

Proposition 1.1.3. *[7] Let $(B_t)_{t\geq 0}$ be a standard Brownian motion*

1. *Self-similarity. For any $a > 0$, $\{a^{-1/2} B_{at}\}$ is Brownian motion.*

2. *Symmetry. $\{-B_t, t \geq 0\}$ is also a Brownian motion.*

3. *Time-inversion. $\{tB_{\frac{1}{t}}, t > 0\}$ is also a Brownian motion.*

4. *If B_t is a Brownian motion on $[0,1]$, then $(t+1)B_{\frac{1}{t+1}} - B_1$ is a Brownian motion on $[0, \infty)$.*

Remark 1.1.2. Observe that a consequence of (3) is the Law of Large Numbers for the Brownian motion, namely $\mathbb{P}[\lim_{t\to+\infty} t^{-1} B_t = 0] = 1$.

1.1.2.1 Non-differentiability of Brownian Motion

Despite being continuous, the random nature of Brownian motion yields many interesting pathological properties. The most prominent example of this is that it is nowhere differentiable.

Lemma 1.1.1. Almost surely

$$\limsup_{n\to+\infty} \frac{B(n)}{\sqrt{n}} = +\infty.$$

And similarly for lim inf.

So $B(t)$ grows slower than t. But this lemma shows that its lim sup grows faster than $\sqrt{t}$.

Proof. By reverse Fatou,

$$\mathbb{P}[B(n) > c\sqrt{n} i.o.] \geq \limsup_{n\to+\infty} \mathbb{P}[B(n) > c\sqrt{n}] = \limsup_{n\to+\infty} \mathbb{P}[B(1) > c] > 0,$$

by the scaling property. Thinking of $B(n)$ as the sum of $X_n = B(n) - B(n-1)$, the event on the LHS is exchangeable and the Hewitt-Savage 0-1 law implies that it has probability 1 (where we used the positive lower bound). ■

Definition 1.1.4. (Upper and lower derivatives) For a function f, we define the upper and lower right derivatives as

$$D^* f(t) = \limsup_{h\downarrow 0} \frac{f(t+h) - f(t)}{h},$$

and

$$D_* f(t) = \liminf_{h\downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

We begin with an easy first result.

Theorem 1.1.5. *Fix $t \geq 0$. Then almost surely Brownian motion is not differentiable at t. Moreover, $D^* B(t) = +\infty$ and $D_* B(t) = -\infty$.*

Proof. Consider the time inversion X. Then

$$D^* X(0) \geq \limsup_{n\to+\infty} \frac{X(n^{-1}) - X(0)}{n^{-1}} = \limsup_{n\to+\infty} B(n) = +\infty,$$

by the lemma above. This proves the result at 0. Then note that $X(s) = B(t+s) - B(s)$ is a standard Brownian motion and differentiability of X at 0 is equivalent to differentiability of B at t.
In fact, we can prove something much stronger. ■

Theorem 1.1.6. *Almost surely, Brownian motion is nowhere differentiable. Furthermore, almost surely, for all t*

$$D^*B(t) = +\infty,$$

or

$$D_*B(t) = -\infty,$$

or both.

Proof. Suppose there is t_0 such that the latter does not hold. By boundedness of BM over $[0,1]$, we have

$$\sup_{h\in[0,1]} \frac{|B(t_0+h)-B(t_0)|}{h} \leq M,$$

for some $M < +\infty$. Assume t_0 is in $[(k-1)2^{-n}, k2^{-n}]$ for some k, n. Then for all $1 \leq j \leq 2^{-n} - k$, in particular, for $j = 1, 2, 3$,

$$\begin{aligned} |B((k+j)2^{-n}) - B((k+j-1)2^{-n})| &\leq |B((k+j)2^{-n}) - B(t_0)| + |B(t_0) - B((k+j-1)2^{-n})| \\ &\leq \left[M(2j+1)2^{-n}\right], \end{aligned}$$

by our assumption. Define the events

$$\Omega_{n,k} = \{|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \leq M(2j+1)2^{-n}, j = 1, 2, 3\}.$$

It suffices to show that $\bigcup_{k=1}^{2^n-3} \Omega_{n,k}$ cannot happen for infinitely many n. Indeed,

$$\mathbb{P}\left[\exists t_0 \in [0,1], \sup_{h\in[0,1]} \frac{|B(t_0+h)-B(t_0)|}{h} \leq M\right]$$

$$\leq \mathbb{P}\left[\bigcup_{k=1}^{2^n-3} \Omega_{n,k} \quad for \quad infinitely \quad many \quad n\right].$$

(Then the result follows by taking all $[k, k+1]$ intervals and all M integers). But by the independence of increments

$$\begin{aligned}
\mathbb{P}[\Omega_{n,k}] &= \prod_{j=1}^{3} \mathbb{P}[|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \le M(2j+1)2^{-n}] \\
&\le \mathbb{P}\left[|B(2^{-n})| \le \frac{7M}{2^n}\right]^3 \\
&= \mathbb{P}\left[\left|\frac{1}{\sqrt{2^{-n}}} B\left(\left[\sqrt{2^{-n}}\right]^2\right)\right| \le \frac{7M}{\sqrt{2^{-n}}.2^n}\right]^3 \\
&= \mathbb{P}\left[|B(1)| \le \frac{7M}{\sqrt{2^n}}\right]^3 \\
&\le \left(\frac{7M}{\sqrt{2^n}}\right)^3,
\end{aligned}$$

because the density of a standard Gaussian is bounded by 1/2. (The choice of 3 comes from summability). Hence

$$\mathbb{P}\left[\bigcup_{k=1}^{2^n-3} \Omega_{n,k}\right] \le 2^n \left(\frac{7M}{\sqrt{2^n}}\right)^3 = (7M)^3 2^{-n/2},$$

which is summable. The result follows from BC. That is, the probability above is 0. ■

1.1.2.2 Brownian paths

Lemma 1.1.2. (Kolmogrov-Chentsov)
Fix a compact interval $\mathbb{T} = [0, T] \subset \mathbb{R}_+$, and let $X = (X_t)_{t\in\mathbb{T}}$ be a centered Gaussian process. Suppose that there exists $C, \eta > 0$ such that, for all $s, t \in \mathbb{T}$,

$$\mathbb{E}[(X_t - X_s)^2] \le C \mid t - s \mid^{\eta} . \tag{1.1}$$

Then, for all $\alpha \in (0, \eta/2)$, there exists a modification Y of X with α-*Hölder* continuous paths. In particular, X admits a continuous modification.

Proof. Fix $t > s$. Since X is Gaussian and centered, we have that

$$X_t - X_s \overset{Law}{=} \sqrt{\mathbb{E}[(X_t - X_s)^2]}G,$$

where $G \sim \mathcal{N}(0, 1)$. We deduce from (1.1) that, for all $p \ge 1$,

$$\mathbb{E}[\mid X_t - X_s \mid^p] \le C^{p/2}\mathbb{E}[\mid G \mid^p] \mid t - s \mid^{\eta p/2} .$$

Therefor, the general version of the classical Kolmogrov-Chentsov lemma is applied and gives the desired result. ■

Proposition 1.1.7. *A Brownian motion has its paths almost surely, locally γ-Hölder continuous for $\gamma \in [0, 1/2)$.*

Proof. Let $T > 0$, $n \in \mathbb{N}$ and $0 \leq s \leq t$. Then we have,

$$\mathbb{E}((B_t - B_s)^{2n}) = \frac{(2n)!}{2^n n!}(t-s)^n.$$

Hence, by using the Kolmogrov-Chentsov lemma 1.1.2, there exists a continuous modification $(\tilde{B}_t)_{0 \leq t \leq T}$ of $(B_t)_{0 \leq t \leq T}$, whose the paths are locally γ-Hölder continuous for $\forall \gamma \in [0, \frac{n-1}{2n})$. Moreover, we have

$$\mathbb{P}(\forall t \in [0, T], B_t = \tilde{B}_t) = 1,$$

because the two processes are continuous, It implies that also almost all the paths of $(B_t)_{0 \leq t \leq T}$ are locally γ-Hölder continuous. ■

Proposition 1.1.8. *[14] The Brownian motion's sample paths are almost surely nowhere differentiable.*

There is an intuitive way to understand this property of Brownian paths. Indeed, consider the increment for $h > 0$, $B_{t+h} - B_t \sim \mathcal{N}(0, h)$. Then we have that $\frac{B_{t+h}}{\sqrt{h}} \sim \mathcal{N}(0, 1)$. But the derivative is defined to be the limit, as h tends to 0, of the quantity $\frac{B_{t+h} - B_t}{h} \sim \mathcal{N}(0, \frac{1}{h})$. It is clear, now, that when we let h tends to 0, we obtain an "infinite" variance so that there would not be a limit.

1.1.2.3 Quadratic variation and Brownian Motion

Definition 1.1.9. (Bounded variation) A function $f : [0, t] \to \mathbb{R}$ is of bounded variation if there is $M < +\infty$ such that

$$\sum_{j=1}^{k} |f(t_j) - f(t_{j-1})| \leq M,$$

for all $k \geq 1$ and all partitions $0 = t_0 < t_1 < \cdots < t_k = t$. Otherwise, we say that it is of bounded variation.

Functions of bounded variation are known to be differentiable. Since Brownian motion is nowhere differentiable, it must have unbounded variation. However, Brownian motion has a finite "quadratic variation."

Theorem 1.1.10. ***(Quadratic variation)*** *Suppose the sequence of partitions*

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k(n)}^{(n)} = t,$$

is nested, that is, at each step one or more partition points are added, and the mesh

$$\Delta(n) = \sup_{1 \leq j \leq k(n)} \{t_j^{(n)} - t_{j-1}^{(n)}\},$$

converges to 0. Then, almost surely

$$\lim_{n \to +\infty} \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2 = t.$$

Proof. By considering subsequences, it suffices to consider the case where one point is added at each step. Let

$$X_{-n} = \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2.$$

Let

$$\mathcal{G}_{-n} = \sigma(X_{-n}, X_{-n-1}, \ldots)$$

and

$$\mathcal{G}_{-\infty} = \bigcap_{k=1}^{\infty} \mathcal{G}_{-k}.$$

For more details of the proof, see([25]). ■

1.1.2.4 Markov property

Theorem 1.1.11. ***(Markov property)****[24] Let $\{B_t : t \geq 0\}$ is a Brownian motion started in $x \in \mathbb{R}^d$. Then the process $\{B_{t+s} - B_s : t, s > 0\}$ is a Brownian motion started at the origin and is independent of $\{B_t : 0 \leq t \leq s\}$.*

Proof. This follows directly from property the independence of increments of Brownian motion. ■

However, this is rather trivial. A preliminary means of making this property slightly stronger is establishing that Brownian motion is independent of information that exists an infinitesimal amount of time into the future.

Definition 1.1.12. The **germ σ-algebra** is defined as $\mathcal{F}^+(0)$, where

$$\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}^0(s)$$

and $\{\mathcal{F}^0 : t \geq 0\}$ is the σ-algebra generated by $\{B_t : 0 \leq s \leq t\}$.

Theorem 1.1.13. *[10] For all $s \geq 0$, the random process $\{B_{t+s} - B_s : t \geq 0\}$ is independent of $\mathcal{F}^+(s)$.*

Proof. By continuity, we can write the following for a strictly decreasing sequence $\{s_n : n \in \mathbb{N}\}$ converging to s:

$$B_{t+s} - B_s = \lim_{n\to\infty} B_{s_n+t} - B_{s_n}$$

However, the Markov property verifies that the right side of the above equation is independent of $\mathcal{F}^+(s)$. ∎

Theorem 1.1.14. *Every stopping time with respect to the filtration $\{\mathcal{F}^+(t) : t \geq 0\}$ is a strict stopping time.*

Proof. First, let us establish the right-continuity of $\{\mathcal{F}^+(t) : t \geq 0\}$ To do this, we can write

$$\mathcal{F}^+(t) = \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty} \mathcal{F}^0(t + \frac{1}{n} + \frac{1}{k}) = \bigcap_{\epsilon>0} \mathcal{F}^+(t+\epsilon).$$

Thus,

$$\{T \leq t\} = \bigcap_{k=1}^{\infty}\{T < t + \frac{1}{k}\} \in \bigcap_{n=1}^{\infty} \mathcal{F}^+(t + \frac{1}{n}) = \mathcal{F}^+(t)$$

∎ □

Theorem 1.1.15. ***(Strong Markov property)****[24] For every almost surely finite stopping time T, the process $\{B_{T+t} - B_T : t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}^+(T)$.*

Proof. Let T be a stopping time. We can then define

$$T_n = (m+1)2^{-n}, \text{ where } m/2^n \leq T < (m+1)/2^n.$$

Consider this as a discrete approximation that ends at the first dyadic rational adjacent to the original. Keeping in mind that this definition indicates that T_n is a stopping time, we define the following:

$$B_k(t) = B_{t+k/2^n} - B_{k/2^n} \text{ and } B_k = \{B_k(t) : t \geq 0\},$$
$$B_*(t) = B_{t+T_n} - B_{T_n} \text{ and } B_* = \{B_*(t) : t \geq 0\}.$$

Now, take $E \in \mathcal{F}^+(T_n)$ and the event $\{B_* \in A\}$. We have

$$\mathbb{P}(\{B_* \in A\}\bigcap E) = \sum_{k=0}^{\infty} \mathbb{P}(\{B_k \in A\}\bigcap E\bigcap\{T_n = k/2^n\}).$$

Note, however, that $E\bigcap\{T_n = k/2^n\} \in \mathcal{F}^+(k/2^n)$, which by theorem1.1.14 is independent of $\{B_k \in A\}$. Consequently, we have

$$\mathbb{P}(\{B_* \in A\}\bigcap E) = \sum_{k=0}^{\infty}\mathbb{P}\{B_k \in A\}\mathbb{P}(E\bigcap\{T_n = k/2^n\}).$$

Now we use the Markov property we see that for all $k \in \mathbb{N}, \mathbb{P}\{B \in A\} = \mathbb{P}\{B_k \in A\}$. This yields

$$\sum_{K=0}^{\infty}\mathbb{P}\{B_k \in A\}\mathbb{P}(E\bigcap\{T_n = k/2^n\}) = \mathbb{P}\{C \in A\}\sum_{k=0}^{\infty}\mathbb{P}(E\bigcap\{T_n = k/2^n\}) = \mathbb{P}\{C \in A\}\mathbb{P}(E).$$

Consequently, B_* is independent of every E and hence independent of $\mathcal{F}^+(T_n)$. Now, recall that the sequence T_n is a uniformly decreasing sequence that converges to T, hence $\mathcal{F}^+(T_n) \subset \mathcal{F}^+(T)$ is independent of the Brownian motion $B_{s+T_n} - B_{T_n}$. Then, the random process $B_{r+T} - B_T$, defined by the increments

$$B_{s+t+T} - B_{t+T} = \lim_{n\to\infty} B_{s+t+T_n} - B_{t+T_n},$$

is independent, $N(0, s)$, and almost surely continuous. Thus, it is a Brownian motion and independent of $\mathcal{F}^+(T)$. ■

1.1.2.5 Martingale Property

The standard Brownian motion and several functions of it are martingales.

Proposition 1.1.16. *[14] Let $(B_t)_{t\in\mathbb{R}_+}$ be a Brownian motion. Then the following processes are $(\mathcal{F}_t^B)$-martingales:*

1. $(B_t)_{t\in\mathbb{R}_+}$,
2. $(B_t^2 - t)_{t\in\mathbb{R}_+}$,
3. *For any* $u \in \mathbb{R}$, $(e^{uB(t)-\frac{u^2}{2}t})_{t\in\mathbb{R}_+}$.

1.2 Fractional Brownian Motion

The fractional Brownian motion (fBm) is a suitable generalization of standard Brownian motion; it is the best-known process that's not a semi-martingale. It is the only Gaussian self-similar stationary process with a long-range dependence property. Due to these interesting properties it enjoyed success as a modeling tool in many fields of applications, including telecommunications, turbulence, and finance; the demand to stochastic calculus with respect to fBm are raised. This process was introduced by Kolmogorov[18] and studied later by Mandelbrot and Van Ness[21] who provided an integral representation of fBm with respect to a standard Brownian motion over a real line time interval.

1.2.1 Definition of Fractional Brownian Motion

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space.

Definition 1.2.1. The fractional Brownian motion (fBm) with Hurst index ($H \in (0,1)$) is a Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$ on $(\Omega, \mathcal{A}, \mathbb{P})$, having the properties:

1. $B_0^H = 0$,
2. $\mathbb{E}(B_t^H) = 0$; $t \in \mathbb{R}$,
3. $cov(B_t^H, B_s^H) = \mathbb{E}(B_t^H B_s^H) = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big)$; $s, t \in \mathbb{R}$.

Remark 1.2.1. Since $\mathbb{E}(B_t^H - B_s^H)^2 = |t-s|^{2H}$ and B_H is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem.

Remark 1.2.2. We have:

- For $H = \frac{1}{2}$, the fBm is the standard Brownian motion.
- For $H = 1$, we set $B_t^H = B_t^1 = t\xi$, where ξ is a standard normal Random variable.

1.2.2 Basic properties

Proposition 1.2.2. *Let B^H be a fractional Brownian motion of Hurst parameter $H \in (0,1)$. Then:*

1. *Selfsimilarity. For all $a > 0$, $(B_{at}^H) \overset{d}{=} (a^H B_t^H)$.*
2. *Stationarity of increments. For all $h > 0$, $(B_{t+h}^H - B_h^H) \overset{d}{=} B_t^H$.*
3. *Hölder continuity. For each $0 < \varepsilon < H$ and each $T > 0$ there exists a random variable $K_{\varepsilon,T}$ such that*

$$\mid B^H(t) - B^H(s) \mid \leq K_{\varepsilon,T} \mid t - s \mid^{H-\varepsilon} .$$

4. *Differentiability. The sample paths of fBm are nowhere differentiable.*

Proof. First, let us prove the selfsimilarity property. We have that

$$\begin{aligned}
\mathbb{E}(B_{at}^H B_{as}^H) &= \frac{1}{2}((at)^{2H} + (as)^{2H} - (a \mid t-s \mid)^{2H}) \\
&= a^{2H}\mathbb{E}(B_t^H B_s^H) \\
&= \mathbb{E}((a^H B_t^H)(a^H B_s^H)).
\end{aligned}$$

Thus, since all processes are centered and Gaussian, it implies that

$$(B_{at}^H) \overset{d}{=} (a^H B_t^H).$$

Seconde, we show that it has stationary increments. Note that for all $h > 0$, we have

$$\begin{aligned}
\mathbb{E}((B_{t+h}^H - B_h^H)(B_{s+h}^H - B_h^H)) &= \mathbb{E}(B_{t+h}^H B_{s+h}^H) - \mathbb{E}(B_{t+h}^H B_h^H) - \mathbb{E}(B_{s+h}^H B_h^H) + \mathbb{E}((B_h^H)^2) \\
&= \frac{1}{2}\left[((t+h)^{2H} + (s+h)^{2H} - \mid t-s \mid^{2H})\right. \\
&\quad \left. - \left((t+h)^{2H} + h^{2H} - t^{2H}\right) - \left((s+h)^{2H} + h^{2H} - s^{2H}\right) + 2h^{2H}\right] \\
&= \frac{1}{2}(t^{2H} + s^{2H} - \mid t-s \mid^{2H}) = \mathbb{E}(B_t^H B_s^H).
\end{aligned}$$

Therefore the fBm is of stationary increments.

For the Hölder continuity it follows from Kolmogrov-Chentsov lemma 1.1.2 and the fact that for any $\alpha > 0$, we have

$$\mathbb{E}(\mid B_t^H - B_s^H \mid^{\alpha}) = \mathbb{E}(\mid B_1^H \mid^{\alpha}) \mid t-s \mid^{2H} .$$

Finally, lets prove the differentiability, indeed for every $t_0 \in [0, \infty]$,

$$\mathbb{P}\left(\limsup_{t \to t_0} \mid \frac{B_t^H - B_{t_0}^H}{t - t_0} \mid = \infty\right) = 1.$$

Let us denote by $\mathfrak{B}_{t,t_0} = \dfrac{B_t^H - B_{t_0}^H}{t - t_0}$, using the selfsimilarity property, we have

$$\mathfrak{B}_{t,t_0} \overset{d}{=} (t - t_0)^{H-1} B_1^H.$$

We define $\mathfrak{u}(t, \omega) = \{ \sup_{0 \le s \le t} \mid \dfrac{B_s^H}{s} \mid > d\}$. Then, for any sequence $(t_n)_{n \in \mathbb{N}}$ decreasing to 0,

we have $\mathfrak{u}(t_n, \omega) \supseteq \mathfrak{u}(t_{n+1}, \omega)$, thus,

$$\mathbb{P}(\lim_{n \to \infty} \mathfrak{u}(t_n)) = \lim_{n \to \infty} \mathbb{P}(\mathfrak{u}(t_n)),$$

and

$$\mathbb{P}(\mathfrak{u}(t_n)) \ge \mathbb{P}(\mid \frac{B_{t_n}^{(H)}}{t_n} \mid > d) = \mathbb{P}(\mid B_1^{(H)} \mid > t_n^{1-H} d) \overset{n \to \infty}{\to} 1.$$

■

1.2.3 Long and Short-Range Dependence

Process with long-range dependence have many application, such as in telecommunication specially in Internet traffic problems. Basically, the notion of long-range dependence is that the variance of the sum of stationary sequence grows non-linearly with respect to n.

Definition 1.2.3. *A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits long-range dependence if $\rho(n) = cov(X_k, X_{k+n})$ satisfies*

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1,$$

for $\alpha \in (0,1)$ and some constant c.

Remark 1.2.3. If a stationary sequence $(X_n)_{n\in\mathbb{N}}$ is long-range dependent, then the dependence between X_k and X_{k+1} decays slowly as n tends to infinity and $\sum_{n=1}^{\infty} \rho(n) = \infty$.

Proposition 1.2.4. The fBm is one of the simplest processes which exhibit long-range dependency.

Proof. let us consider its increments

$$X_k = B_k^H - B_{k-1}^H, \qquad X_{k+1} = B_{k+n}^H - B_{k+n-1}^H.$$

Since the fBm is centered then

$$\begin{aligned}
\rho(n) &= \mathbb{E}(X_k, X_{k+n}) = \mathbb{E}\left[(B_k^H - B_{k-1}^H)(B_{k+n}^H - B_{k+n-1}^H)\right] \\
&= \mathbb{E}\left[(B_{n+1}^H - B_n^H)B_1^H\right] = \mathbb{E}(B_{n+1}^H B_1^H) - \mathbb{E}(B_n^H B_1^H) \\
&= \frac{1}{2}\left[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right] \\
&= \frac{1}{2}n^{2H}\left[(1+\frac{1}{n})^{2H} - 2 + (1-\frac{1}{n})^{2H}\right] \\
&= \frac{n^{2H}}{2}\left[1 + \frac{2H}{n} + \frac{H(2H-1)}{n^2} - 2 + 1 - \frac{2H}{n} + \frac{H(2H-1)}{n^2} + o(\frac{1}{n^2})\right] \\
&= H(2H-1)n^{2H-2} + o(n^{2H-2}).
\end{aligned}$$

It follows that for $H > \frac{1}{2}$, we have

$$\rho(n) > 0 \quad and \quad \sum_n \rho(n) = \infty.$$

And for $H < \frac{1}{2}$, we have

$$\rho(n) < 0 \quad and \quad \sum_n \rho(n) < \infty.$$

Therefore, we say that the fBm has long-range dependence property if and only if $H > \frac{1}{2}$ and for the other case has short-range dependence. ■

1.2.4 Fractional Brownian Motion is not Markovian

Theorem 1.2.5. *Let B^H be a fractional Brownian motion of Hurst index $H \in (0,1) - \{\frac{1}{2}\}$. Then B^H is not a Markov process.*

Since the fBm is a Gaussian centered process, to prove this result we need the next lemma.

Lemma 1.2.1. *If X is a Gaussian centered Markovian process, then for all $s < t < u$*

$$\mathbb{E}(X_t X_s)\mathbb{E}(X_t X_u) = \mathbb{E}(X_t X_t)\mathbb{E}(X_u X_s).$$

Proof. Note that $R_{st} = cov(X_s, X_t)$. Since X is a Markov process then $\forall \ s < t < u$

$$\mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u/X_t) = \mathbb{E}(X_u) + \frac{cov(X_t, X_u)}{var(X_t)}(X_t - \mathbb{E}(X_t)).$$

Therefore,

$$\begin{cases} \mathbb{E}(X_u/X_t) = \dfrac{R_{ut}}{R_{tt}} X_t, \\ \mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u) + \theta_{uv}\theta_v^{-1}(v - \mathbb{E}(v)) \end{cases}$$

where $v = \begin{pmatrix} X_t \\ X_s \end{pmatrix}$ and $\theta_{uv} = \mathbb{E}[X_u v^t]$, $\theta_v = \mathbb{E}(v^t v)$

We have that,

$$\theta_{uv} = (R_{ut} R_{us}) \ and \ \theta_v = \begin{pmatrix} R_{tt} & R_{ts} \\ R_{st} & R_{ss} \end{pmatrix}$$

$$\theta_v^{-1} v = \frac{1}{R_{tt}R_{ss} - R_{ts}^2} \begin{pmatrix} R_{ss}X_t - R_{ts}X_s \\ R_{tt}X_s - R_{st}X_t \end{pmatrix}$$

We observe that,

$$\begin{aligned} \mathbb{E}(X_u/X_t, X_s) &= \theta_{uv}\theta_v^{-1} v \\ &= \frac{1}{R_{tt}R_{ss} - R_{ts}^2}(R_{ut}R_{ss}X_t - R_{ut}R_{ts}X_s - R_{us}R_{st}X_t + R_{us}R_{tt}X_s). \end{aligned}$$

Hence, $\mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u/X_t)$ we have

$$\frac{R_{ut}}{R_{tt}}X_t = \frac{1}{R_{tt}R_{ss} - R_{ts}^2}(R_{ut}R_{ss}X_t - R_{ut}R_{ts}X_s - R_{us}R_{st}X_t + R_{us}R_{tt}X_s).$$

Moreover,

$$\begin{aligned} X_t(R_{tt}R_{ut}R_{ss} - R_{tt}R_{ut}R_{ss} - R_{ut}R_{st}^2 + R_{tt}R_{us}R_{st}) + X_s(R_{tt}R_{ut}R_{st} - R_{tt}^2R_{us}) &= 0 \\ R_{st}X_t(R_{tt}R_{us} - R_{ut}R_{st}) - R_{tt}X_s(R_{tt}R_{us} - R_{ut}R_{st}) &= 0. \end{aligned}$$

Or,

$$(R_{tt}R_{us} - R_{ut}R_{st})(R_{st}X_t - R_{tt}X_s) = 0,$$

then,

$$R_{tt}R_{us} - R_{ut}R_{st} = 0.$$

Which is the result. ■

Proof of theorem 1.2.5 We proceed by contradiction. Assume that B^H is a Markov process. Since it is a Gaussian process as well, by the previous lemma we have, for $s = 1 < t = 2 < u = 3$

$$\mathbb{E}(B_1^H B_2^H)\mathbb{E}(B_2^H B_3^H) = \mathbb{E}(B_2^H B_2^H)\mathbb{E}(B_1^H B_3^H).$$

So,

$$\begin{aligned} \frac{1}{4}(1 + 2^{2H} - 1)(2^{2H} + 3^{2H} - 1) &= 2^{2H}\frac{1}{2}(1 + 3^{2H} - 2^{2H}) \\ 2^{2H}(2^{2H} + 3^{2H} - 1) &= 2^{2H}[2(1 + 3^{2H} - 2^{2H})], \end{aligned}$$

by differentiating

$$\begin{aligned} 3 + 3^{2H} + 3(2^{2H}) &= 0 \\ 1 + 3^{2H-1} + 2^{2H} &= 0. \end{aligned}$$

We deduce that, $1 + 3^{2H-1} + 2^{2H} = 0$ only if $H = \frac{1}{2}$ which leads to a contradiction. ■

1.2.5 Fractional Brownian Motion is not a semi-martingale

The fact that the fBm is not a semi-martingale for $H \neq \frac{1}{2}$ has been proved by several authors. In order to verify that B^H is not a semi-martingale for $H \neq \frac{1}{2}$, it is sufficient to compute the p-variation of B^H.

Definition 1.2.6. Let $(X(t))_{t\in[0,T]}$ be a stochastic process and consider a partition $\pi = \{0 = t_0 < t_1 < ...... < t_n = T\}$. Put

$$\mathcal{S}_p(x,\pi) := \sum_{i=1}^{n} |X(t_i) - X(t_{i-1})|^p$$

The p-variation of X over the interval $[0,T]$ is defined as

$$\mathcal{V}_p(X,[0,T]) := sup_\pi \mathcal{S}_p(X,\pi),$$

where π is a finite partition of $[0,T]$. The index of p-variation of a process is defined as

$$I(X,[0,T]) := \inf\{p > 0; \mathcal{V}_p(X,[0,T]) < \infty\}.$$

We claim that

$$I(B^H,[0,T]) = \frac{1}{H}.$$

In fact, consider for $p > 0$,

$$Y_{n,p} = n^{pH-1}\sum_{i=1}^{n}\left|B^H(\frac{i}{n}) - B^H(\frac{i-1}{n})\right|^p.$$

Since B^H has the self-similarity property, the sequence $Y_{n,p}, n \in N$ has the same distribution as

$$\tilde{Y}_{n,p} = n^{-1}\sum_{i=1}^{n}\left|B^H(i) - B^H(i-1)\right|^p.$$

By the Ergodic theorem (see, for example,[11]) the sequence $\tilde{y}_{n,p}$ converges almost surely and in L^1 to $\mathbb{E}\left[|B^H(1)|^p\right]$ as n tends to infinity. It follows that

$$V_{n,p} = \sum_{i=1}^{n}\left|B^H(\frac{i}{n}) - B^H(\frac{i-1}{n})\right|^p$$

converges in probability respectively to 0 if $pH > 1$ and to infinity if $pH < 1$ as n tends to infinity. Thus we can conclude that $I(B^H,[0,T]) = \frac{1}{H}$. Since for every semimartingale X, the index $I(X,[0,T])$ must belong to $[0,1]\cup\{2\}$, the fBm B^H cannot be a semimartingale unless $H = \frac{1}{2}$.

1.2.6 Representation of Fractional Brownian Motion

There are some representations of the fractional Brownian motion as a Wiener integral.

1.2.6.1 Lévy-Hida Representation

Let B^H be a fractional Brownian motion with parameter $H \in (0,1)$. The fBm admits a representation as a Wiener integral of the form

$$B^H = \int_0^t K_H(t,s)dWs,$$

where $W = (W_t)_{t\in T}$ is a Wiener process, and $K_H(t,s)$ is the kernel

$$K_H(t,s) = d_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right),$$

d_H being a constant and

$$F_1(z) = d_H\left(\frac{1}{2} - H\right)\int_0^{z-1} \theta^{H-\frac{3}{2}}\left(1 - (\theta+1)^{H-\frac{1}{2}}\right) d\theta.$$

If $H > \dfrac{1}{2}$,the kernel K_H has the simpler expression

$$K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $t > s$ and $c_H = \left(\dfrac{H(H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$. The fact that the process B^H is a fBm follows is from the equality

$$\int_0^{t\wedge s} K_H(t,u)K_H(s,u)du = R_H(t,s).$$

The kernel K_H satisfies the condition

$$\frac{\partial K_H}{\partial t}(t,s) = d_H\left(H - \frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}.$$

1.2.6.2 Moving Average Representation

FBm can be represented as an integral with respect to a standard Brownian motion on the whole real line. Let $(B_s)_{s\in\mathbb{R}}$ be a standard Brownian motion. Then

$$B_t^H = \frac{1}{C(H)}\int_{\mathbb{R}}\left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] dB_s, \tag{1.2}$$

with $C(H) > 0$ an explicit normalizing constant, is a fractional Brownian motion.

1.2.6.3 Harmonizable Representation

There is another representation which uses the complex-valued Brownian motion (but the fBm is real-valued). In fact, for a fBm $(B_t^H)_{t\in\mathbb{R}}$, we obtain

$$B_t^H = \frac{1}{C_2(H)}\int_{\mathbb{R}} \frac{e^{itx}-1}{ix}|x|^{-(H-\frac{1}{2})}d\tilde{B}_x, \quad t\in\mathbb{R},$$

where $(\tilde{B}_t)_{t\in\mathbb{R}}$ is a complex Brownian measure and

$$C_2(H) = \left(\frac{\pi}{H\Gamma(2H)sin(H\pi)}\right)^{1/2}.$$

Let us note that the complex Brownian measure on $\mathbb{R}$ can be splitted as $\tilde{B} = B_1 + iB_2$ and is such that $B_1(A) = B_1(-A)$, $B_2(A) = -B_2(-A)$ and $\mathbb{E}(B_1(A))^2 = \frac{\mid A\mid}{2}$, $\forall A \in \mathcal{B}(\mathbb{R})$. We also call this representation, the spectral representation

Chapter 2

Mixed Fractional Brownian Motion and its Properties

This chapter introduces a significant extension of fractional Brownian motion: the mixed fractional Brownian motion. After defining this process and examining its properties, we aim to construct a representation of mixed fractional Brownian motion in the white noise space. We demonstrate that this process is differentiable in the sense of distributions. Additionally, we explore the transformed characteristics of this process. This investigation leads us to our primary objective, which is stochastic analysis of mixed fractional Brownian motion.

Mixed fractional Brownian motion with parameter H is a stochastic process that was introduced by Cheridito[10], to model a financial phenomenon by the stochastic process $(X_t^H(a,b))_{t\in[0,1]}$ given by:

$$X_t^H(a,b) = X_0^H(a,b)e^{\nu t+\sigma M_t^H(a,b)}.$$

The authors took ν, σ, two constants, and $a > 0, b = 1$.

2.1 Definition of Mixed Fractional Brownian Motion

Definition 2.1.1. A mixed fractional Brownian motion with parameters a, b and H is a process $M^H = \{M_t^H(a,b), \forall t \geq 0\} = \{M_t^H, \forall t \geq 0\}$ defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ as:

$$\forall t \in \mathbb{R}_+, \qquad M_t^H = aB_t + bB_t^H,$$

where $(B_t^H)_{t\geq 0}$ is a fractional Brownian motion with parameter H independent of B and $(B_t)_{t\geq 0}$ is Brownian motion.

2.2 Basic properties of Mixed Fractional Brownian Motion

Proposition 2.2.1. *[39] The following properties are satisfied by the mixed fractional Brownian motion:*

1. *M^H is a centered Gaussian process.*
2. *$\forall t \in \mathbb{R}_+, \qquad \mathbb{E}((M_t^H(a,b))^2) = a^2 t + b^2 t^{2H}$.*
3. *Its covariance function is given by*
$$Cov(M_t^H(a,b), M_s^H(a,b)) = a^2 \min(t,s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \qquad \forall t,s \geq 0.$$
4. *The increments of mixed fractional Brownian motion are stationary.*
5. *For all $H \in (0,1)\backslash\{\frac{1}{2}\}, a \in \mathbb{R}, b \in \mathbb{R}, \quad (M_t^H(a,b))_{t\geq 0}$ is not a Markov process.*
6. *For all $\alpha > 0$, $\left(M_{\alpha t}^H(a,b)\right)_{t\geq 0} = \left(M_t^H(a\alpha^{\frac{1}{2}}, b\alpha)\right)_{t\geq 0}$, this property is called mixed self-similarity.*

2.2.1 Correlation between the increments

Notation 2.2.1. Assume that X and Y are two random variables that are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficient of correlation is noted $\rho(X,Y)$, as follows:
$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}}.$$

Lemma 2.2.1. $\forall s \in \mathbb{R}_+, \forall t \in \mathbb{R}_+, \forall h \in \mathbb{R}_+, 0 \leq h \leq t-s$
$$\rho(M_{t+h}^H - M_t^H, M_{s+h}^H - M_s^H) = \frac{b^2}{2(a^{2h+b^2h^{2H}})}\left[(t-s+h)^{2H} - 2(t-s)^{2H} + (t-s-h)^{2H}\right].$$

Corollary 2.2.1. For all $a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\{0\}$, the increments of $(M_t^H(a,b))_{t\in\mathbb{R}_+}$ are positively correlated if $\frac{1}{2} < H < 1$, negatively correlated if $0 < H < \frac{1}{2}$, and no correlated if $H = \frac{1}{2}$.

Proof.
If $H < \frac{1}{2}$, by the concavity of the function $x \longmapsto x^{2H}$, one derives
$$\forall x \in \mathbb{R}_+, \forall h \in \mathbb{R}_+\backslash\{0\}, \quad (x+h)^{2H} - 2x^{2H} + (x-h)^{2H} < 0.$$

If $H > \frac{1}{2}$, by the convexity of the function $x \longmapsto x^{2H}$, one derives

$$\forall x \in \mathbb{R}_+, \forall h \in \mathbb{R}_+ \backslash \{0\}, \quad (x+h)^{2H} - 2x^{2H} + (x-h)^{2H} > 0.$$

consequently, using the lemma (2.2.1),

$$\begin{cases} \text{If } H < \frac{1}{2}, & \rho(M_{t+h}^H - M_t^H, M_{s+h}^H - M_s^H) < 0. \\ \text{If } H > \frac{1}{2}, & \rho(M_{t+h}^H - M_t^H, M_{s+h}^H - M_s^H) > 0. \\ \text{If } H = \frac{1}{2}, & \rho(M_{t+h}^H - M_t^H, M_{s+h}^H - M_s^H) = 0. \end{cases}$$

■

Remark 2.2.1. By using corollary (2.2.1) and lemma (2.2.1), we get

i) If $H > \frac{1}{2}$ (respectivelly $H < \frac{1}{2}$), if $a \neq 0, b_1$, and b_2 are two reel constants such that $|b_1| \leq |b_2| (resp, |b_1| \geq |b_2|)$, since

$$\forall s \in \mathbb{R}_+, \quad \forall t \in \mathbb{R}_+, \quad \forall h \in \mathbb{R}_+, \quad 0 \leq h \leq t-s$$

$$(M_{t+h}^H(a,b_1) - M_t^H(a,b_1), M_{s+h}^H(a,b_1) - M_s^H(a,b_1))$$

$$\leq (M_{t+h}^H(a,b_2) - M_t^H(a,b_2), M_{s+h}^H(a,b_2) - M_s^H(a,b_2)).$$

Then, if $H > \frac{1}{2}$ (*resp.* $H < \frac{1}{2}$)

1. While |b| is great (resp. small), the increments are more correlated.
2. While |b| is small (resp. great), the increments are less correlated.

ii) If $H > \frac{1}{2}$ ($H < \frac{1}{2}$), if $b \neq 0, a_1$, and a_2 are two real constants such that $|a_1| \leq |a_2| (resp, |a_1| \geq |a_2|)$, since

$$\forall s \in \mathbb{R}_+, \quad \forall t \in \mathbb{R}_+, \quad \forall h \in \mathbb{R}_+, \quad 0 \leq h \leq t-s$$

$$(M_{t+h}^H(a_2,b) - M_t^H(a_2,b), M_{s+h}^H(a_2,b) - M_s^H(a_2,b))$$

$$\leq (M_{t+h}^H(a_1,b) - M_t^H(a_1,b), M_{s+h}^H(a_1,b) - M_s^H(a_1,b)).$$

Then, if $H > \frac{1}{2}$ (*resp.,* $H < \frac{1}{2}$), we have

1. While |a| is great (resp. small), the increments are less correlated.
2. While |a| is small (resp. great), the increments are more correlated.

In practical application, we can select H, a, b such that the $M_t^H(a, b)$ would be a suitable model for a given phenomenon.

2.2.2 Long and short term dependence

Lemma 2.2.2. The increments of mixed fractional Brownian motion are long-term dependent if and only if $H > \frac{1}{2}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\{0\}$.

Proof.
For all $n \in \mathbb{N}^*$,

$$\begin{aligned} r(n) &= \mathbb{E}\left((M_{n+1}^H - M_n^H)M_1^H\right) = \frac{b^2}{2}\left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right] \\ &= b^2 H(H-1)n^{2H-2}\epsilon(n), \end{aligned}$$

where $\lim_{n \to +\infty} \epsilon(n) = 0$.

Observing that $\sum_{n \in \mathbb{N}^*} r(n) = +\infty$, we can conclude that $H > \frac{1}{2}$ if and only if $2H-2 > -1$.■

2.2.3 Hölderian Continuity and differentiability

Lemma 2.2.3. The mixed fractional Brownian motion has a modification with trajectories that are $\gamma-$Hölder continuous in $[0, T]$ for all $T > 0$ and $\gamma < \frac{1}{2} \wedge H$.

Proof.
Using *Kolmogorov*'s theorem, it is sufficient to demonstrate that

$$\forall \alpha > 0, \exists C_\alpha, \forall (s,t) \in [0,T]^2, \qquad \mathbb{E}\left(\left|M_t^H - M_s^H\right|^\alpha\right) \le C_\alpha |t-s|^{\alpha(\frac{1}{2} \wedge H)}.$$

Based on the increments of M_t^H, we get the stationarity (2.2.1) and mixed autosimilarity (2.2.1).

$$\begin{aligned} \mathbb{E}\left(\left|M_t^H - M_s^H\right|^\alpha\right) &\le \mathbb{E}\left(\left|M_{t-s}^H\right|^\alpha\right) \\ &\le \mathbb{E}\left(\left|M_1^H(a(t-s)^{\frac{1}{2}-H}), b(t-s)H\right|^\alpha\right). \end{aligned}$$

Two positive constants, C_1 and C_2, depending on α, exist if $H \le \frac{1}{2}$, such that

$$\begin{aligned}\mathbb{E}\left(\left|M_t^H - M_s^H\right|^\alpha\right) &\leq (t-s)^{\alpha H}\mathbb{E}\left(\left|M_1^H(a(t-s)^{\frac{1}{2}}, b)\right|^\alpha\right)\\ &\leq (t-s)^{\alpha H}\left[C_1|a|^\alpha(t-s)^{\alpha((\frac{1}{2})-H)}\mathbb{E}(|B_1|^\alpha) + C_2|b|^\alpha\mathbb{E}(|B_1^H|^\alpha)\right]\\ &\leq C_\alpha(t-s)^{\alpha H},\end{aligned}$$

where

$$C_\alpha = C_1|a|^\alpha T^{\alpha((\frac{1}{2})-H)}\mathbb{E}(|B_1|^\alpha) + C_2|b|^\alpha\mathbb{E}(|B_1^H|^\alpha).$$

Two positive constants, C_1' and C_2', depending on α, exist if $H > \dfrac{1}{2}$. These constants ensure

$$\begin{aligned}\mathbb{E}\left(\left|M_t^H - M_s^H\right|^\alpha\right) &\leq (t-s)^{\frac{\alpha}{2}}\mathbb{E}\left(\left|M_1^H(a, b(t-s)^{H-\frac{1}{2}})\right|^\alpha\right)\\ &\leq (t-s)^{\frac{\alpha}{2}}\left[C_1'|a|^\alpha\mathbb{E}(|B_1|\alpha) + C_2'|b|^\alpha(t-s)^{\alpha(H-(\frac{1}{2}))}\mathbb{E}(|B_1^H|^\alpha)\right]\\ &\leq C_\alpha(t-s)^{\frac{\alpha}{2}},\end{aligned}$$

where

$$C_\alpha = C_1'|a|^\alpha\mathbb{E}(|B_1|\alpha) + C_2'|b|^\alpha T^{\alpha(H-(\frac{1}{2}))}\mathbb{E}(|B_1^H|^\alpha).$$

The concepts presented by *Kolwankar* and *Gangal* were followed by *Ben Adda* [19] and *Cresson* [6] in their analysis, as per the findings of *Mounir Zili* [39]. ■

Definition 2.2.2. allow f be a continuous function on $[a, b]$, and allow $\alpha \in]0, 1[$. $\alpha-$local fractional derivative of f in $t_0 \in [a, b]$ is what we refer to it as. $d_\sigma^\alpha f(t_0)$ provided by

$$d_\sigma^\alpha f(t_0) = \Gamma(1+\alpha)\, \underset{t\to t_0^\sigma}{lim}\, \frac{\sigma(f(t) - f(t_0))}{|t-t_0|^\alpha},$$

for $\sigma = +(resp, \sigma = -)$, where the *Euler*-function is denoted by Γ.

Definition 2.2.3. Let $\alpha \in]0, 1[$, and let f be a continuous function on $[a, b]$. In $t_0 \in [a, b]$, the function f is $\alpha-$differentiable. assuming the existence and equality of $d_+^\alpha f(t_0)$ and $d_-^\alpha f(t_0)$.
In this instance, $d^\alpha f(t_0)$ is the $\alpha-$derivative of f in t_0.

Theorem 2.2.4. *For all $t_0 \geq 0$, the trajectories of mixed fractional Brownian motion are nearly certainly $\alpha-$differentiable for any $\alpha \in]0, \dfrac{1}{2} \wedge H[$; furthermore,*

$$\forall t_0 \geq 0, \qquad \mathbb{P}(d^\alpha M_{t_0}^H = 0) = 1.$$

Proof.
The evidence $\sigma = +$ is provided.(The same proof applies to $\sigma = -$).
By employing mixed stationarity and auto-similarity of mixed fractional Brownian motion increases, we have

$$\begin{aligned}\frac{M_t^H - M_{t_0}^H}{(t-t_0)^\alpha} &\overset{\mathfrak{L}}{=} (t-t_0)^{-\alpha} M_1^H\left(a(t-t_0)^{\frac{1}{2}}, b(t-t_0)^H\right)\\ &\overset{\mathfrak{L}}{=} a(t-t_0)^{\frac{1}{2}-\alpha}B_1 + b(t-t_0)^{H-\alpha}B_1^H.\end{aligned}$$

Thus, if $0 < \alpha < \dfrac{1}{2} \wedge H$,

$$\begin{aligned}\mathbb{P}\left(d_+^\alpha M_{t_0}^H = 0\right) &= \mathbb{P}\left(\lim_{t\to t_0} \frac{M_t^H - M_{t_0}^H}{(t-t_0)^\alpha} = 0\right)\\ &= \mathbb{P}\left(\lim_{t\to t_0} a(t-t_0)^{\frac{1}{2}-\alpha}B_1 + b(t-t_0)^{H-\alpha}B_1^H = 0\right) = 1.\end{aligned}$$

■

Theorem 2.2.5. *For all $\alpha \in]\frac{1}{2} \wedge H, 1[$, the trajectories of the mixed fractional Brownian motion are not $\alpha-$differentiables almost surely.*

Proof.
When $d > 0$, the event is defined as

$$A(t) = \left\{\sup_{0\le s\le t}\left|\frac{M_s^H(a,b)}{s^\alpha}\right| > d\right\}.$$

For every sequence $t_n \searrow 0$, we have

$$A(t_{n+1}) \subset A(t_n).$$

So,

$$\mathbb{P}(\lim_{t\to+\infty} A(t_n)) = \lim_{t\to+\infty} \mathbb{P}(A(t_n)),$$

By employing M^H's mixed self-similarity, we have

$$\begin{aligned}\mathbb{P}(A(t_n)) &\ge \mathbb{P}\left(\left|\frac{M_{t_n}^H(a,b)}{t_n^\alpha}\right| > d\right)\\ &= \mathbb{P}\left(|at_n^{\frac{1}{2}-\alpha}B_1 + bt_n^{H-\alpha}B_1^H| > d\right).\end{aligned}$$

i) If $H < \dfrac{1}{2}$, in this case $\alpha > \dfrac{1}{2}$

$$\mathbb{P}(A(t_n)) \ge \mathbb{P}\left(|at_n^{\frac{1}{2}-H}B_1 + bB_1^H| > t_n^{\alpha-H}d\right),$$

$$\lim_{n\to+\infty} \mathbb{P}\left(|at_n^{\frac{1}{2}-H}B_1 + bB_1^H| > t_n^{\alpha-H}d\right) = \mathbb{P}\left(|B_1^H| \ge 0\right) = 1.$$

ii) If $H = \frac{1}{2}$, in this case $\alpha > H$ and $\alpha > \frac{1}{2}$

$$\mathbb{P}(A(t_n)) \geq \mathbb{P}\left(|aB_1 + bB_1^H| > t_n^{\alpha-H} d\right),$$

so,

$$\lim_{n\to+\infty} \mathbb{P}\left(|aB_1 + bB_1^H| > t_n^{\alpha-H} d\right) = \mathbb{P}\left(|aB_1^H| + bB_1^H \geq 0\right) = 1.$$

iii) If $H > \frac{1}{2}$, in this case $\alpha > \frac{1}{2}$

$$\mathbb{P}(A(t_n)) \geq \mathbb{P}\left(|aB_1 + bt_n^{H-\frac{1}{2}} B_1^H > t^{\alpha-\frac{1}{2}}| > d\right),$$

so,

$$\lim_{n\to+\infty} \mathbb{P}\left(|aB_1 + bt_n^{H-\frac{1}{2}} B_1^H > t^{\alpha-\frac{1}{2}}| > d\right) = \mathbb{P}\left(|aB_1| \geq 0\right) = 1.$$

We deduce that for every $t_0 \geq 0$, for all $\alpha \in]\frac{1}{2} \wedge H, 1[$,

$$\mathbb{P}\left(\lim\sup_{t\to t_0} \left|\frac{M_t^H - M_{t_0}}{(t-t_0)^\alpha}\right| = +\infty\right) = 1.$$

■

2.2.4 Semimartingale property

The classical notion of semimartingale has emerged from a sequence of generalizations of Brownian motion, each extending the class of stochastic processes that can serve as integrators in Itô's stochastic integration framework [28]. A stochastic process X_t that is adapted to a filtration $\mathbb{F}$ is called an $\mathbb{F}$-semimartingale if it satisfies the following condition:

$$I_X(\beta(\mathbb{F})) \quad \text{is bounded in } L^0 \tag{2.1}$$

where

$$\beta(\mathbb{F}) = \left\{ \sum_{j=0}^{n-1} f_j \mathbb{1}_{(t_j,t_{j-1})} \; n \in \mathbb{N}, 0 \leq t_0, \ldots, t_n \leq 1, \forall j, f_j \quad \text{is } \mathbb{F}-\text{measurable and} |f_j| \leq 1 \quad a.s. \right\}$$

and

$$I_X(\vartheta) = \sum_{j=0}^{n-1} f_j (X_{t_{j+1}} - X_{t_j}) \qquad pour \qquad \vartheta = \sum_{j=0}^{n-1} f_j \mathbb{1}_{(t_j,t_{j+1}]} \in \beta(\mathbb{F})$$

Cheridito [10] suggested using a less detailed definition of semimartingale in their work. Actually, he defines a less robust semimartingale formula.

Definition 2.2.6. If a stochastic process X_t is $\mathbb{F}$-adapted and satisfies (2.1), then it is weak $\mathbb{F}$-semimartingale.
If X is a weak $\mathcal{F}$-semimartingale, then we say that it is a weak semimartingale. If X is a weak $\bar{\mathbb{F}}$-semimartingale, then we call it a semimartingale.

Example
Confirming that the deterministic process

$$X_t = \begin{cases} 0, & \text{for } t \in [0, \frac{1}{2}] \\ 1, & \text{for } \quad t \in (\frac{1}{2}, 1]. \end{cases}$$

Is a weak semimartingale. But, it is not a semimartingale because it is not continuous on the right a.s.

Lemma 2.2.4. [10] Consider a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Every random process that is a weak $\mathbb{F}$-semimartingale satisfies certain conditions with respect to this filtration. Now, let $(X_t)_{t\geq 0}$ be a process that is right-continuous almost surely. Specifically, X is a weak semimartingale with respect to the completed filtration $\bar{\mathbb{F}}$. In other words, X is a $\bar{\mathbb{F}}$-semimartingale if it is right-continuous almost surely.

It can be deduced from lemma (2.2.4) that for any filtration $\mathbb{F}$, a right-continuous weak $\mathbb{F}$-semimartingale also qualifies as a $\bar{\mathbb{F}}$-semimartingale.

Determining whether the mixed fractional Brownian motion is a $\mathbb{F}$-semimartingale becomes straightforward when $H = \frac{1}{2}$. It becomes evident that the expression

$$\frac{1}{\sqrt{1+\alpha^2}} M^{\frac{1}{2},\alpha}$$

represents a Brownian motion, and specifically, it serves as a $\bar{\mathbb{F}}^{M^{\frac{1}{2},\alpha}}$-semimartingale. Consequently, $M^{\frac{1}{2},\alpha}$ qualifies as a semimartingale under this condition. For other scenarios, we refer to the primary findings outlined in the ensuing theorem.

Theorem 2.2.7. *[10] M^H is not a weak semimartingale if $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$, it is equivalent to $\sqrt{1+\alpha^2} B_t$ if $H = \frac{1}{2}$ and equivalent to Brownian motion if $H \in (\frac{3}{4}, 1]$.*

Different techniques were employed by *Cheridito* [10] to prove this theorem. The demonstration relies on the fact that the quadratic variation of fractional Brownian motion for $H \in (0, \frac{1}{2})$ is not finite. Therefore, for $H < \frac{1}{2}$, the mixed fractional Brownian motion will exhibit an infinite quadratic variation.

In the case of $H > \frac{1}{2}$, the opposite holds. In fact, the process's quadratic variation is the same as the Brownian motion's. In this instance, however, the mixed fractional Brownian motion is equal to the Brownian motion for $H \in (\frac{3}{4}, 1)$ rather than being a weak semimartingale for $H \in (\frac{1}{2}, \frac{3}{4}]$.

Proof.

Using *Stricker* [34]'s theorem, the proof is abridging only for the case $H \in (\frac{1}{2}, \frac{3}{4})$. First, we introduce the *Stricker* theorem. Since the processes are indexed on $[0, 1]$, we operate on a complete probability space. Let $(X_t)_{t\in[0,1]}$ be a stochastic process such that a Gaussian space containing all the possible combinations of the random variables $\mathbb{E}(X_t/\mathcal{F}_s), s, t \in [0, 1]$ exists. Remember that the fact that the collection $I_X(\beta)$ is confined in L^0 has been used to characterize a semimartingale. ■

Theorem 2.2.8. ***Stricker 1983[34].*** *Suppose we have a Gaussian process $(X_t)_{t\in[0,1]}$ with natural filtration. If $I_X(\beta)$ is bounded in L^0, then, it is bounded in L^2.*

Definition 2.2.9. A stochastic process $(X_t)_{t\in[0,1]}$ is a quasi-martingale if

$$X_t \in L^1, \quad \forall t \in [0, 1],$$

and

$$\sup_{\tau} \sum_{j=0}^{n-1} \left\| \mathbb{E}(X_{t_{j+1}} - X_{t_j}/\mathcal{F}^X_{t_j}) \right\|_1 < \infty,$$

where τ is the set of all finite partitions of $[0, 1]$.

Remark 2.2.2. $(X_t)_{t\in[0,1]}$ is a quasi-martingale since $I_X(\beta)$ is bounded in L^2.

Theorem 2.2.10. *It is not a weak semimartingale if $(M^H_t)_{t\in[0,1]}$ is not a quasi-martingale.*

Proof.

Assume M^H is a weak semimartingale. According to Stricker's theorem (Theorem 2.2.8), $I_{M^H}(\beta(\mathcal{F}^{M^H}))$ is bounded in L^2. Consequently, it is also bounded in L^1.For all partitions $0 = t_0 < t_1 \ldots < t_n = 1$

$$\sum_{j=0}^{n-1} sign\left(\mathbb{E}\left[M^H_{t_{j-1}} - M^H_{t_j}/\mathcal{F}_{t_j}\right]\right) \mathbb{1}_{(t_j, t_{j-+1}]} \in \beta(\mathbb{F}^{M^H}),$$

and

$$\left\| I_{M^H}\left(\sum_{j=0}^{n-1} sign\left(\mathbb{E}\left[M^H_{t_{j-1}} - M^H_{t_j}/\mathcal{F}_{t_j}\right]\right)\mathbb{1}_{(t_j,t_{j-+1}]}\right)\right\|_1$$

$$\geq \mathbb{E}\left[I_{M^H}\left(\sum_{j=0}^{n-1} sign\left(\mathbb{E}\left[M^H_{t_{j-1}} - M^H_{t_j}/\mathcal{F}_{t_j}\right]\right)\mathbb{1}_{(t_j,t_{j-+1}]}\right)\right]$$

$$= \sum_{j=0}^{n-1}\left\|\mathbb{E}\left[M^H_{t_{j-1}} - M^H_{t_j}/\mathcal{F}_{t_j}\right]\right\|_1.$$

It follows that M^H is a quasi-martingale. Consequently, if M^H is not a quasi-martingale, it cannot be a weak semimartingale.

It remains to prove that M^H is not a quasi-martingale. We will demonstrate this for $H \in \left(\frac{1}{2}, \frac{3}{4}\right]$ by calculating

$$\sum_{j=0}^{n-1}\left\|\mathbb{E}\left(\Delta^n_{j+1}M^H\middle|\mathcal{F}^{M^H_{\frac{j}{n}}}_{t_j}\right)\right\|_1$$

and showing that this quantity tends to infinity as $n \to \infty$. This proves that $(M_t^{\frac{3}{4}})_{t\in[0,1]}$ is not a quasi-martingale.

Cheridito [10] obtained a remarkable result by demonstrating that the mixed fractional Brownian motion is a semimartingale for $H \in \left]\frac{3}{4}, 1\right)$. Specifically, he showed that the sum of two independent centered Gaussian processes, the first being Brownian motion and the second being fractional Brownian motion with parameter H is a semimartingale if $H \in \left]\frac{3}{4}, 1\right)$. This result leads us to consider examples where the sum of two independent centered Gaussian processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ forms a semimartingale, despite at least one of the processes not being a semimartingale individually. ■

Example:

We examine the Brownian bridge $(\eta_u(t), t \leq u)$ over the interval $[0, u]$, defined as the Brownian motion process $(B_t, t \leq u)$ conditioned on $B_u = 0$. It's worth recalling that $\eta_u(t)$ can be expressed as $\eta_u(t) = B_t - \frac{t}{u}B_u$, ensuring its independence from B_u. Its canonical decomposition is represented by

$$\eta_u(t) = \beta_t - \int_0^t ds\frac{\eta_u(s)}{u-s}, \quad t \leq u,$$

where β_t stands for the Brownian motion within the filtration $\{\mathcal{P}_t^u, t \leq u\}$ of $\eta_u(t)$. Additionally, we present the following proposition.

Proposition 2.2.11. *Let $f \in L^2([0,u])$, then*

1. *The process*

$$\int_0^t f(s)\eta_u(s) = \int_0^t f(s)d\beta_s - \int_0^t dsf(s)\frac{\eta_u(s)}{u-s}$$

is well defined for $t \leq u$, with

$$\int_0^u f(s)d\eta_u(t) = \lim_{t\to u}\int_0^t f(s)d\eta_u(s) \ p.s \ dans \quad L^2.$$

2. *$(\int_0^t f(s)\eta_u(s))$ is a semimartingale w.r.t $\{\mathcal{P}_t^u, t \leq u\}$ if and only if*

$$\int_0^t ds|f(s)|\frac{1}{\sqrt{u-s}} < \infty.$$

Now, let $u \in]0,1]$ and $\alpha \in]\frac{1}{2}, 1]$ and let the function

$$\psi(s) = \frac{1}{\sqrt{u-s}}|\log(u-s)|^{-\alpha}\mathbb{1}_{\frac{u}{2}<s<u},$$

satisfies

$$\int_0^u ds\psi^2(s) < \infty \quad \text{but} \quad \int_0^u ds\psi(s)\frac{1}{\sqrt{u-s}} = \infty.$$

To accomplish our objective, we decompose the Brownian motion $(B_t)_{t\geq 0}$ as follows

$$B_t = \eta_u(t) + \frac{t}{u}B_u, \quad t \leq u.$$

We consider $g \in L^2([0,u])$ such that

$$\int_0^t ds|g(s)|\frac{1}{\sqrt{u-s}} = \infty, \text{ et } g(s) \neq 0, \text{ for all } s.$$

Then, we take

$$X_t = \int_0^t g(s)d\eta_u(s), \quad \text{and} \quad Y_t = \frac{B_u}{u}\int_0^t g(s)ds.$$

Given that X and Y are independent and $X_t + Y_t = \int_0^t g(s)dB_s$, it follows that $X_t + Y_t$ constitutes a martingale.

In a broader context, let $u \in (0,1)$, employing a similar approach. Initially, we decompose the process $(B_t)_{t\geq 0}$ into $\eta_u(t) + \frac{t}{u}B_u$.

Then $\hat{B}_t = B_{t+u} - B_u, t \leq 1-u$ in $\hat{\eta}_{1-u}(t) + \frac{t}{1-u}\hat{B}_{1-u}$.

After, for $f \in L^2([0,1])$, we write

$$\begin{aligned}\int_0^t f(s)dB_s &= \int_0^t f(s)\mathbb{1}_{(s\leq u)}dB_s + \mathbb{1}_{(u<t)}\int_u^t f(s)dB_s \\ &= \int_0^t f(s)\mathbb{1}_{(s\leq u)}d\eta_u(s) + \frac{B_u}{u}\int_0^t f(s)\mathbb{1}_{(s\leq u)}ds \\ &+\mathbb{1}_{(u<t)}\int_u^t f(s)d\hat{\eta}_{1-u}(s-u) + \mathbb{1}_{(u<t)}\frac{B_1 - B_s}{1-u}\int_u^t f(s)ds.\end{aligned}$$

Now, we choose g such that

$$\int_0^t |g(s)|\frac{ds}{\sqrt{u-s}} = \infty, \quad \int_u^1 |g(s)|\frac{ds}{\sqrt{1-s}} = \infty, \quad \text{and for all} \quad s < 1.$$

Then, we have:

$$X_t = \int_0^t g(s)\mathbb{1}_{(s\leq u)}d\eta_u(s) + \mathbb{1}_{(u<t)}\frac{B_1 - B_s}{1-u}\int_u^t g(s)ds,$$

and

$$Y_t = \mathbb{1}_{(u<t)}\int_u^t g(s)d\hat{\eta}_{1-u}(s-u) + \frac{B_u}{u}\int_0^t g(s)\mathbb{1}_{(s\leq u)}ds$$

These are two independent Gaussian processes. Furthermore, their sum $X_t + Y_t = \int_0^t g(s)dB_s$ constitutes a martingale.

We can confirm that neither Y nor X qualifies as a semimartingale by applying proposition (2.2.11).

Chapter 3

Stochastic Differential Equations Driven by Mixed Fractional Brownian Motion

In this chapter, we study stochastic differential equations (SDEs) driven by mixed fractional Brownian motion (MFBM). The importance of considering mixed-driven SDEs stems from the need to model real-world phenomena that exhibit both short-term random fluctuations and long-range memory effects. Classical Brownian motion captures only short-range dependence, while fractional Brownian motion exhibits long-range dependence but lacks certain analytical properties required for classical stochastic calculus.

The mixed fractional Brownian motion, denoted $M_t^H = aB_t + bB_t^H$, naturally combines these two regimes, allowing us to:

1. Capture both short-term volatility (via the Brownian component aB_t)
2. Model persistent correlations and long-memory effects (via bB_t^H)
3. Retain analytical tractability by leveraging classical Itô calculus for the Brownian part
4. Incorporate fractional dynamics through appropriate integration techniques.

We work on a complete, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions.

Notation 1. *Let the following denote standard spaces and norms:*

- $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$*: Space of bounded linear operators from Hilbert space* $\mathcal{H}_1$ *to* $\mathcal{H}_2$
- $L^2_{\mathcal{F}}([0,T];\mathcal{H})$*: Adapted square-integrable processes with values in* $\mathcal{H}$
- $C([0,T];L^2(\Omega;\mathcal{H}))$*: Continuous adapted processes with finite second moment*
- $\|\cdot\|_{\mathcal{H}}$*: Norm in Hilbert space* $\mathcal{H}$

- $\|\cdot\|_{L^2}$: *Norm in* $L^2(\Omega)$
- $W_p^{1,2}$: *Sobolev space with integrable weak derivatives*

3.1 Stochastic Integration with Respect to Mixed Fractional Brownian Motion

To define and analyze SDEs driven by MFBM, we must first establish the appropriate integration frameworks. Since MFBM is a sum of two independent components with different regularity properties, we decompose the integral accordingly.

3.1.1 Integration with Respect to the Brownian Part

3.1.1.1 Itô Integral

Definition 3.1.1 (Itô Integral)**.** Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Let $u = (u_t)_{t\in[0,T]}$ be a predictable process such that

$$\mathbb{E}\left[\int_0^T u_t^2\, dt\right] < \infty.$$

The Itô integral is defined:

$$\int_0^T u_t\, dB_t = \lim_{|\pi|\longrightarrow 0} \sum_{i=0}^{n-1} u_{t_i}(B_{t_{i+1}} - B_{t_i}),$$

where the limit is taken in $L^2(\Omega)$ over partitions $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$.

Theorem 3.1.2 (Properties of the Itô Integral)**.** *Let u_t be an adapted process with* $\mathbb{E}[\int_0^T u_t^2 dt] < \infty$. *Then:*

1. ***Linearity****: For constants λ, μ and integrands u, v,*

$$\int_0^T (\lambda u_t + \mu v_t)\, dB_t = \lambda \int_0^T u_t\, dB_t + \mu \int_0^T v_t\, dB_t.$$

2. ***Zero mean property****:*

$$\mathbb{E}\left[\int_0^T u_t\, dB_t\right] = 0.$$

3. ***Itô isometry****:*

$$\mathbb{E}\left[\left(\int_0^T u_t\, dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T u_t^2\, dt\right].$$

4. ***Martingale property***: *The process* $M_t = \int_0^t u_s\, dB_s$ *is a* $(\mathcal{F}_t)$*martingale.*

5. ***Quadratic variation***: *If* $X_t = \int_0^t u_s\, dB_s$, *then*

$$[X]_t = \int_0^t u_s^2\, ds \quad \textit{(in probability).}$$

6. ***Continuity***: *The process* $t \mapsto \int_0^t u_s\, dB_s$ *admits a continuous modification almost surely.*

7. ***Burkholder-Davis-Gundy inequality***: *For* $p \geq 1$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t u_s\, dB_s\right|^p\right] \leq C_p \mathbb{E}\left[\left(\int_0^T u_s^2\, ds\right)^{p/2}\right],$$

where C_p *is a constant depending only on* p.

3.1.2 Integration with Respect to the Fractional Part

Unlike the Brownian component, the fractional Brownian motion is not a semimartingale (except when $H = 1/2$), so the Itô integral does not apply directly. Instead, we use alternative integration frameworks.

3.1.2.1 Young Integral

The Young integral is useful when both the integrand and integrator have sufficient Hölder regularity.

Definition 3.1.3 (Young Integral [38]). Let $H > 1/2$, and let $f : [0,T] \longrightarrow \mathbb{R}$ be a function that is γ-Hölder continuous and X_t be a process with α-Hölder continuous paths, where $\alpha + \gamma > 1$. The Young integral

$$\int_0^T f(t)\, dX_t$$

is defined as the limit

$$\int_0^T f(t)\, dX_t = \lim_{|\pi|\longrightarrow 0} \sum_{i=0}^{n-1} f(t_i)[X_{t_{i+1}} - X_{t_i}],$$

where the limit is taken over partitions of $[0,T]$.

Theorem 3.1.4 (Properties of the Young Integral [38]). *Suppose $H > 1/2$, f is γ-Hölder continuous, and (X_t) is α-Hölder continuous with $\alpha + \gamma > 1$. Then:*

1. *The Young integral $\int_0^T f(t)\,dX_t$ exists and is well-defined.*
2. *The Young integral satisfies the following estimate:*

$$\left|\int_0^T f(t)\,dX_t\right| \leq C\|f\|_\infty \cdot Var_\alpha(X),$$

where C is a constant and $Var_\alpha(X)$ is the α-variation of X.

3. *If (X_t) is a Gaussian process with Hölder continuous paths of order $> 1/2$, then for deterministic f,*

$$\mathbb{E}\left[\int_0^T f(t)\,dX_t\right] = \int_0^T f(t)\,d\mathbb{E}[X_t].$$

3.1.2.2 Skorokhod Integral (Malliavin Calculus)

For more general cases (including $H < 1/2$), we employ the Skorokhod integral from Malliavin calculus.

Definition 3.1.5 (Skorokhod Integral [26]). Let $(B_t^H)_{t\geq 0}$ be a fractional Brownian motion. An adapted process $u = (u_t)_{t\in[0,T]}$ is Skorokhod-integrable with respect to B_t^H if there exists a square-integrable random variable $\delta(u)$ such that

$$\int_0^T u_t\,dB_t^H = \delta(u) \in L^2(\Omega)$$

satisfies the duality relationship:

$$\mathbb{E}\left[\left(\int_0^T u_t\,dB_t^H\right)\phi\right] = \mathbb{E}\left[\int_0^T D_s u_t \mathbb{1}_{\{s\leq t\}}\,dB_s^H\,d\phi\right]$$

for all square-integrable functionals ϕ with square-integrable Malliavin derivative $D_s\phi$.

Remark 3.1.6. [26] The Skorokhod integral can be decomposed as:

$$\int_0^T u_t\,dB_t^H = \int_0^T u_t\,dB_t + \int_0^T u_t\,d\tilde{B}_t^H,$$

where the first part is classical Itô integration and the second part is purely fractional. For adapted processes, the Skorokhod integral coincides with the Itô integral when $H = 1/2$.

3.1.2.3 Itô's Formula for Fractional Brownian Motion

The classical Itô formula must be modified for fractional Brownian motion due to its non-semimartingale nature.

Theorem 3.1.7 (Itô's Formula for fBm - Young Integral Version)**.** *Let $H > 1/2$, and suppose $f \in C^2(\mathbb{R})$ with bounded second derivative. Let (Y_t) be a process satisfying*

$$dY_t = \mu(t, Y_t)\,dt + \sigma(t, Y_t)\,dB_t^H,$$

where σ is γ-Hölder continuous with $\gamma > 1 - H$. Then

$$f(Y_t) = f(Y_0) + \int_0^t f'(Y_s)\,dY_s + \int_0^t f''(Y_s)(\sigma(s, Y_s))^2\,H(2H-1)s^{2H-2}\,ds + R_t,$$

where R_t is a negligible remainder term.

3.2 Stochastic Differential Equations Driven by Mixed Fractional Brownian Motion

3.2.1 General Formulation

Definition 3.2.1. Stochastic differential equations driven by MFBM generally take the following form:

$$dX_t = \mu(t, X_t)\,dt + \sigma_1(t, X_t)\,dB_t + \sigma_2(t, X_t)\,dB_t^H, \quad X_0 = x_0, \tag{3.1}$$

or equivalently,

$$X_t = x_0 + \int_0^t \mu(s, X_s)\,ds + \int_0^t \sigma_1(s, X_s)\,dB_s + \int_0^t \sigma_2(s, X_s)\,dB_s^H, \tag{3.2}$$

where:

- $X_t \in \mathbb{R}^d$ is the state process
- B_t is a standard Brownian motion
- B_t^H is a fractional Brownian motion with Hurst index $H \in (0, 1)$, independent of B_t
- μ, σ_1, σ_2 are measurable coefficient functions
- The second integral is an Itô integral
- The third integral is a Young or Skorokhod integral (depending on H)

Remark 3.2.2. Special cases include:

- If $\sigma_2 = 0$ we recover classical Brownian SDEs.
- If $\sigma_1 = 0$ we obtain SDEs driven purely by fractional Brownian motion.
- If $H = 1/2$ the fractional part reduces to Brownian motion, giving a classical SDE with correlated components.

3.2.2 SDEs in the Sense of Wiener Integral (Classical Case)

Theorem 3.2.3. *When $H = 1/2$, equation (3.1) becomes*

$$dX_t = \mu(t, X_t)\,dt + (\sigma_1(t, X_t) + \sigma_2(t, X_t))\,dW_t,$$

where $W_t = B_t + B_t^{1/2}$ is a standard Brownian motion. This is a classical Itô SDE.

3.2.3 SDEs in the Sense of Young Integral ($H > 1/2$)

When $H > 1/2$, the sample paths of the fractional Brownian motion B^H are almost surely Hölder continuous of any order strictly less than H. This regularity makes it possible to define the stochastic integral with respect to B^H in the sense of Young [38, 26]. In this framework, the mixed stochastic differential equation is interpreted as (3.2) where the last integral is understood in the Young sense,and the existence and uniqueness of solutions to mixed stochastic differential equations can be established by a fixed-point argument; see [17, 7].

Theorem 3.2.4. *Let $H \in (1/2, 1)$. Assume that the coefficients satisfy the following conditions:*

- *$\mu(t, x)$ is Lipschitz continuous with respect to x, uniformly in t, and satisfies a linear growth condition:*
$$|\mu(t, x)| \le K(1 + |x|).$$
- *$\sigma_1(t, x)$ is Lipschitz continuous with respect to x and satisfies*
$$|\sigma_1(t, x)| \le K(1 + |x|).$$
- *$\sigma_2(t, x)$ is γ-Hölder continuous with respect to x for some $\gamma > 1 - H$, and satisfies*
$$|\sigma_2(t, x)| \le K(1 + |x|).$$

Then there exists a unique strong solution to (3.2) above on $[0, T]$;see [26, 7]. Moreover, the solution is continuous and adapted to the natural filtration.

Proof of Theorem 3.2.4. We use a fixed-point argument. Let

$$\mathcal{B} = \left\{ X \in C([0,T]; L^2(\Omega; \mathbb{R}^d)) : \ \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2 \right] < \infty \right\},$$

equipped with the norm

$$\|X\|_{\mathcal{B}} = \left(\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2 \right] \right)^{1/2}.$$

Define the operator Φ on $\mathcal{B}$ by

$$\Phi(X)_t = x_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma_1(s, X_s)\, dB_s + \int_0^t \sigma_2(s, X_s)\, dB_s^H.$$

We proceed in several steps.

Step 1: Well-definedness of Φ. The first integral is classical and well defined under the linear growth assumption on μ. The second integral is an Itô integral, and it is well defined since $\sigma_1(\cdot, X.)$ is progressively measurable and square-integrable. The third integral is a Young integral, which exists because B^H is almost surely Hölder continuous of order less than H, and the condition $\gamma > 1 - H$ guarantees the Young integrability criterion.

Step 2: A priori estimates. Using the Burkholder–Davis–Gundy inequality and the Itô isometry, we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \sigma_1(s, X_s)\, dB_s \right|^2 \right] \le C\, \mathbb{E}\left[\int_0^T |\sigma_1(s, X_s)|^2\, ds \right].$$

By the linear growth of σ_1, this term is bounded by a constant multiple of

$$1 + \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2 \right].$$

For the Young integral, the Young inequality yields

$$\left| \int_0^t f(s)\, dB_s^H \right| \le C\, \|f\|_\alpha\, \|B^H\|_\beta,$$

for some $\alpha, \beta > 0$ such that $\alpha + \beta > 1$. Since β may be chosen arbitrarily close to H and $\gamma > 1 - H$, the integral is well defined and satisfies a similar estimate.

Step 3: Contraction estimate. Let $X, Y \in \mathcal{B}$. Using the Lipschitz property of μ and σ_1, together with the continuity estimate for the Young integral, we obtain

$$\|\Phi(X) - \Phi(Y)\|_{\mathcal{B}} \le \rho\, \|X - Y\|_{\mathcal{B}},$$

for some $\rho < 1$, provided that the time interval $[0, T]$ is sufficiently small.

Step 4: Existence and uniqueness. By Banach's fixed-point theorem, Φ admits a unique fixed point on a small time interval $[0, T_*]$. This fixed point is the unique solution of the mixed stochastic differential equation on $[0, T_*]$.

Step 5: Extension to $[0, T]$. Using a standard continuation argument and the linear growth of the coefficients, the local solution can be extended step by step to the whole interval $[0, T]$.

Therefore, there exists a unique strong solution to the equation on $[0, T]$ [26, 7].

□

3.2.4 SDEs in the Sense of Skorokhod Integral $(H < \frac{1}{2})$

When $H < \frac{1}{2}$, the sample paths of the fractional part B^H are too rough to be treated by the Young integral framework. In this case, the stochastic integral with respect to B^H is interpreted in the sense of Skorokhod, which is defined through Malliavin calculus. This approach is suitable for anticipating integrands and allows one to study mixed stochastic differential equations driven by both standard Brownian motion and fractional Brownian motion.

Theorem 3.2.5. *Let $H \in (0, \frac{1}{2})$. Assume that the coefficients μ, σ_1, and σ_2 satisfy suitable regularity, Lipschitz, and linear growth conditions. Then the corresponding mixed stochastic differential equation (3.2) admits a weak solution in the Skorokhod integral sense [26, 7].*

Proof of Theorem 3.2.5. Let $(B_t^{H,n})_{n \geq 1}$ be a sequence of smooth processes approximating B_t^H; see [26, 7].

$$B_t^{H,n} \longrightarrow B_t^H \quad \text{in an appropriate sense as } n \longrightarrow \infty.$$

For each n, consider the approximating equation

$$\begin{aligned} X_t^n = x_0 &+ \int_0^t \mu(s, X_s^n)\, ds \\ &+ \int_0^t \sigma_1(s, X_s^n)\, dB_s \\ &+ \int_0^t \sigma_2(s, X_s^n)\, dB_s^{H,n}. \end{aligned}$$

Since the coefficients satisfy the assumed regularity and growth conditions, this equation admits a unique solution X^n.

Moreover, there exists a constant $C > 0$, independent of n, such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t^n|^2\right] \le C.$$

This uniform estimate implies that the family of laws $\{\mathcal{L}(X^n)\}_{n\ge1}$ is tight in

$$C([0,T];\mathbb{R}^d).$$

Hence, by Prohorov's theorem [8], there exists a subsequence, still denoted by (X^n), and a process X such that

$$X^n \Rightarrow X \quad \text{in } C([0,T];\mathbb{R}^d).$$

Passing to the limit in the approximating equations, we obtain that X satisfies

$$\begin{aligned} X_t = x_0 &+ \int_0^t \mu(s, X_s)\, ds \\ &+ \int_0^t \sigma_1(s, X_s)\, dB_s \\ &+ \int_0^t \sigma_2(s, X_s)\, \delta B_s^H, \end{aligned}$$

where δB_s^H denotes the Skorokhod integral.

Therefore, X is a weak solution of the mixed stochastic differential equation in the Skorokhod sense [26, 7]. Under additional assumptions, uniqueness in law may also be established. □

3.2.5 Stochastic Differential Equations Driven by Hölder Paths

Definition 3.2.6. An SDE driven by Hölder paths is an equation of the form (3.1) where the driving processes are replaced by general Hölder continuous paths X_t with Hölder exponent $\alpha > 0$ [38, 5].

Theorem 3.2.7 (SDE Driven by Hölder Paths)**.** *Let (X_t) be a continuous path with α-Hölder regularity for some $\alpha > 0$. Consider the SDE*

$$dY_t = f(t, Y_t)\, dX_t, \quad Y_0 = y_0,$$

where f is β-Hölder continuous in x with $\alpha + \beta > 1$. Then this SDE admits a unique solution in the Young integral sense, which is also $(\alpha \wedge \beta)$-Hölder continuous [38, 5].

3.3 Existence and Uniqueness of Solutions

3.3.1 Existence of Strong Solutions

Definition 3.3.1. A strong solution of the SDE (3.1) is an adapted process $(X_t)_{t\in[0,T]}$ defined on the given probability space such that X_t is continuous, and (3.2) holds almost surely for all $t \in [0,T]$.

Theorem 3.3.2 (Existence of Strong Solution)**.** *Let $H \in (1/2, 1)$. Assume:*

1. *$\mu(t,x)$ is Lipschitz in x uniformly in t: $|\mu(t,x) - \mu(t,y)| \leq L|x-y|$*
2. *$\sigma_1(t,x)$ is Lipschitz: $|\sigma_1(t,x) - \sigma_1(t,y)| \leq L|x-y|$*
3. *$\sigma_2(t,x)$ is γ-Hölder for some $\gamma > 1-H$: $|\sigma_2(t,x) - \sigma_2(t,y)| \leq L|x-y|^\gamma$*
4. *All coefficients have linear growth: $|\mu(t,x)| + |\sigma_i(t,x)| \leq L(1+|x|)$*
5. *$x_0 \in L^2(\Omega;\mathbb{R}^d)$ is $\mathcal{F}_0$-measurable*

Then there exists a unique strong solution $(X_t)_{t\in[0,T]}$ that is continuous, adapted, and satisfies

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right] < \infty.$$

Proof of the theorem (3.3.2)*.* Let

$$\Phi(X)_t = x_0 + \int_0^t \mu(s,X_s)\,ds + \int_0^t \sigma_1(s,X_s)\,dB_s + \int_0^t \sigma_2(s,X_s)\,dB_s^H, \qquad t \in [0,T].$$

We consider the Banach space

$$\mathcal{B} = \left\{X : \Omega \times [0,T] \longrightarrow \mathbb{R}^d \text{ adapted and continuous} : \|X\|_{\mathcal{B}} < \infty\right\},$$

where

$$\|X\|_{\mathcal{B}} = \left(\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right]\right)^{1/2}.$$

Step 1: Well-definedness of the map Φ. For $X \in \mathcal{B}$, the drift term is well defined by the linear growth of μ. The Brownian term is well defined since $\sigma_1(\cdot, X_\cdot)$ is progressively measurable and square-integrable, and by the Itô isometry and BDG inequality we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \sigma_1(s,X_s)\,dB_s\right|^2\right] \leq C\,\mathbb{E}\left[\int_0^T |\sigma_1(s,X_s)|^2\,ds\right].$$

For the fractional Brownian term, since $H > 1/2$, the paths of B^H are almost surely Hölder continuous of any order strictly smaller than H, so the Young integral is well defined provided the integrand has Hölder exponent β such that $\beta + H > 1$; this is ensured by the assumption on σ_2 together with the regularity of X [38, 7, 27].

Step 2: A priori estimate. Using the linear growth assumptions, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |\Phi(X)_t|^2\right] \le C\left(1 + \mathbb{E}\left[\sup_{0\le t\le T} |X_t|^2\right]\right).$$

Indeed, the drift term is bounded by

$$\sup_{0\le t\le T}\left|\int_0^t \mu(s, X_s)\,ds\right| \le \int_0^T |\mu(s, X_s)|\,ds \le C\int_0^T (1 + |X_s|)\,ds,$$

and the Brownian term is controlled by BDG. For the Young term, the classical estimate gives

$$\left|\int_s^t f(r)\,dB^H_r - f(s)(B^H_t - B^H_s)\right| \le C\,\|f\|_\beta\,\|B^H\|_\alpha\,|t-s|^{\alpha+\beta},$$

whenever $\alpha + \beta > 1$. This yields the required bound for the mapping Φ [38, 5].

Step 3: Contraction estimate. Let $X, Y \in \mathcal{B}$. Then

$$\begin{aligned}\Phi(X)_t - \Phi(Y)_t &= \int_0^t \big(\mu(s, X_s) - \mu(s, Y_s)\big)\,ds \\ &+ \int_0^t \big(\sigma_1(s, X_s) - \sigma_1(s, Y_s)\big)\,dB_s \\ &+ \int_0^t \big(\sigma_2(s, X_s) - \sigma_2(s, Y_s)\big)\,dB^H_s.\end{aligned}$$

By the Lipschitz property of μ,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t (\mu(s, X_s) - \mu(s, Y_s))\,ds\right|^2\right] \le C\,T\,\|X - Y\|^2_{\mathcal{B}}.$$

By BDG and the Lipschitz continuity of σ_1,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t (\sigma_1(s, X_s) - \sigma_1(s, Y_s))\,dB_s\right|^2\right] \le C\,T\,\|X - Y\|^2_{\mathcal{B}}.$$

For the Young term, the continuity of the Young integral with respect to the integrand yields

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t (\sigma_2(s, X_s) - \sigma_2(s, Y_s))\,dB^H_s\right|^2\right] \le C\,T^\theta\|X - Y\|^2_{\mathcal{B}},$$

for some $\theta > 0$ depending on H and the Hölder regularity of σ_2 [38, 7, 27].

Therefore,

$$\|\Phi(X) - \Phi(Y)\|_{\mathcal{B}} \leq \rho \|X - Y\|_{\mathcal{B}},$$

where $\rho = C(T + T^{1/2} + T^{\theta})$, and $\rho < 1$ for $T > 0$ sufficiently small.

Step 4: Existence and uniqueness. By Banach's fixed-point theorem [15], Φ admits a unique fixed point on a sufficiently small interval $[0, T_*]$. This fixed point is the unique strong solution on $[0, T_*]$.

Step 5: Extension to the whole interval. Using the linear growth of the coefficients, the solution can be extended step by step from $[0, T_*]$ to $[0, T]$. The moment estimate prevents explosion, so the continuation procedure yields a unique adapted continuous solution on the whole interval.

Hence, there exists a unique strong solution $(X_t)_{t\in[0,T]}$ satisfying

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right] < \infty.$$

□

3.3.2 Uniqueness of Strong Solutions

Theorem 3.3.3 (Uniqueness of Strong Solution)**.** *Under the assumptions of Theorem 3.3.2, the strong solution of the mixed stochastic differential equation is unique. More precisely, if $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ are two strong solutions defined on the same probability space with the same Brownian motions and the same initial condition, then*

$$\mathbb{P}\left(\sup_{t\in[0,T]} |X_t - Y_t| = 0\right) = 1.$$

Proof of the theorem (3.3.3)*.* Set

$$Z_t = X_t - Y_t, \qquad t \in [0, T].$$

Then

$$\begin{aligned} Z_t = &\int_0^t \big(\mu(s, X_s) - \mu(s, Y_s)\big)\, ds \\ &+ \int_0^t \big(\sigma_1(s, X_s) - \sigma_1(s, Y_s)\big)\, dB_s \\ &+ \int_0^t \big(\sigma_2(s, X_s) - \sigma_2(s, Y_s)\big)\, dB_s^H. \end{aligned}$$

Taking the supremum over $[0,t]$ and using the elementary inequality

$$|Z_t|^2 \le 3\left|\int_0^t \big(\mu(s,X_s)-\mu(s,Y_s)\big)\,ds\right|^2 + 3\left|\int_0^t \big(\sigma_1(s,X_s)-\sigma_1(s,Y_s)\big)\,dB_s\right|^2 + 3\left|\int_0^t \big(\sigma_2(s,X_s)\sigma_2(s,Y_s)\big)\,dB_s^H\right|^2,$$

we obtain, after taking expectations,

$$\mathbb{E}\left[\sup_{0\le r\le t}|Z_r|^2\right] \le I_1(t)+I_2(t)+I_3(t),$$

where

$$I_1(t) = 3\,\mathbb{E}\left[\sup_{0\le r\le t}\left|\int_0^r \big(\mu(s,X_s)-\mu(s,Y_s)\big)\,ds\right|^2\right],$$

$$I_2(t) = 3\,\mathbb{E}\left[\sup_{0\le r\le t}\left|\int_0^r \big(\sigma_1(s,X_s)-\sigma_1(s,Y_s)\big)\,dB_s\right|^2\right],$$

$$I_3(t) = 3\,\mathbb{E}\left[\sup_{0\le r\le t}\left|\int_0^r \big(\sigma_2(s,X_s)-\sigma_2(s,Y_s)\big)\,dB_s^H\right|^2\right].$$

By the Lipschitz property of μ, we have

$$I_1(t) \le C\int_0^t \mathbb{E}\left[\sup_{0\le u\le s}|Z_u|^2\right]ds.$$

For the Itô integral, the Burkholder–Davis–Gundy inequality yields

$$I_2(t) \le C\,\mathbb{E}\left[\int_0^t |\sigma_1(s,X_s)-\sigma_1(s,Y_s)|^2\,ds\right] \le C\int_0^t \mathbb{E}\left[\sup_{0\le u\le s}|Z_u|^2\right]ds.$$

For the fractional Brownian term, the Young estimate gives

$$I_3(t) \le C\int_0^t \mathbb{E}\left[\sup_{0\le u\le s}|Z_u|^2\right]ds,$$

since σ_2 is assumed to satisfy the appropriate Hölder continuity condition and $H > \frac{1}{2}$ ensures the Young integral is well defined.

Therefore, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{0\le r\le t}|Z_r|^2\right] \le C\int_0^t \mathbb{E}\left[\sup_{0\le u\le s}|Z_u|^2\right]ds.$$

By Gronwall's lemma,

$$\mathbb{E}\left[\sup_{0\le r\le t}|Z_r|^2\right]=0 \qquad \text{for all } t\in[0,T].$$

Hence,

$$\mathbb{P}\left(\sup_{t\in[0,T]}|X_t-Y_t|=0\right)=1.$$

This proves uniqueness of the strong solution. □

3.3.3 Weak Solutions

Definition 3.3.4. A *weak solution* of (3.1) is a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ together with processes $(\tilde{B}_t)_{t\in[0,T]}$, $(\tilde{B}^H_t)_{t\in[0,T]}$, $(\tilde{X}_t)_{t\in[0,T]}$ defined on this space such that

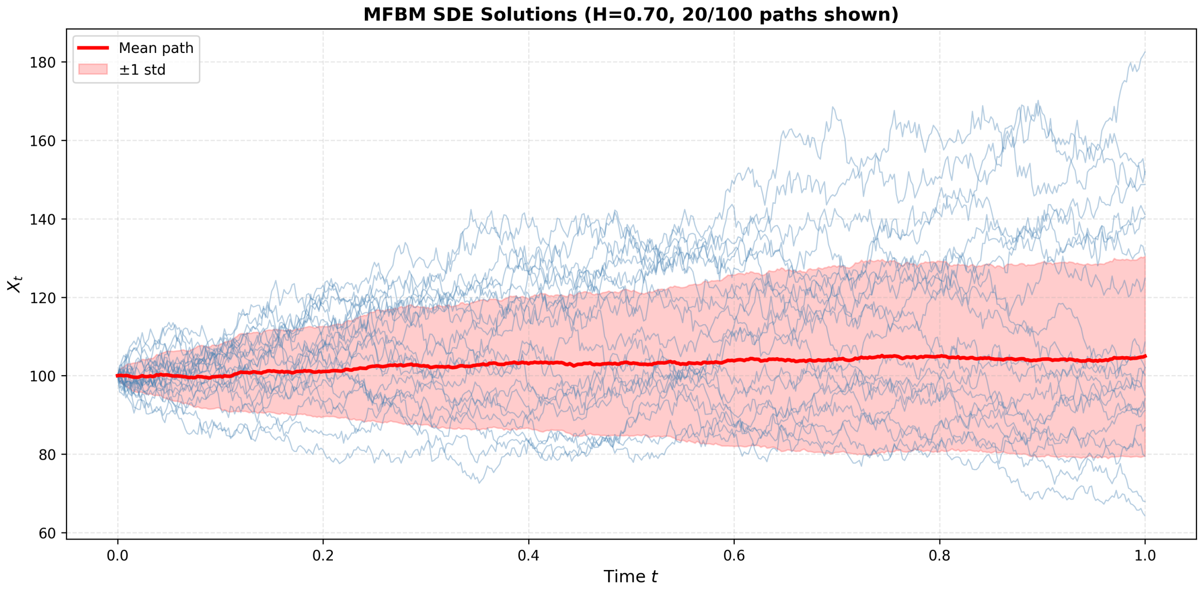
- $\tilde{B}_t$ is a standard Brownian motion,
- $\tilde{B}^H_t$ is a fractional Brownian motion with Hurst parameter H, independent of $\tilde{B}_t$,
- $\tilde{X}_t$ satisfies (3.2) almost surely under $\tilde{\mathbb{P}}$.

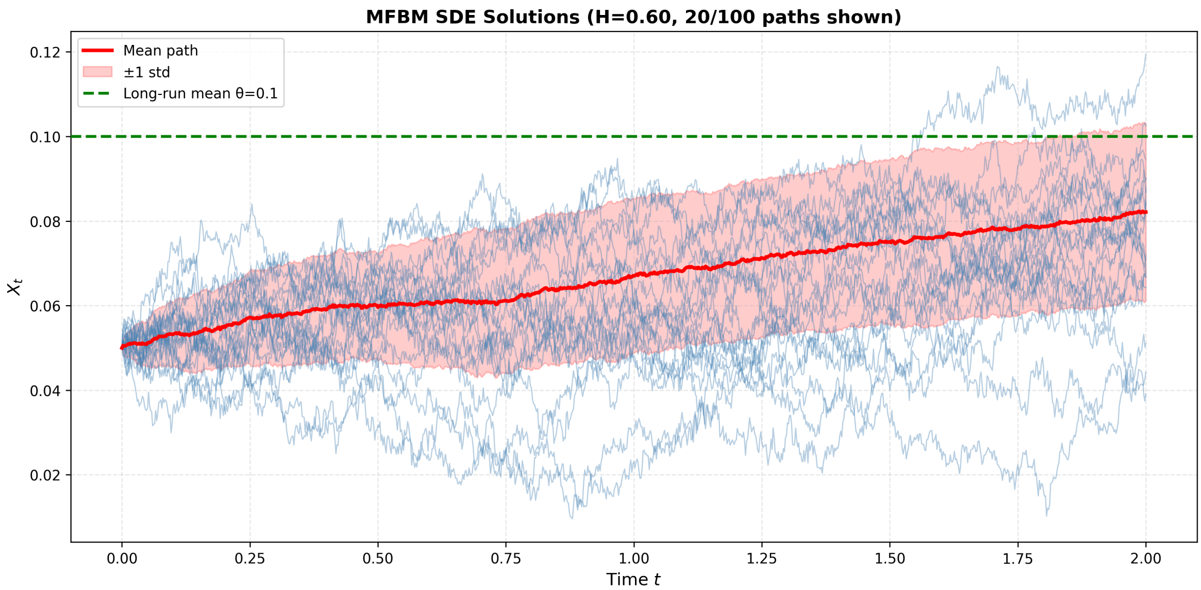
Theorem 3.3.5 (Existence of Weak Solution)**.** *Assume that:*

1. *the coefficients* $\mu(t,x)$, $\sigma_2(t,x)$ *are bounded and measurable;*
2. *There exists a constant* $\sigma_0>0$ *such that* $\sigma_1(t,x)\ge\sigma_0$ *for all* $(t,x)\in[0,T]\times\mathbb{R}$ *non-degeneracy.*

Then the mixed fractional SDE (3.1) *admits a weak solution on* $[0,T]$.

3.4 Numerical solutions





The first figure displays sample paths for a geometric-type mixed fractional SDE with Hurst parameter (H = 0.7).

Sample paths of a geometric-type mixed fractional SDE with Hurst parameter (H = 0.7) are presented in the first figure.

The pink band reflects one empirical standard deviation around the mean, the red curve is the empirical mean path, and the thin blue curves show individual simulated paths.

Our model's bounded drift and non-degenerate diffusion are in line with our findings that the paths' dispersion expands as time passes while the mean path remains mostly constant.

Sample paths of a mean-reverting mixed fractional SDE with Hurst parameter (H = 0.6) are shown in the second figure. Furthermore, the pink band shows one standard deviation around the empirical mean, the red curve symbolizes the empirical mean, and the blue curves show individual paths. The long-run mean level ($\theta = 0.1$) is represented by the horizontal green dashed line.

We can see that the process gradually converges to the long-run mean, while fractional noise induces long-range dependence and results in smoother trajectories as opposed to the purely Brownian case.

Chapter 4

Controllability of Stochastic Differential Equations Driven by Mixed Fractional Brownian Motion

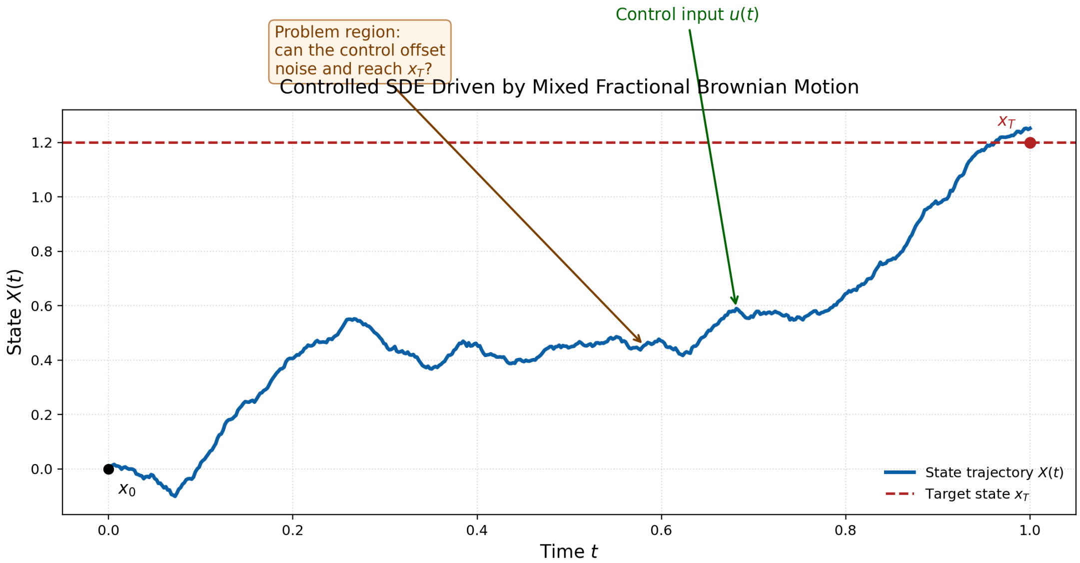


Figure 4.1: Schematic trajectory of a controlled stochastic system driven by mixed fractional Brownian motion, illustrating the controllability problem.

Figure 4.1 provides a qualitative illustration of the problem addressed in this chapter. The state trajectory $X(t)$ is driven by a mixed fractional Brownian noise, while the control input $u(t)$ attempts to steer the system from the initial state x_0 towards the target state x_T.

In the previous chapters, we introduced the stochastic framework associated with mixed fractional Brownian motion and established the existence and uniqueness of (strong) solutions to the corresponding stochastic differential equations. In the present chapter, we address the controllability problem for the same class of systems. The main objective is to determine whether an admissible control can steer the system toward a prescribed terminal state under suitable hypotheses on the linear operator, the control operator, and the nonlinear coefficients [10, 30].

Stochastic systems have a fundamentally different controllability behavior compared to deterministic systems because:

1. The presence of random noise introduces inherent uncertainty.
2. We cannot prescribe the exact trajectory but only the probability distribution.
3. Different notions of controllability apply (null controllability, approximate controllability, exact controllability).
4. The memory effects in fractional noise complicate the analysis significantly.

In this framework, we consider controlled stochastic differential equations of the form

$$dX_t = \big(AX_t + \mu(X_t) + Bu_t\big)\,dt + \sigma_1(X_t)\,dB_t + \sigma_2(X_t)\,dB_t^H, \qquad X_0 = x_0, \tag{4.1}$$

where

- $X_t \in \mathbb{R}^n$ is the state,
- $u_t \in \mathbb{R}^m$ is the control input,
- B_t is a standard Brownian motion,
- B_t^H is a fractional Brownian motion with Hurst parameter $H \in (0,1)$,
- A is a linear operator and μ is a nonlinear drift,
- B is the control operator,
- σ_1, σ_2 are diffusion coefficients.

The central question of this chapter is: under what conditions is the controlled mixed fractional Brownian motion system (4.1) controllable on a finite time interval?

4.1 Preliminaries and Functional Setting

In this section, we introduce the notation, assumptions, and analytical framework used throughout the controllability analysis. The goal is to place the controlled mixed fractional Brownian motion system in a setting where semigroup methods, stochastic integration, and fixed-point arguments can be applied in a rigorous way, building on the existence and uniqueness results of Section 3.3 and the framework of [30].

4.1.1 Notations and assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be the complete filtered probability space introduced in Section 3.3, and let $\mathcal{H}$ denote the separable Hilbert space used there as the state space of the solution. We denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$. For a separable Hilbert space $\mathcal{H}$, we write

$$L^2_{\mathcal{F}}([0,T];\mathcal{H})$$

for the space of $\mathcal{H}$-valued, $(\mathcal{F}_t)$-adapted, square-integrable processes on $[0,T]$, and

$$C([0,T];L^2(\Omega;\mathcal{H}))$$

for the space of continuous processes with finite second moment. The norm in $\mathcal{H}$ is denoted by $\|\cdot\|_{\mathcal{H}}$, while $\|\cdot\|_{L^2}$ denotes the usual norm in $L^2(\Omega)$ [30].

Assumption 4.1.1 (Linear operator)**.** The linear operator $A : D(A) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ generates a C_0-semigroup $(e^{At})_{t\geq 0}$. Moreover, there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|e^{At}\| \leq Me^{\omega t}, \qquad t \geq 0,$$

and the semigroup is exponentially stable if $\omega < 0$ [30].

Assumption 4.1.2 (Control operator)**.** The control operator $B : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is bounded and linear. We assume that the control acts effectively on the system so that the associated controllability operator is well defined on $[0,T]$; in particular, the pair (A,B) will later be required to satisfy a suitable controllability condition on the linear reference system, in the spirit of [30].

Assumption 4.1.3 (Nonlinear drift)**.** The drift $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is Lipschitz continuous and satisfies a linear growth condition, that is, there exist constants $L_\mu > 0$ and $C_\mu > 0$ such that

$$\|\mu(x) - \mu(y)\| \leq L_\mu \|x - y\|, \qquad \|\mu(x)\| \leq C_\mu(1 + \|x\|),$$

for all $x, y \in \mathbb{R}^n$. These assumptions coincide with those used in the existence and uniqueness analysis of Section 3.3 [30].

Assumption 4.1.4 (Diffusion coefficients)**.** The diffusion functions $\sigma_1, \sigma_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$ are measurable and satisfy suitable Lipschitz and growth conditions ensuring that the stochastic integrals with respect to the Brownian and fractional Brownian parts are well defined. In particular, we assume that σ_1 is non-degenerate and that σ_2 has sufficient Hölder regularity, as in the mixed fractional framework discussed in Chapter 2 and in [30].

4.1.2 Mixed fractional Brownian motion and controlled system

Recall from Chapter 2 that the mixed fractional Brownian motion is given by

$$M_t^H = aB_t + bB_t^H, \qquad t \in [0, T],$$

where B_t is a standard Brownian motion, B_t^H is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$, and $a, b \in \mathbb{R}$. The Brownian part captures short-range random fluctuations, while the fractional part introduces memory and persistence effects [10].

Proposition 4.1.5 (*MFBM Properties for Control* 2.2.1)**.** *The mixed fractional Brownian motion has the following properties relevant to controllability:*

1. ***Covariance Structure****: Cov*$(M_t^H, M_s^H) = a^2 \min(t, s) + \frac{b^2}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$
2. ***Memory Effects****: For* $H > 1/2$*, long-range dependence means that control actions have persistent effects*
3. ***Regularity****: Hölder continuity of order* $\gamma < \min(1/2, H)$ *ensures the system remains well-defined*
4. ***Independence****: The Brownian component* B_t *and fractional component* B_t^H *are independent*

Within this framework, we study the controlled stochastic differential equation (4.1) where $X_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the control input, and B_t, B_t^H are independent Brownian and fractional Brownian motions, respectively.

Definition 4.1.6 (Admissible control)**.** A control $u : [0, T] \times \Omega \longrightarrow \mathbb{R}^m$ is called admissible if it is progressively measurable and satisfies

$$\mathbb{E} \int_0^T \|u_t\|^2 \, dt < \infty.$$

The set of all admissible controls is denoted by $\mathcal{U}_{ad}[0, T]$.

Remark 4.1.7. This notion of admissibility is consistent with the one used in the previous chapters and in [30], where square-integrability and adaptedness ensure that the corresponding stochastic integrals in (4.1) are well defined.

Under Assumptions 4.1.1, 4.1.4, the existence and uniqueness of a (strong) solution to (4.1) follow from the results of Section 3.3. In the next section, we reformulate this solution in the semigroup setting and introduce the mild solution, which will be the natural object for the controllability analysis.

4.2 Formulation of the Controlled Stochastic Differential Equation

In this section, we write the controlled mixed fractional Brownian motion system in a form that is suitable for the controllability analysis. The starting point is the stochastic differential equation (4.1) where the assumptions of Section 4.1 are in force.

4.2.1 General form of the controlled system

Formally, the dynamics (4.1) can be decomposed into three parts:

- the deterministic evolution generated by the semigroup $(e^{At})_{t\geq 0}$,
- the action of the control input u,
- and the random perturbations driven by the Brownian and fractional Brownian components.

Under Assumptions 4.1.1, 4.1.4, the existence and uniqueness of a (strong) solution to (4.1) follow from the results of Section 3.3. In the present chapter, we reinterpret this solution in the semigroup framework and work with its mild formulation, which will be the natural object for controllability.

4.3 Mild Solution and Control System

4.3.1 Definition of mild solution

Assume that A generates a C_0-semigroup $(e^{At})_{t\geq 0}$ on $\mathcal{H}$. Then the variation-of-constants formula leads to the following notion of mild solution for the controlled system (4.1).

Definition 4.3.1 (Mild solution)**.** Let $u \in \mathcal{U}_{ad}[0,T]$. A process $X = \{X_t\}_{t\in[0,T]}$ is called a mild solution of (4.1) if it is adapted, continuous, and satisfies

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}\big(\mu(X_s) + Bu_s\big)\,ds$$

$$+ \int_0^t e^{A(t-s)}\sigma_1(X_s)\,dB_s + \int_0^t e^{A(t-s)}\sigma_2(X_s)\,dB_s^H, \qquad t \in [0,T], \tag{4.2}$$

almost surely.

Remark 4.3.2. The mild formulation (4.2) is the natural setting for controllability. The semigroup term $e^{At}x_0$ describes the deterministic evolution of the uncontrolled system, the integral involving Bu_s represents the action of the control input, and the stochastic integrals describe the random perturbations coming from the Brownian and fractional Brownian components. In the sequel, controllability will be studied through this mild formulation and through the associated control-to-state map, in the spirit of [30].

4.3.2 Equivalent integral form

For later use, it is convenient to rewrite the mild solution (4.2) in an explicit integral form that separates the contributions of the control, the drift, and the noise:

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_s\,ds + \int_0^t e^{A(t-s)}\mu(X_s)\,ds$$

$$+ \int_0^t e^{A(t-s)}\sigma_1(X_s)\,dB_s + \int_0^t e^{A(t-s)}\sigma_2(X_s)\,dB_s^H, \qquad t \in [0,T]. \tag{4.3}$$

This representation makes explicit how the control influences the terminal state: the term

$$\int_0^t e^{A(t-s)}Bu_s\,ds$$

is linear in u and will give rise to the control-to-state operator used in the characterization of the reachable set.

4.3.3 Control-to-state operator

For a fixed time horizon $T > 0$ and an admissible control $u \in \mathcal{U}_{ad}[0,T]$, we define the control-to-state operator by

$$\mathcal{T}u := \int_0^T e^{A(T-s)}Bu_s\,ds. \tag{4.4}$$

This operator represents the direct contribution of the control to the terminal state of the mild solution. The stochastic terms in (4.3) act as perturbations around this controlled deterministic evolution, and the structure of $\mathcal{T}$ will play a central role in the controllability analysis developed in the next sections.

4.4 Controllability Concepts

In this section, we introduce the notions of controllability that will be used in the sequel. Since our main focus is on the ability of an admissible control to steer the system toward a prescribed terminal state, we concentrate on the classical controllability framework for the controlled mixed fractional system and mention approximate controllability only as a weaker motivating concept [30].

4.4.1 Controllability of the controlled system

We recall that, for a given admissible control $u \in \mathcal{U}_{ad}[0,T]$ and initial state $x_0 \in \mathbb{R}^n$, the corresponding mild solution of the controlled system (4.1) is denoted by $X^{u,x_0} = \{X_t^{u,x_0}\}_{t\in[0,T]}$.

Definition 4.4.1 (Controllability)**.** The controlled system (4.1) is said to be (exactly) controllable on $[0, T]$ if for every initial state $x_0 \in \mathbb{R}^n$ and every target state $x_T \in \mathbb{R}^n$, there exists an admissible control $u \in \mathcal{U}_{ad}[0, T]$ such that the corresponding mild solution satisfies

$$X_T^{u,x_0} = x_T \quad \text{almost surely.}$$

Remark 4.4.2. Because the system is stochastic, controllability is understood with respect to the terminal random state generated by the admissible control. In practice, one often studies first the controllability of an associated linear reference system and then extends the result to the nonlinear case by a fixed-point argument [30].

Definition 4.4.3 (Reachable set)**.** For a fixed initial state $x_0 \in \mathbb{R}^n$, the reachable set at time T is defined by

$$R_T(x_0) := \left\{X_T^{u,x_0} : u \in \mathcal{U}_{ad}[0, T]\right\}.$$

The system is controllable on $[0, T]$ if and only if $R_T(x_0) = \mathbb{R}^n$ for every $x_0 \in \mathbb{R}^n$.

4.4.2 Approximate and null controllability

Although the main result of this chapter concerns exact controllability in the sense of Definition 4.4.1, it is useful to recall weaker notions that are often considered in the literature.

Definition 4.4.4 (Approximate controllability)**.** The controlled system (4.1) is said to be approximately controllable on $[0, T]$ if for every $x_0, x_T \in \mathbb{R}^n$ and every $\varepsilon > 0$, there exists $u \in \mathcal{U}_{ad}[0, T]$ such that

$$\mathbb{E}\left\|X_T^{u,x_0} - x_T\right\|^2 < \varepsilon.$$

Equivalently, the reachable set $R_T(x_0)$ is dense in $\mathbb{R}^n$.

Definition 4.4.5 (Null controllability)**.** The controlled system (4.1) is said to be null controllable on $[0, T]$ if for every initial state $x_0 \in \mathbb{R}^n$, there exists an admissible control $u \in \mathcal{U}_{ad}[0, T]$ such that

$$\mathbb{E}\left\|X_T^{u,x_0}\right\|^2 = 0,$$

that is, the origin is reachable (in the sense of the terminal random state) from every initial state.

Remark 4.4.6. For linear systems, these notions are related by classical implications: exact controllability implies null controllability, and null controllability together with linearity often implies exact controllability. Approximate controllability is a weaker property, corresponding to the density of the reachable set, and is sometimes more natural in stochastic settings where exact pathwise control may be too strong [30].

4.4.3 Control-to-state operator and linear reference system

For the controllability analysis, it is convenient to isolate the linear reference dynamics associated with (4.1), namely

$$dX_t = (AX_t + Bu_t)\,dt + \sigma_1(X_t)\,dB_t + \sigma_2(X_t)\,dB_t^H, \qquad X_0 = x_0. \tag{4.5}$$

In this setting, the control-to-state operator $\mathcal{T}$ defined in (4.4) maps an admissible control $u \in \mathcal{U}_{ad}[0,T]$ to the deterministic contribution

$$\mathcal{T}u = \int_0^T e^{A(T-s)} Bu_s\,ds.$$

The range of $\mathcal{T}$ characterizes the controllable directions of the linear reference system, and the associated controllability Gramian will be used in the next section to formulate a sufficient condition for controllability of the full nonlinear mixed fractional system, following the approach of [30].

4.5 Main Controllability Result

In this section, we establish the main controllability theorem for the controlled mixed fractional Brownian motion system (4.1). The strategy follows the approach of [30]: first we characterize the controllability of the associated linear reference system via the controllability Gramian, and then we extend the result to the nonlinear system by a fixed-point argument.

4.5.1 Hypotheses for controllability

We recall the control-to-state operator $\mathcal{T} : \mathcal{U}_{ad}[0,T] \longrightarrow \mathbb{R}^n$ defined by

$$\mathcal{T}u = \int_0^T e^{A(T-s)} Bu_s\,ds,$$

and introduce the associated controllability Gramian

$$W_T = \int_0^T e^{As} BB^* e^{A^* s}\,ds. \tag{4.6}$$

We impose the following hypotheses, which are in line with those used in [30].

Assumption 4.5.1 (Invertibility of the controllability Gramian)**.** The controllability Gramian W_T associated with the linear reference system is invertible on $\mathbb{R}^n$. In particular, there exists a constant $C_W > 0$ such that $\|W_T^{-1}\| \le C_W$.

Assumption 4.5.2 (Compatibility of the nonlinear terms)**.** The nonlinear drift μ and the diffusion coefficients σ_1, σ_2 satisfy the Lipschitz and growth conditions stated in Assumptions4.1.3 and4.1.4, and the associated constants are sufficiently small (or the time horizon T is sufficiently short) so that the fixed-point operator defined in the proof below is a contraction on the chosen state space.

Assumption 4.5.3 (Regularity of the control map)**.** For every admissible control $u \in \mathcal{U}_{ad}[0,T]$, the control-to-state operator $\mathcal{T}$ defined by (4.4) is bounded from $\mathcal{U}_{ad}[0,T]$ into $\mathbb{R}^n$.

4.5.2 Main theorem

We can now state the main result of this chapter.

Theorem 4.5.4 (Controllability of the controlled MFBM system)**.** *Assume that Assumptions 4.1.1, 4.1.4 and 4.5.1, 4.5.3 hold. Then the controlled stochastic system* (4.1) *is controllable on* $[0,T]$ *in the sense of Definition 4.4.1. That is, for every initial state* $x_0 \in \mathbb{R}^n$ *and every target state* $x_T \in \mathbb{R}^n$*, there exists an admissible control* $u \in \mathcal{U}_{ad}[0,T]$ *such that the corresponding mild solution satisfies*

$$X_T^{u,x_0} = x_T \quad \textit{almost surely.}$$

4.5.3 Outline of the proof

We briefly describe the main steps of the proof, following the strategy of [30].

Proof outline. Let $x_0 \in \mathbb{R}^n$ and $x_T \in \mathbb{R}^n$ be given. By the mild formulation (4.2), the terminal value X_T^{u,x_0} corresponding to an admissible control $u \in \mathcal{U}_{ad}[0,T]$ can be written as

$$\begin{aligned} X_T^{u,x_0} &= e^{AT}x_0 + \int_0^T e^{A(T-s)}\mu(X_s^{u,x_0})\,ds + \mathcal{T}u \\ &\quad + \int_0^T e^{A(T-s)}\sigma_1(X_s^{u,x_0})\,dB_s + \int_0^T e^{A(T-s)}\sigma_2(X_s^{u,x_0})\,dB_s^H. \end{aligned}$$

We aim to construct a control u such that $X_T^{u,x_0} = x_T$ almost surely. The proof proceeds in three main steps.

Step 1: Linear reference system. Consider first the linear reference system obtained by setting $\mu \equiv 0$ and freezing the diffusion coefficients. Under Assumption4.5.1, the controllability Gramian W_T is invertible, and the classical linear control theory ensures that for every $y \in \mathbb{R}^n$ there exists a control $u^0 \in \mathcal{U}_{ad}[0,T]$ such that

$$\mathcal{T}u^0 = y.$$

In particular, choosing $y = x_T - e^{AT}x_0$, we can find a control u^0 that steers the linear system from x_0 to x_T (up to the contribution of the nonlinear and stochastic terms).

Step 2: Nonlinear perturbation. We now consider the full nonlinear system and write the mild solution as

$$X^{u,x_0} = Z^{u^0,x_0} + Y,$$

where Z^{u^0,x_0} is the mild solution of the linear reference system with control u^0, and Y is the correction term accounting for the nonlinear drift and the dependence of the diffusion coefficients on the state. Using the Lipschitz and growth conditions in Assumption4.5.2, together with the semigroup estimates in Assumption4.1.1, one defines a suitable operator on a closed subset of $C([0,T];L^2(\Omega;\mathbb{R}^n))$ and shows that it is a contraction for T small enough (or for sufficiently small Lipschitz constants). By the Banach fixed-point theorem, there exists a unique correction Y, and hence a unique mild solution X^{u,x_0} associated with a control of the form $u = u^0 + v$, where v is a feedback term depending on Y.

Step 3: Choice of the target and conclusion. By construction of u^0 and the fixed point Y, the terminal value X_T^{u,x_0} coincides with the prescribed target x_T. Therefore, for every $x_0, x_T \in \mathbb{R}^n$, there exists an admissible control $u \in \mathcal{U}_{ad}[0,T]$ such that $X_T^{u,x_0} = x_T$ almost surely. This proves the controllability of the system on $[0,T]$. □

Proof. Let $x_0, x_T \in \mathbb{R}^n$ be given and fix $T > 0$. For $u \in \mathcal{U}_{ad}[0,T]$, denote by X^{u,x_0} the mild solution of (4.1) given by (4.2). We split the argument into two steps.

Step 1: *Linear reference system and choice of a preliminary control.* Consider first the linear reference system obtained by setting $\mu \equiv 0$ and freezing the diffusion coefficients:

$$dZ_t = (AZ_t + Bu_t)\,dt, \qquad Z_0 = x_0. \tag{4.7}$$

Its mild solution is

$$Z_t^{u,x_0} = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_s\,ds, \qquad t \in [0,T].$$

Let W_T be the controllability Gramian defined in (4.6). Under Assumption4.5.1, the operator

$$\mathcal{T} : \mathcal{U}_{ad}[0,T] \longrightarrow \mathbb{R}^n, \qquad \mathcal{T}u = \int_0^T e^{A(T-s)}Bu_s\,ds,$$

is onto. In particular, there exists a control $u^0 \in \mathcal{U}_{ad}[0,T]$ such that

$$\mathcal{T}u^0 = x_T - e^{AT}x_0.$$

For this control, the terminal value of the linear reference solution satisfies

$$Z_T^{u^0,x_0} = e^{AT}x_0 + \mathcal{T}u^0 = x_T.$$

Step 2: *Nonlinear stochastic system and fixed-point argument.* We now consider the full nonlinear system (4.1). For an admissible control $u \in \mathcal{U}_{ad}[0,T]$, the mild solution can be written as

$$X_t^{u,x_0} = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_s\,ds + \int_0^t e^{A(t-s)}\mu(X_s^{u,x_0})\,ds + \int_0^t e^{A(t-s)}\sigma_1(X_s^{u,x_0})\,dB_s + \int_0^t e^{A(t-s)}\sigma_2(X_s^{u,x_0})\,dB_s^H. \tag{4.8}$$

We look for a control of the form

$$u = u^0 + v,$$

where u^0 is the linear control constructed in Step 1 and $v \in \mathcal{U}_{ad}[0,T]$ is a correction term to be determined. Substituting into (4.8) and using the linearity of $\mathcal{T}$, we obtain

$$X_T^{u,x_0} = e^{AT}x_0 + \mathcal{T}u^0 + \mathcal{T}v + \int_0^T e^{A(T-s)}\mu(X_s^{u,x_0})\,ds + \int_0^T e^{A(T-s)}\sigma_1(X_s^{u,x_0})\,dB_s + \int_0^T e^{A(T-s)}\sigma_2(X_s^{u,x_0})\,dB_s^H.$$

Since $e^{AT}x_0 + \mathcal{T}u^0 = x_T$, this reduces to

$$X_T^{u,x_0} - x_T = \mathcal{T}v + \int_0^T e^{A(T-s)}\mu(X_s^{u,x_0})\,ds + \int_0^T e^{A(T-s)}\sigma_1(X_s^{u,x_0})\,dB_s + \int_0^T e^{A(T-s)}\sigma_2(X_s^{u,x_0})\,dB_s^H. \tag{4.9}$$

We want to choose v such that the right-hand side of (4.9) vanishes almost surely.

To this end, we define an operator

$$\Phi : C([0,T]; L^2(\Omega;\mathbb{R}^n)) \longrightarrow C([0,T]; L^2(\Omega;\mathbb{R}^n))$$

as follows: for a given process Y, we set $u = u^0 + v(Y)$, where $v(Y)$ is chosen so that

$$\mathcal{T}v(Y) = -\int_0^T e^{A(T-s)}\mu(Y_s)\,ds - \int_0^T e^{A(T-s)}\sigma_1(Y_s)\,dB_s - \int_0^T e^{A(T-s)}\sigma_2(Y_s)\,dB_s^H.$$

This is possible because $\mathcal{T}$ is onto by Assumption4.5.1. We then define $\Phi(Y)$ to be the mild solution of (4.1) associated with the control $u = u^0 + v(Y)$. By construction, any fixed point Y^* of Φ satisfies

$$X_T^{u^*,x_0} = x_T,$$

where $u^* = u^0 + v(Y^*)$.

By Assumption 4.5.2, the Lipschitz constants of μ, σ_1, σ_2 and the semigroup bounds imply that there exists $T^* > 0$ such that, for $T \le T^*$, the operator Φ is a contraction on a suitable closed ball of $C([0,T]; L^2(\Omega; \mathbb{R}^n))$. More precisely, for Y^1, Y^2 in this ball, using Assumptions4.1.1, 4.1.3 and 4.1.4, one can show that

$$\sup_{t\in[0,T]} \mathbb{E}\big\|\Phi(Y^1)_t - \Phi(Y^2)_t\big\|^2 \le q \sup_{t\in[0,T]} \mathbb{E}\big\|Y_t^1 - Y_t^2\big\|^2,$$

with some $q \in (0,1)$ when T (or the Lipschitz constants) is sufficiently small. Hence, by the Banach fixed-point theorem, Φ admits a unique fixed point Y^*, and the associated control $u^* = u^0 + v(Y^*)$ is admissible by construction.

For this control u^*, the corresponding mild solution X^{u^*,x_0} satisfies $X_T^{u^*,x_0} = x_T$ almost surely, as follows from (4.9) and the definition of $v(Y^*)$. Since $x_0, x_T \in \mathbb{R}^n$ were arbitrary, the system is controllable on $[0,T]$ in the sense of Definition4.4.1. □

Remark 4.5.5. The proof above follows the same general philosophy as in [30]: controllability is first established for the linearized system via the invertibility of the Gramian W_T, and then the nonlinear and stochastic terms are treated as perturbations absorbed by a fixed-point argument. The mixed fractional noise affects the sample paths but does not destroy controllability, provided its coefficients satisfy the regularity assumptions stated earlier and the linear control pair (A, B) remains effective.

4.6 Illustrative Example

In this section, we present a finite-dimensional example to illustrate the controllability result of Theorem4.5.4. The aim is to show how the abstract hypotheses translate into explicit conditions on a simple linear system perturbed by mixed fractional Brownian noise.

4.6.1 Example setting

We consider a two-dimensional controlled system of the form

$$\begin{cases} dX_t^{(1)} = \left(-\alpha X_t^{(1)} + u_t\right) dt + \sigma_{11} X_t^{(1)}\, dB_t + \sigma_{12} X_t^{(1)}\, dB_t^H, \\ dX_t^{(2)} = \left(\beta X_t^{(1)} - \gamma X_t^{(2)}\right) dt + \sigma_{21} X_t^{(2)}\, dB_t + \sigma_{22} X_t^{(2)}\, dB_t^H, \\ X_0 = x_0 \in \mathbb{R}^2, \end{cases} \tag{4.10}$$

where $\alpha, \beta, \gamma > 0$ and $\sigma_{ij} \in \mathbb{R}$ are constants. Here $u_t \in \mathbb{R}$ denotes the scalar control input, B_t is a standard Brownian motion, and B_t^H is an independent fractional Brownian motion with Hurst parameter $H \in (0,1)$.

In matrix form, (4.10) can be written as

$$dX_t = (AX_t + Bu_t)\,dt + \Sigma_1 X_t\,dB_t + \Sigma_2 X_t\,dB_t^H,$$

with

$$A = \begin{pmatrix} -\alpha & 0 \\ \beta & -\gamma \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Sigma_1 = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{21} \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} \sigma_{12} & 0 \\ 0 & \sigma_{22} \end{pmatrix}.$$

We now check that the assumptions of Theorem 4.5.4 are satisfied for (4.10) under suitable conditions on the parameters.

Linear operator and semigroup. The matrix A has eigenvalues $-\alpha$ and $-\gamma$, which are both negative. Hence A generates an exponentially stable C_0-semigroup $(e^{At})_{t\geq 0}$ on $\mathbb{R}^2$ with

$$\|e^{At}\| \leq Me^{-\delta t}, \qquad t \geq 0,$$

for some constants $M \geq 1$ and $\delta = \min\{\alpha, \gamma\} > 0$, so Assumption 4.1.1 holds.

Control operator and Gramian. The pair (A, B) is controllable in the classical linear sense if and only if the Kalman rank condition holds:

$$\operatorname{rank}\big[B, AB\big] = 2.$$

Here

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad AB = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix},$$

so

$$\det\big[B, AB\big] = \det \begin{pmatrix} 1 & -\alpha \\ 0 & \beta \end{pmatrix} = \beta.$$

Thus, if $\beta \neq 0$, the rank condition is satisfied and the controllability Gramian W_T given by (4.6) is invertible for every $T > 0$. Hence Assumption 4.5.1 holds.

Nonlinear drift and diffusion. In this example, the drift is linear,

$$\mu(X_t) = AX_t,$$

and the diffusion coefficients are of the form $\sigma_1(X_t) = \Sigma_1 X_t$, $\sigma_2(X_t) = \Sigma_2 X_t$. These functions are globally Lipschitz with linear growth, so Assumptions 4.1.3 and 4.1.4 are satisfied. Moreover, for sufficiently small values of $|\sigma_{ij}|$, the constants in the Lipschitz and growth bounds can be made small enough to ensure that the fixed-point operator in the proof of Theorem 4.5.4 is a contraction on a suitable time interval.

Admissible controls. Any square-integrable adapted control $u \in \mathcal{U}_{ad}[0,T]$ is admissible by Definition 4.1.6. The control-to-state operator $\mathcal{T}$ is bounded from $\mathcal{U}_{ad}[0,T]$ into $\mathbb{R}^2$ because A generates a bounded semigroup and B is constant, so Assumption 4.5.3 holds. Under the conditions

$$\alpha > 0, \quad \gamma > 0, \quad \beta \neq 0, \quad \text{and} \quad |\sigma_{ij}| \text{ small enough},$$

all assumptions of Theorem 4.5.4 are satisfied for the system (4.10). Therefore, the controlled mixed fractional Brownian motion system (4.10) is controllable on $[0,T]$: for every initial state $x_0 \in \mathbb{R}^2$ and every target state $x_T \in \mathbb{R}^2$, there exists an admissible control $u \in \mathcal{U}_{ad}[0,T]$ such that the corresponding mild solution X^{u,x_0} satisfies

$$X_T^{u,x_0} = x_T \quad \text{almost surely.}$$

This simple example illustrates how the abstract controllability conditions translate into concrete algebraic conditions (such as $\beta \neq 0$) and regularity bounds on the mixed fractional noise coefficients.

4.6.2 Numerical illustration

In order to illustrate the main controllability result obtained in this chapter, we consider the following two-dimensional stochastic system driven by a mixed fractional Brownian motion (MFBM):

$$dX_t = AX_t\,dt + Bu(t)\,dt + \Sigma_1 X_t\,dM_t,$$

where

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}, \quad B = (1,0)^\top,$$

and $M_t = B_t + B_t^H$ is a mixed fractional Brownian motion with Hurst parameter $H = 0.7$. We choose the initial condition $X_0 = (0, 0.5)^\top$ and the target state at the terminal time $T = 1$ as $x_T = (1,0)^\top$. For the control input we consider a linear state-feedback law

$$u(t, X_t) = k^\top (x_T - X_t),$$

with gain vector $k = (2,0)^\top$, which is consistent with the controllability of the underlying linear pair (A,B). The mixed noise is simulated by combining standard Brownian increments with an approximate fractional Brownian component based on a spectral approximation of fractional Gaussian noise, and the corresponding SDE is discretized in time by an Euler-type scheme applied to the drift and to the multiplicative mixed noise term. Figure 4.2 displays several sample trajectories of the components $X_1(t)$ and $X_2(t)$ under this feedback, together with the reference levels of the target state, and shows that the controlled trajectories are effectively steered in the direction of x_T in the presence of mixed noise. Figure 4.3 presents the corresponding phase-plane representation of initial and final states for a collection of sample paths; the clustering of end points around the target x_T provides a numerical illustration of the controllability behavior established in the theoretical part of this chapter.

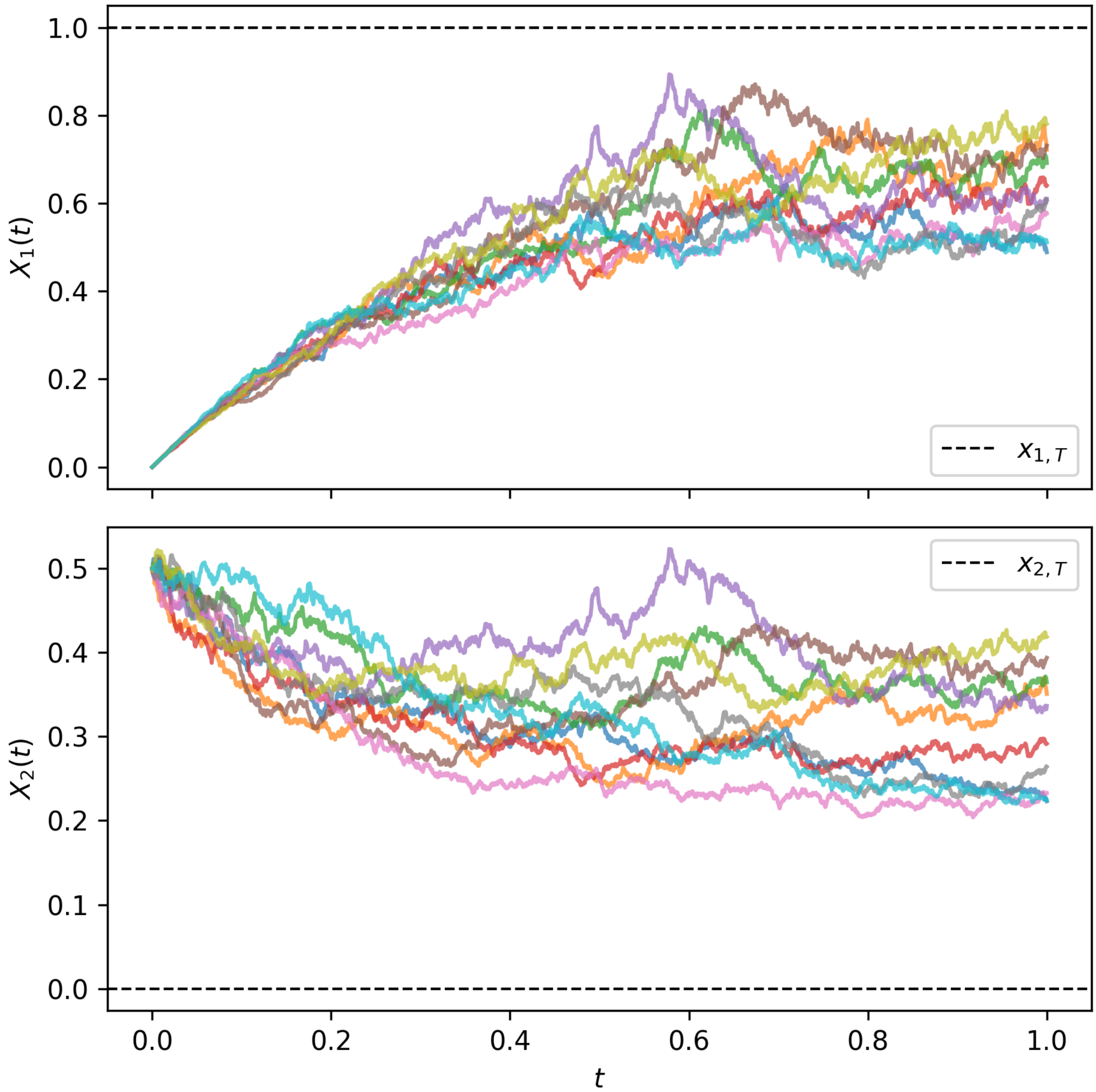


Figure 4.2: Sample trajectories of the components $X_1(t)$ (top) and $X_2(t)$ (bottom) for the controlled two-dimensional system driven by a mixed fractional Brownian motion. The dashed horizontal lines indicate the target levels of the corresponding components of the terminal state x_T.

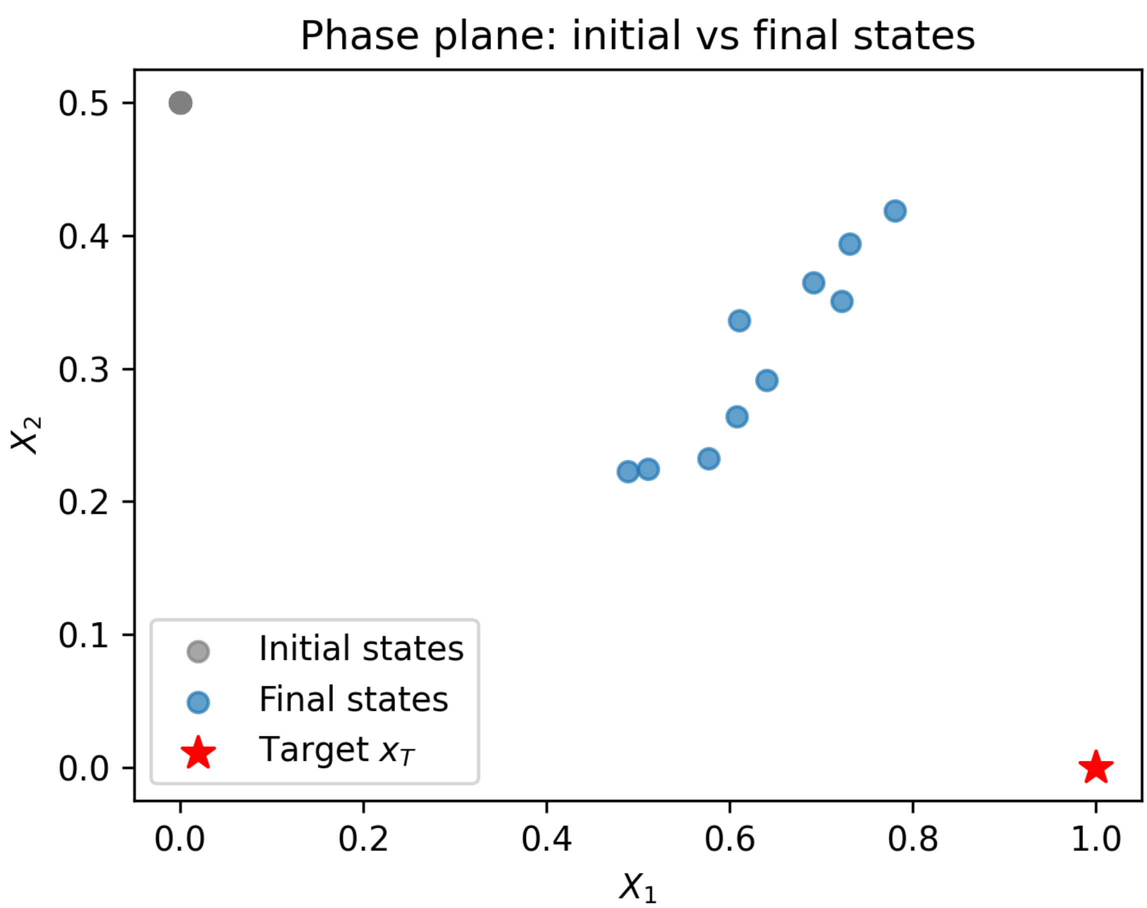


Figure 4.3: Phase-plane representation of the initial (grey dots) and final (blue dots) states of the controlled mixed fractional system, together with the target state x_T (red star). The clustering of final states around x_T provides a numerical illustration of the controllability behaviour established in this chapter.

Conclusion

We looked at stochastic differential equations driven by mixed fractional Brownian motion theoretically in this study, focusing on the controllability of the associated stochastic systems as well as the probabilistic foundations of the driving noise. From the traditional framework of Brownian motion and fractional Brownian motion to the advanced setting of mixed fractional Brownian motion, the work progresses to the mild formulation of controlled stochastic evolution equations.

The first part of the thesis revisits the essential properties of Brownian motion and fractional Brownian motion, such as Gaussianity, self-similarity, Hölder regularity, dependence structure, and the (non-)Markov and (non-)semimartingale character of these processes. This preliminary analysis clarifies the limitations of standard Itô calculus in the fractional context and motivates the introduction of alternative integration tools, including Young and Skorohod integrals, as well as more general pathwise approaches.

Building on this, we have introduced mixed fractional Brownian motion as a flexible hybrid model that combines a standard Brownian component with a fractional Brownian component. We have recalled and used key structural features of this process, such as its covariance function, correlation of increments, long- and short-range dependence, Hölder continuity, and the dependence of its semimartingale property on the Hurst parameter. These properties show that mixed fractional Brownian motion can simultaneously capture short-term randomness and long-memory effects, while preserving analytical tractability in certain parameter regimes.

In the subsequent analysis of stochastic differential equations driven by mixed fractional Brownian motion, we have defined appropriate stochastic integrals with respect to both the Brownian and fractional parts of the noise. Under suitable Lipschitz and growth conditions on the coefficients, we established existence and uniqueness results for strong (and, in some cases, weak) solutions. The combination of classical Itô techniques with fractional and Malliavin-type tools demonstrates that well-posedness can still be achieved in the presence of mixed noise, despite the partial loss of semimartingale structure.

Finally, we have addressed the controllability of stochastic systems driven by mixed fractional Brownian motion in an abstract semigroup framework. By formulating the controlled equation in mild form and introducing the control-to-state operator, we derived sufficient conditions ensuring controllability on a finite time interval. These conditions involve both the properties of the underlying deterministic operator and the structure of the control and diffusion terms. An illustrative example shows how the abstract hypotheses translate into explicit algebraic criteria, connecting the general theory to concrete finite-dimensional systems.

Overall, this thesis contributes to the growing literature on stochastic systems with memory by providing a coherent framework that links the probabilistic analysis of mixed fractional Brownian motion with the theory of stochastic differential equations and controllability. Several directions for future research naturally arise, including the study of approximate and null controllability in infinite dimensions, the development of optimal control and stabilization strategies for mixed fractional systems, and the extension of the present results to more general non-Gaussian noises, delay equations, and systems with state or control constraints.

Referances

[1] S. H. Abid, S. Q. Hasan, and U. J. Quaez. Approximate controllability of fractional stochastic integro-differential equations driven by mixed fractional brownian motion. *American Journal of Mathematics and Statistics*, 5(2):72–81, 2015.

[2] E. Alos, O. Mazet, and D. Nualart. Stochastic calculus with respect to gaussian processes. *Annals of Probability*, 2001.

[3] Anonymous. A modified maximum principle for control systems driven by mixed fractional brownian motion. *arXiv preprint arXiv:2312.11893*, 2023.

[4] L. Bachelier. Théorie de la spéculation. *Annales Scientifiques de l'École Normale Supérieure*, 17:21–86, 1900.

[5] F. Baudoin. Stochastic differential equations driven by fractional brownian motions. Lecture notes, Finnish summer school on Probability Theory and Statistics, 2012. Lecture 7: Young's integral.

[6] F. Ben Adda and J. Cresson. About non-differentiable functions. *Journal of Mathematical Analysis and Applications*, 263(2):721–737, 2001.

[7] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang. *Stochastic Calculus for Fractional Brownian Motion and Applications*. Springer, 2008.

[8] P. Billingsley. *Convergence of Probability Measures*. Wiley, 2 edition, 1999.

[9] R. Brown. A brief account of microscopical observations made in the months of june, july and august 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *Philosophical Magazine*, 4(21):161–173, 1828.

[10] P. Cheridito. Mixed fractional brownian motion. *Bernoulli*, 2001.

[11] G. Da Prato and J. Zabczyk. *Ergodicity for Infinite-Dimensional Systems*. Cambridge University Press, 1996.

[12] L. Decreusefond and O. Ştên. Stochastic integration with respect to fractional brownian motion. *Theory of Probability and Its Applications*, 1999.

[13] A. Einstein. On the motion of particles suspended in resting liquids required by the molecular kinetic theory of heat. *Annalen der Physik*, 322(8):549–560, 1905.

[14] J. François. *Mouvement Brownien, martingales et calcul stochastique.* Springer-Verlag Berlin Heidelberg, 2013.

[15] A. Granas and J. Dugundji. *Fixed Point Theory.* Springer, 2003.

[16] J. Guerra and D. Nualart. Stochastic differential equations driven by fractional brownian motion and standard brownian motion. *Stochastic Analysis and Applications*, 26(5):1053–1075, 2008. Existence and uniqueness for mixed Brownian/fractional Brownian-driven SDEs.

[17] Y. Hu and D. Nualart. Differential equations driven by fractional brownian motion and the existence of global solutions. *Stochastic Analysis and Applications*, 26(2):365–395, 2008.

[18] A. N. Kolmogorov. Wienersche spiralen und einige andere interessante kurven im hilbertschen raum. *Doklady Akademii Nauk SSSR*, 26:115–118, 1940.

[19] K. Kolwankar and A. D. Gangal. Local fractional derivatives and fractal functions of several variables. In *Proceedings of the International Conference on Fractals in Engineering*, Arcachon, 1997.

[20] E. H. Lakhel. Controllability of fractional stochastic neutral functional differential equations driven by fractional brownian motion with infinite delay. *arXiv preprint arXiv:1604.04079*, 2016.

[21] B. Mandelbrot and J. W. Van Ness. Fractional brownian motions, fractional noises and applications. *SIAM Review*, 10(4):422–437, 1968.

[22] Y. Mishura. *Stochastic Calculus for Fractional Brownian Motion and Related Processes.* Springer, 2008.

[23] Y. Mishura and G. Shevchenko. Mixed stochastic differential equations with long-range dependence: Existence, uniqueness and convergence of solutions. *Computers & Mathematics with Applications*, 2016.

[24] P. Mörters and Y. Peres. *Brownian Motion.* Cambridge University Press, 2010.

[25] Y. J. Nahmani. Introduction to stochastic integration with respect to fbm. Institute of Mathematics, 2009.

[26] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer, 2006.

[27] D. Nualart and A. Răsçanu. Differential equations driven by fractional brownian motion. *Collectanea Mathematica*, 53(1):55–81, 2002.

[28] P. E. Protter. *Stochastic Integration and Differential Equations*. Springer, 2003.

[29] D. Ruan and J. Luo. The existence, uniqueness, and controllability of neutral stochastic delay partial differential equations driven by standard brownian motion and fractional brownian motion. *Discrete Dynamics in Nature and Society*, 2018:1–11, 2018.

[30] D. Ruan and J. Luo. The existence, uniqueness, and controllability of neutral stochastic delay partial differential equations driven by standard brownian motion and fractional brownian motion. *Discrete Dynamics in Nature and Society*, 2020.

[31] F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probability Theory and Related Fields*, 1993.

[32] T. Sathiyaraj and P. Balasubramaniam. Controllability of fractional higher order stochastic integrodifferential systems with fractional brownian motion. *ISA Transactions*, 82:107–119, 2018.

[33] M. Smoluchowski. On the kinetic theory of brownian molecular motion and suspensions. *Annals of Physics*, 326(6):756–780, 1906.

[34] C. Stricker. Quelques remarques sur les semimartingales gaussiennes et le problème de l'innovation. *Filtering and Control of Random Processes*, 1983.

[35] T. N. Thiele. On the basis of probability in connection with counting operations. Writings of the Scandinavian Society of Sciences and Letters, Series 5, 1880.

[36] N. Wiener. Differential space. *Journal of Mathematics and Physics*, 2(1):131–174, 1923.

[37] D. Williams. *Probability with Martingales*. Cambridge University Press, 1991.

[38] L. C. Young. An inequality of the hölder type, connected with stieltjes integration. *Acta Mathematica*, 1936.

[39] M. Zili. On the mixed fractional brownian motion. *Journal of Mathematical Analysis and Applications*, pages 1–9, 2006.