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Recursive estimation for functional data and some convergence results

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Dedication

I dedicate this humble work to my beloved parents,
whose unconditional love, constant support, patience, and countless sacrifices have been the light guiding me throughout every stage of my life.
Your prayers and encouragement have always been my source of strength, hope, and determination to continue moving forward.
No words, no matter how beautiful, can truly express the depth of my gratitude and appreciation for everything you have done for me.

To my beloved son,
the greatest blessing in my life and the light that brightens my path,
whose innocent smile gave me courage during moments of exhaustion and filled my heart with hope in times of doubt.
You are the inspiration behind every dream I strive to achieve.

To my dear brothers and sisters,
thank you for your love, encouragement, and constant presence by my side.
Your support has been a source of comfort and motivation throughout this academic journey.

I also dedicate this work to all my respected teachers,
and to everyone who contributed, directly or indirectly, to the completion of this work.
To all of you, I express my sincere gratitude, appreciation, and respect.

Finally, I dedicate this humble work to everyone who believes in the value of knowledge and perseverance,
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Finally, I would like to thank all those who supported me and believed in my abilities during this academic journey.

General Introduction

Functional data analysis has become one of the most active and important areas in modern statistics due to the increasing availability of complex data observed continuously over time or space. Unlike classical statistical data represented by scalar or finite-dimensional vectors, functional data are generally viewed as curves, surfaces, or trajectories evolving in an infinite-dimensional space. Such data naturally arise in many scientific fields including climatology, medicine, economics, chemometrics, engineering, and environmental sciences.

The rapid development of computational technologies and data acquisition systems has considerably increased the interest in statistical methods adapted to functional observations. In this context, nonparametric approaches play a fundamental role because they provide flexible modeling techniques without imposing restrictive parametric assumptions on the underlying stochastic process.

Among the different nonparametric methods, recursive estimation procedures have received considerable attention in recent years. Recursive methods offer important computational advantages since they allow estimators to be updated sequentially when new observations become available without repeating all previous calculations. Consequently, these procedures are particularly suitable for large datasets, online learning problems, and dependent data structures where computational efficiency and memory reduction are essential.

The study of conditional models in the functional framework has also attracted growing interest in the statistical literature. In particular, the estimation of the conditional cumulative distribution function and the conditional density function constitutes an important problem because of its numerous theoretical and practical applications, especially in prediction, classification, reliability analysis, and risk modeling.

The main objective of this dissertation is to study recursive nonparametric estimation methods for functional data. More precisely, we focus on the construction of recursive kernel estimators for the conditional distribution function and the conditional density function when the explanatory variable takes its values in a semi-metric functional space. We also investigate the asymptotic properties of these estimators under both independent and strong mixing assumptions.

This dissertation is organized into three chapters.

The first chapter is devoted to the presentation of the general framework of functional statistics and conditional models. We introduce the main concepts related to functional variables, recursive procedures, and strong mixing conditions. Some asymptotic tools and preliminary probabilistic results used throughout this work are also presented.

In the second chapter, we study recursive nonparametric estimators of the conditional distribution function and the conditional density function in the independent case. We establish several asymptotic properties of the proposed estimators, including almost sure convergence and convergence in quadratic mean under suitable assumptions.

The third chapter is dedicated to recursive estimation methods under strong mixing conditions. We construct recursive estimators adapted to dependent functional data and investigate their asymptotic behavior through convergence results and theoretical analysis.

The obtained results highlight the efficiency of recursive estimation techniques in the functional framework and emphasize the importance of nonparametric approaches for the analysis of infinite-dimensional statistical models.

Chapter 1

Functional variables and conditional models

1.1 Functional Variables

Statistical issues related to the modeling and analysis of functional random variables have attracted considerable interest in statistics. Early studies in this field were mainly based on the discretization of functional observations in order to adapt classical multivariate statistical techniques. However, with the rapid development of computational tools and the increasing availability of large datasets, a new approach has emerged that treats this type of data in its natural infinite-dimensional framework while preserving its functional nature.

In fact, since the 1960s, the analysis of observations represented as trajectories has been investigated in several scientific areas. Among the pioneering works, we can mention the studies of Obhukov [44] and Holmstrom [31] in climatology, Deville [16] in econometrics, as well as the contributions of Molenaar and Boosma [42], followed later by Kirkpatrick and Heckman [31] in genetics.

During recent years, functional regression models, whether parametric or nonparametric, have received particular attention. In the linear setting, the work of James Ramsay and Bernard Silverman [50, 44] provided an important collection of statistical methodologies dedicated to functional variables. Moreover, Denis Bosq [7] made significant contributions to the development of statistical procedures in the context of functional linear autoregressive processes.

Using functional principal component analysis, F.Cardot et al. [44] proposed an estimator for the Hilbertian linear regression model, similar to the approach developed by Denis Bosq [7] for Hilbertian autoregressive processes. This estimator is constructed through the spectral properties of the empirical covariance operator associated with the functional explanatory variable. The authors established convergence in probability in certain situations and almost complete convergence of the proposed estimator in other cases.

1.1.1 Concrete problem in statistics for functional variables

Functional data arise in many scientific fields and have motivated the development of functional statistical methods for solving various practical problems.

• In biology, some of the earliest studies dealing with functional data appeared in 1958 through the analysis of growth curves. More recently, several works have focused on the study of knee angle variations during walking, notably by Ramsay and Silverman [44], as well as on knee movements during constrained exercise. In animal biology, functional approaches have also been used to study medfly oviposition curves, where the data are represented by trajectories describing the number of eggs laid over time.

• Chemometrics is another important field where functional statistical methods are widely applied. Several studies have considered the analysis of spectrometric curves and laser intensity measurements.

Researchers investigated the relationship between the fat content of meat samples and their infrared absorption curves, where the explanatory variable is functional in nature.

• Environmental applications also represent an important area of functional statistics. In particular, pollution forecasting problems have been studied using functional data methods. These studies generally involve predicting daily ozone pollution peaks from pollutant concentration curves and meteorological variables observed over time.

• Climatology is an area where functional data appear naturally. A study of the phenomenon El Niño (hot current in Pacific Ocean) has been realized by Besse *and al.* (2000)[22]; Ramsay and Silverman (2005)[45], Ferraty *a*nd al. (2005)[22] and Hall and Vial (2006)[28].

1.2 Conditional models

The study of conditional models in the functional framework has attracted considerable attention in recent years. One of the first contributions in this area was proposed by Ferraty *et al.* (2006)[22], who introduced a double kernel estimator for the conditional distribution function in the functional setting. They also established the rate of almost complete convergence of this estimator in the case of independent and identically distributed observations.

Later, the same problem was extended to dependent data. In particular, Ferraty *et al.* (2005b) [21]studied the conditional distribution estimator under α-mixing conditions and obtained important asymptotic results.

Several authors investigated the estimation of the conditional distribution function as a preliminary step for conditional quantile estimation. For instance, Ezzahrioui and Ould Said (2005, 2006)[49, 40] studied the asymptotic normality of the estimator in both the independent and the α-mixing cases.

We also refer to Cardot *et al.* (2004)[14], who developed a linear approach for conditional quantile estimation in functional statistics.

Concerning the conditional density function, Ferraty *et al.* (2006)[22] introduced kernel estimators of the conditional density and its derivatives in the functional framework. They established almost complete convergence in the independent case. Since then, numerous works have been devoted to conditional estimation and its derivatives, particularly for estimating the conditional mode.

In the dependent framework, Ferraty *et al.* (2005b)[21] established the almost complete convergence of the kernel estimator of the conditional mode obtained by maximizing the conditional density estimator under α-mixing assumptions

1.2.1 Recursive models

The idea of recursive methods is to use the estimates calculated on the basis of the initial data and to update them with only new observations arriving in the database. A major advantage of these methods is that it is not necessary to restart all the calculations of the model parameters whenever the database is completed by new observations. In general, the advantage of these methods is to take into account the successive arrival of the data and to refine, as time goes by, the estimation algorithms implemented, the applications of Such an approach are numerous. The gain in terms of computation time can be very interesting.

Historically, the recursive estimation with rate was introduced by Wolverton and Wagner (1969)[52]. Later, Baltagi and Li (1994)[5] proposed a simple recursive estimation method for linear regression models with AR(p) disturbances. As a recent application of recursive methods we cite Amiri and Thiam (2018)[2] who studied regression estimation by local polynomial fitting for multivariate data streams. The objective of our work is to propose a parametric family of recursive kernel estimator of the conditional distribution function *(cdf)* by adopting to functional case the result given by Roussas (1992)[48].

The estimate of the *cdf* in a functional setting has been introduced by Ferraty *and al.* (2006)[22]. The authors built a double kernel estimator for the *cdf* and they established the almost complete convergence rate of the estimator when observations are independent and identically distributed (i.i.d). The case of α-mixing observations has been studied earlier by Ferraty *and al.*(2005)[20, 21]. The first uniforms results available in the literature on the estimation of the distribution function conditionally to a functional variable were established in Ferraty *and al.* (2010)[25]. More recently, Amiri and Kherdani (2019)[3] who studied a recursive kernel regression method adapted to censored data, the asymptotic normality of the kernel estimator of the *cdf* was studied by Bouadjemi Abdelkader (2014)[12], the author introduced a new nonparametric estimator of the *cdf* of a scalar response variable Y given a functional random variable X. This estimate was based on recursive approach. Under certain terms and conditions, he proved the asymptotic normality of the built model. Keddani *and al.* (2018)[29] built an estimator of the *cdf* when the explanatory variable takes its values in a functional space by using the recursive estimation method when the sample is considered as an i.i.d sequence. Authors proposed a technique based on a multivariate counterpart of the stochastic approximation method for successive experiments for the local polynomial estimation issue.

1.3 Definitions and tools

1.3.1 Types of convergence

Throughout this chapter, $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ denote sequences of real-valued random variables, while $(u_n)_{n\in\mathbb{N}}$ represents a deterministic sequence of positive real numbers. In addition, the notation $(Z_n)_{n\in\mathbb{N}}$ will be used to designate a sequence of independent centered random variables.

The following definitions and results can be found in (Ferraty *and Vieu.*[22])

Definition 1.3.1. *One says that $(X_n)_{n\in\mathbb{N}}$ converges almost completely (a.co.) to some r.r.v. X, if and only if*

$$\forall\varepsilon > 0, \quad \sum_{n\in\mathbb{N}} \mathbb{P}\left(|X_n - X| > \varepsilon\right) < \infty,$$

and the almost complete convergence of $(X_n)_{n\in\mathbb{N}}$ to X is denoted by

$$\lim_{n\to\infty} X_n = X, \ a.co.$$

Definition 1.3.2. *One says that the rate of almost complete convergence of $(X_n)_{n\in\mathbb{N}}$ to X is of order u_n if and only if*

$$\exists\varepsilon_0 > 0, \qquad \sum_{n\in\mathbb{N}} \mathbb{P}\left(|X_n - X| > \varepsilon_0 u_n\right) < \infty,$$

and we write

$$X_n - X = O_{a.co.}(u_n)$$

Proposition 1.3.1. *Assume that $\lim_{n\to\infty} u_n = 0$, $X_n = O_{a.co.}(u_n)$ and $\lim_{n\to\infty} Y_n = l_0$, a.co., where l_0 is a deterministic real number.*

i) We have $X_nY_n = O_{a.co.}(u_n)$;

ii) We have $\dfrac{X_n}{Y_n} = O_{a.co.}(u_n)$ as long as $l_0 \neq 0$.

Remark 1.3.1. *The almost complete convergence of Y_n to l_0 implies that there exists some $\delta > 0$ such that*

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left(|Y_n| > \delta\right) < \infty.$$

Now , suppose that $Z_1, ..., Z_n$ are independent real random variable (r.r.v). with zero mean. As can be seen throughout this part, the statement of almost complete convergence properties needs to find an upper bound for some probabilities involving sum of r.r.v. such as

$$\mathbb{P}\left(\Big|\sum_{i=1}^{n} Z_i\Big| > \varepsilon\right),$$

where, eventually, the positive real ε decreases with n. In this context, there exists powerful probabilistic tools, generically called Exponential Inequalities. The literature provides sieveral versions of exponential inequalities. These inequalities differ according to the various hypotheses checked by the variables Z_i's. We focus here on the so-called Bernstein's inequality. This choice was made because the form of Bernstein's inequality is the easiest for the theoretical developments on functional statistics that are developed throughout our thesis. Other forms of exponential inequality can be found in (see Nagaev ([36],[37])).

Proposition 1.3.2. *Assume that*

$$\forall m \geq 2,\ |\mathbb{E}Z_i^m| \leq (m!/2)(a_i)^2 b^{m-2},$$

and let $(A_n)^2 = (a_1)^2 + \ldots + (a_n)^2$. *Then, we have:*

$$\forall \varepsilon \geq 0,\quad \mathbb{P}\left(\Big|\sum_{i=1}^{\infty} Z_i\Big| \geq \varepsilon A_n\right) \leq 2\exp\left\{-\frac{\varepsilon^2}{2\left(1+\frac{\varepsilon b}{A_n}\right)}\right\}.$$

Corollary 1.3.1. *i) If* $\forall m \geq 2, \exists C_m > 0,\ \mathbb{E}|Z_1^m| \leq C_m a^{2(m-1)}$, *we have*

$$\forall \varepsilon \geq 0,\quad \mathbb{P}\left(\Big|\sum_{i=1}^{\infty} Z_i\Big| \geq n\varepsilon\right) \leq 2\exp\left\{-\frac{n\varepsilon^2}{2a^2(1+\varepsilon)}\right\}.$$

ii) Assume that the variables depend on n (that is, $Z_i = Z_{i,n}$*). If* $\forall m \geq 2, \exists C_m > 0,\ \mathbb{E}|Z_1^m| \leq C_m a^{2(m-1)}$, *and if* $u_n = n^{-1}a_n^2 \log n$ *verifies* $\lim_{n\to\infty} u_n = 0$, *we have:*

$$\frac{1}{n}\sum_{i=1}^{n} Z_i = O_{a.co.}\left(\sqrt{u_n}\right).$$

Remark 1.3.2. *By applying Proposition 1.3.2 with* $A_n = a\sqrt{u_n}$, $b = a^2$ *and taking* $\varepsilon = \varepsilon_0\sqrt{u_n}$, *we obtain for some* $C' > 0$*:*

$$\mathbb{P}\left(\frac{1}{n}\Big|\sum_{i=1}^{\infty} Z_i\Big| > \varepsilon_0\sqrt{u_n}\right) \leq 2\exp\left\{-\frac{\varepsilon_0^2 \log n}{2(1+\varepsilon_0\sqrt{u_n})}\right\} \leq 2n^{-C'\varepsilon_0^2}.$$

Corollary 1.3.2. *i) If* $\exists M < \infty,\ |Z_1| \leq M$, *and denoting* $\sigma^2 = \mathbb{E}Z_1^2$, *we have*

$$\forall \varepsilon \geq 0,\quad \mathbb{P}\left(\Big|\sum_{i=1}^{n} Z_i\Big| \geq n\varepsilon\right) \leq 2\exp\left\{-\frac{n\varepsilon^2}{2\sigma^2(1+\varepsilon\frac{M}{\sigma^2})}\right\}.$$

ii) Assume that the variables depend on n (that is, $Z_i = Z_{i,n}$*) and are such that* $\exists M = M_n < \infty$, $|Z_1| \leq M$ *and define* $\sigma_n^2 = \mathbb{E}Z_1^2$. *If* $u_n = n^{-1}\sigma_n^2 \log n$ *such that* $\lim_{n\to\infty} u_n = 0$, *and if* $M/\sigma_n^2 < C < \infty$, *then we have:*

$$\frac{1}{n}\sum_{i=1}^{n} Z_i = O_{a.co.}\left(\sqrt{u_n}\right).$$

Remark 1.3.3. *By applying Proposition 1.3.2 with* $a_i^2 = \sigma^2$, $A_n = n\sigma^2$, *and by choosing* $\varepsilon = \varepsilon_0\sqrt{u_n}$, *we obtain for some* $C' > 0$*:*

$$\mathbb{P}\left(\frac{1}{n}\Big|\sum_{i=1}^{\infty} Z_i\Big| > \varepsilon_0\sqrt{u_n}\right) \leq 2\exp\left\{-\frac{\varepsilon_0^2 \log n}{2(1+\varepsilon_0\sqrt{v_n})}\right\} \leq 2n^{-C'\varepsilon_0^2}.$$

where $v_n = \frac{Mu_n}{\sigma_n^2}$

1.3.2 properties of kernel

Definition 1.3.3. *i) A function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called a kernel of type I if there exist two real constants $0 < C_1 < C_2 < \infty$ such that:*

$$C_1 \mathbf{1}_{[0,1]} \leq K \leq C_2 \mathbf{1}_{[0,1]}.$$

ii) A function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called a kernel of type II if its support is $[0,1]$ and if its derivative K' exists on $[0,1]$ and satisfies for two real constants $-\infty < C_2 < C_1 < 0$:

$$C_2 \leq K' \leq C_1.$$

The first kernel family contains the usual discontinuous kernels such as the asymmetrical box one while the second family contains the standard asymmetrical continuous ones (as the triangle, quadratic, ...). Finally, to be in harmony with this definition and simplify our purpose, for local weighting of real random variables we just consider the following kernel-type.

Definition 1.3.4. *A function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ with compact support $[-1,1]$ and such that $\forall u \in (0,1),\ K(u) > 0$ is called a kernel of type 0.*

We can now build the bridge between local weighting and the notation of small ball probabilities. To fix the ideas, consider the simplest kernel among those of type I namely the asymmetrical box kernel. Let x be f.r.v. valued in $\mathcal{F}$ and x be again a fixed element of $\mathcal{F}$. We can write:

$$\mathbb{E}\left(\mathbf{1}_{[0,1]}\left(\frac{d(x,X)}{h}\right)\right) = \mathbb{E}(\mathbf{1}_{B(x,h)}(X)) = \mathbb{P}(X \in B(x,h)).$$

The probability of the ball $B(x,h)$ appears clearly in the normalization. At this stage it is worth telling why we are saying *small* ball probabilities. In fact, as we will see later on, the smoothing parameter h (also called the *bandwith*) decreases with the size of the sample of the functional variables (more precisely, h tends to zero when n tends to ∞). Thus, when we take n very large, h is close to zero and then $B(x,h)$ is considered as a small ball and $\mathbb{P}(X \in B(x,h))$ as a small ball probability.
From now, for all x in $\mathcal{F}$ and for all positive real h, we will use the notation:

$$\phi_x(h) = \mathbb{P}(X \in B(x,h)).$$

This notion of small ball probabilities will play a major role both from theoretical and practical points of view. Because the notion of ball is strongly linked with the semi-metric d, the choice of this semi-metric will become an important stage.
Now, let X be a f.r.v. taking its values in the semi-metric space $(\mathcal{F}, d)$, let x be a fixed element of $\mathcal{F}$, let h be a real positive number and let K be a kernel function.

Lemma 1.3.1. *If K is a kernel of type I, then there exist nonnegative finite real constant C and C' such that:*

$$C\phi_x(h) \leq \mathbb{E}K\left(\frac{d(x,X_h)}{h}\right) \leq C'\phi_x(h).$$

Lemma 1.3.2. *If K is a kernel of type II and if $\phi_x(.)$ satisfies*

$$\exists C_3 > 0,\ \exists \epsilon_0,\ \forall \epsilon < \epsilon_0,\ \int_0^\epsilon \phi_x(u)du > C_3 \epsilon \phi_x(\epsilon),$$

then there exist nonnegative finite real constant C and C' such that, for h small enough:

$$C\phi_x(h) \leq \mathbb{E}K\left(\frac{d(x,X}{h}\right) \leq C'\phi_x(h).$$

Lemma 1.3.3. *[43] We have*

$$\frac{1}{F(h)}\int_0^1 tK(t)\,d\mathbb{P}^{\|x-x_i\|/h}(t) \longrightarrow M_0 \text{ as } n\longrightarrow\infty;$$

$$\frac{1}{F(h)}\int_0^1 K(t)\,d\mathbb{P}^{\|x-x_i\|/h}(t) \longrightarrow M_1 \text{ as } n\longrightarrow\infty;$$

$$\frac{1}{F(h)}\int_0^1 K^2(t)\,d\mathbb{P}^{\|x-x_i\|/h}(t) \longrightarrow M_2 \text{ as } n\longrightarrow\infty.$$

Proof.
We note that

$$tK(t) = K(1) - \int_t^1 (sK(s))'\,ds.$$

Applying Fubini's Theorem, we get

$$\begin{aligned}\int_0^1 tK(t)\,\mathrm{dP}^{\|x-x_i\|/h}(t) &= K(1)F(h) - \int_0^1\left(\int_t^1 (sK(s))'\,ds\right)\mathrm{dP}^{\|x-x_i\|/h}(t)\\ &= K(1)F(h) - \int_0^1 (sK(s))'\,F(hs)\,ds.\end{aligned}$$

Similarly, we have

$$\int_0^1 K(t)\,\mathrm{dP}^{\|x-x_i\|/h}(t) = K(1)F(h) - \int_0^1 (K(s))'\,F(hs)\,ds$$

and

$$\int_0^1 K^2(t)\,\mathrm{dP}^{\|x-x_i\|/h}(t) = K^2(1)F(h) - \int_0^1 \left(K^2(s)\right)'F(hs)\,ds.$$

This proof is finished by applying Lebesgue's dominated convergence theorem.

Lemma 1.3.4. ***Toeplitz's Lemma****[11] Let $(a_{n,k})_{n\geq 1,k\geq 1}$ be a real sequence and $(w_n)_{n\geq 1}$ a sequence which converges to w. On suppose that:*

(i) for any $k\geq 1$ $\lim_{n\to\infty} a_{n,k}=0$;

(ii) $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k} = A<\infty$;

(iii) there exists a constant $C>0$ such that for any $n>1$, $\sum_{k=1}^{\infty}|a_{n,k}|<C<\infty$.

Thus we have:

$$\sum_{k=1}^{\infty} a_{n,k}w_k \longrightarrow \infty,$$

as $n\longrightarrow\infty$.

1.3.3 Approximation theorem

The following theorem allows to approximate independent random variables using Brownian motion to exploit the Law of the Iterated Logarithm checked by Brownian motion (see Bosq[11]) then we will give the strongly mixing conditions.

Theorem 1.1. *Let X_n a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that for any $n \geq 0$, EX_n^2 exists and $EX_n = 0$.*

Let:

$$S_n = \sum_{i=1}^{n} X_i,\ S_0 = 0\ and\ V_n = \sum_{i=1}^{n} EX_i^2\ if n \geq 1,\ V_0 = 0.$$

for any $\alpha \geq 0$, suppose that $V_n \longrightarrow \infty$ and:

$$\sum_{k=1}^{\infty} \frac{(\ln_2 V_k)^{\alpha}}{V_k} E\left(X_k^2 1_{\left\{X_k^2 > \frac{V_k}{\ln V_k (\ln_2 V_k)^{2(\alpha+1)}}\right\}} \right) < \infty.$$

Let S a random function defined on $[0, +\infty[$ such that:

$$\forall\, t \epsilon\, [V_n,\, V_{n+1}[\,,\ S\,(t) = S_n.$$

So, defining $\{S\,(t)\,,\ t \geq 0\}$ if necessary on a new probability space, there exists Brownian motion ζ such that

$$|S\,(t) - \zeta\,(t)| = o\left(t^{\frac{1}{2}}\,(\ln \ln t)^{\frac{1-\alpha}{2}} \right).$$

law of iterated logarithm for Brownian motion

Theorem 1.2. *If ζ is Brownian motion, then we have:*

$$\overline{\lim_{t \to \infty}} \frac{\zeta\,(t)}{\sqrt{2t\ \ln \ln t}} = 1\, a.s.$$

1.3.4 The mixing conditions

The $\alpha - mixing$ or strong mixing notion witch is one of the most general among the different mixing structures introduced in literature (see Ferraty and Vieu[22] for definitions of various other mixing structures and link between them the strong mixing notion is defined in the following way:

We consider a sequence of random variables $(\Delta_n)_{n \in \mathbb{N}}$ defined on probabilistic space $(\omega, \mathcal{F}, \mathbb{B})$ in some space $(\omega, \mathcal{F}')$. let us denote for $-\infty \leq j \leq k \leq +\infty$ and for $\mathcal{F}_j^k$ the σ algebra generated by the random variables $(\Delta_i, j \leq i \leq k)$.

The strong mixing coefficients are defined by the following quantities

$$\alpha(n) = \sup_{k \in \mathbb{Z}} \sup_{A \in \mathcal{F}_{-\infty}^k} \sup_{\mathcal{F}_{k+n}^{+\infty}} \left| \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \right|$$

Definition 1.3.5. *The sequence $(\Delta_n)_{n \in \mathbb{Z}}$ is said $\alpha-$mixing (or strongly mixing) if*

$$\lim_{n \longrightarrow \infty} \alpha(n) = 0$$

Definition 1.3.6. *The sequence $(\Delta_n)_{n \in \mathbb{Z}}$ is said arithmetically (or algebraically) $\alpha-$mixing with rate $\alpha > 0$ if:*

$$\exists C > 0, \quad \alpha(n) \leq Cn^{-\alpha}.$$

it is called geometrically $\alpha-$mixing if

$$\exists C > 0, \quad \exists t \in (0,1), \alpha(n) \leq Ct^n.$$

Lemma 1.3.5. *Let* $\Delta_{i i\in\mathbb{N}}$ *the family of random variables valued in* $\mathbb{R}$ *that satisfy the strong mixing condition we put:*

$$S_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov(\Delta_i,\Delta_j)\right|$$

If $\|\Delta\| \leq \infty, \forall i \in \mathbb{N}$ *then they are for all* $\varepsilon > 0$ *and for all* $x > 1$

$$\mathbb{P}\big(|\sum \Delta_i| > 4\varepsilon\big) \leq \big(1+\frac{\varepsilon^2}{rS_n^2}\big)^{\frac{-r}{2}} + 2nCr^{-1}\big(\frac{2r}{\varepsilon}\big)^{a+1}$$

$a > 0$ *and* $(\alpha = n^{-a})$

Lemma 1.3.6. *We consider a family of random variables* $\Delta_{i\in\mathbb{N}}$ *valued in* $\mathbb{R}$*. If the condition of strongly mixing is verified and if* $\|\Delta\| < \infty$ *there are for all* $i \neq j$

$$\left|Cov(\Delta_i,\Delta_j)\right| \leq 4\alpha(|i-j|).$$

Chapter 2

Recursive kernel estimation of conditional distribution function in the independent case

In this chapter, we introduce a recursive estimator for the conditional cumulative distribution function and the conditional density function in the functional data framework under the independent case. The recursive estimation method is of particular interest because it allows the estimators to be updated sequentially as new observations become available. We also investigate the asymptotic properties of these estimators by studying their almost sure convergence and convergence in quadratic mean under appropriate assumptions.

2.1 model and the estimates

Let $(X_i, Y_i)_{i\geq 1}$ be a sequence of independent pairs identically distributed as (X, Y), where (X, Y) is a random pair valued in $\mathcal{F} \times \mathbb{R}$ and $(\mathcal{F}, d(;))$ is a semi-metric space.

The conditional distribution function is defined by

$$\widehat{F}^{[x]}(y) = \frac{\displaystyle\sum_{i=1}^{n} \frac{1}{(F(h_i))^l} K\left(\frac{\|x - X_i\|}{h_i}\right) H\left(\frac{y - Y_i}{h_i}\right)}{\displaystyle\sum_{i=1}^{n} \frac{1}{(F(h_i))^l} K\left(\frac{\|x - X_i\|}{h_i}\right)}.$$

where K is a kernel, H a distribution function, (h_n) a sequence of positive reals, l a parameter belonging to $[0, 1]$, $d(x, X_i) = \|x - X_i\|$ and $F(h_i) = \mathbb{P}(\|x - X_i\| \leq h_i)$.

Our family of estimators is a recursive modification of the estimate defined above and can be written as

$$\widehat{F}_{n+1}^{[x,l]}(y) = \frac{\left[\displaystyle\sum_{i=1}^{n}(F(h_i))^{1-l}\right]\varphi_n^{[l]}(y) + \left[\displaystyle\sum_{i=1}^{n+1}(F(h_i))^{1-l}\right] H\left(\frac{y - Y_{n+1}}{h_{n+1}}\right) K_{n+1}^{[l]}(\|x - X_{n+1}\|)}{\left[\displaystyle\sum_{i=1}^{n}(F(h_i))^{1-l}\right] f_n^{[l]}(x) + \left[\displaystyle\sum_{i=1}^{n+1}(F(h_i))^{1-l}\right] K_{n+1}^{[l]}(\|x - X_{n+1}\|)},$$

with

$$\varphi_n^{[l]}(x, y) = \frac{\displaystyle\sum_{i=1}^{n} \frac{H\left(\frac{y-Y_i}{h_i}\right)}{(F(h_i))^l} K\left(\frac{\|x - X_i\|}{h_i}\right)}{\displaystyle\sum_{i=1}^{n}(F(h_i))^{1-l}}, \qquad f_n^{[l]}(x) = \frac{\displaystyle\sum_{i=1}^{n} \frac{1}{(F(h_i))^l} K\left(\frac{\|x - X_i\|}{h_i}\right)}{\displaystyle\sum_{i=1}^{n}(F(h_i))^{1-l}},$$

and

$$K_i^{[l]}(.) = \frac{1}{(F(h_i))^l \sum_{j=1}^{i} (F(h_j))^{1-l}} K\left(\frac{.}{h_i}\right).$$

The conditional density, defined as the derivative of the conditional distribution, is given by

$$\widehat{f}^{[x]}(y) = \frac{\sum_{i=1}^{n} \frac{1}{h_i (F(h_i))^l} K\left(\frac{\|x - X_i\|}{h_i}\right) H'\left(\frac{y - Y_i}{h_i}\right)}{\sum_{i=1}^{n} \frac{1}{(F(h_i)^l} K\left(\frac{\|x - X_i\|}{h_i}\right)}.$$

This estimator can also be computed recursively by

$$\widehat{f}_{n+1}^{[x,l]}(y) = \frac{\left[\sum_{i=1}^{n} (F(h_i))^{1-l}\right] \phi_n^{[l]}(x,y) + (h_{n+1})^{-1} \left[\sum_{i=1}^{n+1} (F(h_i))^{1-l}\right] H'\left(\frac{y - Y_{n+1}}{h_{n+1}}\right) K_{n+1}^{[l]}(\|x - X_{n+1}\|)}{\left[\sum_{i=1}^{n} (F(h_i)^{1-l}\right] f_n^{[l]}(x) + \left[\sum_{i=1}^{n+1} (F(h_i))^{1-l}\right] K_{n+1}^{[l]}(\|x - X_{n+1}\|)}.$$

with

$$\phi_n^{[l]}(x,y) = \frac{\sum_{i=1}^{n} \frac{1}{h_i (F(h_i))^l} K\left(\frac{\|x - X_i\|}{h_i}\right) H'\left(\frac{y - Y_i}{h_i}\right)}{\sum_{i=1}^{n} (F(h_i))^{1-l}}, \qquad f_n^{[l]}(x) = \frac{\sum_{i=1}^{n} \frac{1}{(F(h_i))^l} K\left(\frac{\|x - x_i\|}{h_i}\right)}{\sum_{i=1}^{n} (F(h_i))^{1-l}},$$

and

$$K_i^{[l]}(.) = \frac{1}{(F(h_i))^l \sum_{j=1}^{i} (F(h_j))^{1-l}} K\left(\frac{.}{h_i}\right).$$

Finally, we introduce the following notation (Ferraty et al., 2007):

$$M_0 = K(1) - \int_0^1 (sK(s))' \tau_0(s) \, ds$$

$$M_1 = K(1) - \int_0^1 K'(s) \tau_0(s) \, ds$$

$$M_2 = K^2(1) - \int_0^1 (K^2(s))' \tau_0(s) \, ds$$

To establish a recursive estimation of the conditional distribution function, we assume throughout this thesis that the following assumptions hold.

2.2 Hypotheses

(H1) K is a bounded kernel on the compact support $[0, 1]$ such that

$$0 < c_1(t) < K(t) < c_2(t) < \infty.$$

(H2) (i) The sequence of bandwidths $\{h_i, i \geq 1\}$ satisfies $0 < h_i \downarrow 0$ as $i \to \infty$.

(ii) If $h_n \to 0$, then $F(h_n) \to F(0) = 0$ as $n \to \infty$, and for all $s \in [0,1]$,

$$\tau_h(s) = \frac{F(hs)}{F(h)} \to \tau_0(s) < \infty \quad \text{as } h \to 0.$$

(H3) (i) $h_n \to 0$; $nF(h_n) \to \infty$; and

$$A_{n,l} = \frac{1}{n}\sum_{i=1}^{n} \frac{h_i}{h_n}\left(\frac{F(h_i)}{F(h_n)}\right)^{1-l} \longrightarrow \alpha_{[l]} < \infty, \quad \text{as } n \to \infty;$$

(ii) $\forall r \leq 2$,

$$B_{n,r} = \frac{1}{n}\sum_{i=1}^{n} \left(\frac{F(h_i)}{F(h_n)}\right)^{r} \longrightarrow \beta_{[r]} < \infty, \quad \text{when } n \to \infty.$$

(H4)

$$\lim_{n\to\infty} \frac{nF(h_n)(\ln n)^{-1-\frac{2}{\mu}}}{(\ln\ln n)^{2(\alpha+1)}} = \infty \quad \text{and} \quad \lim_{n\to\infty} (\ln n)^{\frac{2}{\mu}} F(h_n) = 0.$$

α is a real positive.

(H5) (i)

$$\int_{\mathbb{R}} [H(t)]^2 dt < \infty; \qquad \int_{\mathbb{R}} [H(t)]^2 |t|^{\beta_2} dt < \infty; \qquad \int_{\mathbb{R}} H(t) dt < \infty;$$

$$\int_{\mathbb{R}} H'(t)\, dt = 1, \qquad \int_{\mathbb{R}} [H'(t)] |t|^{\beta_2} dt < \infty.$$

(ii) For any $y \in \mathbb{R}$, $\forall (x_1, x_2) \in N_x^2$,

$$\left| F^{[x_1]}(y_1) - F^{[x_2]}(y_2) \right| \leq (d(x_1,x_2)^{\beta_1} + |y_1 - y_2|^{\beta_2}).$$

(iii) For any $y \in \mathbb{R}$, $\forall (x_1, x_2) \in N_x^2$,

$$\left| f^{[x_1]}(y_1) - f^{[x_2]}(y_2) \right| \leq (d(x_1,x_2)^{\beta_1} + |y_1 - y_2|^{\beta_2}),$$

with $\beta_1 > 0$, $\beta_2 > 0$, $d(x_1,x_2) = \|x_1 - x_2\|$, and N_x^2 a fixed neighborhood of x.

(H6) (i)

$$C_{n,l} = \frac{1}{n}\sum_{i=1}^{n} h_i^{2\beta_1}\left[\frac{F(h_i)}{F(h_n)}\right]^{1-l} \longrightarrow 0, \quad \text{as } n \to \infty$$

(ii) H is a square integrable function as

$$\sigma^2_{\varepsilon_i}(X) = \operatorname{Var}\left[H\left(\frac{y - Y_i}{h_i}\right) \,\Big|\, X \right] \longrightarrow \sigma^2_{\varepsilon}(X) = F^{[x]}(y)\big(1 - F^{[x]}(y)\big) \text{ when } i \longrightarrow \infty.$$

(iii) The function φ is derivable at 0.

$$\varphi(\|x - X_i\|) = \mathbb{E}\left\{ \left[\int_{\mathbb{R}} H'(t)\, F^{[x]}(y - h_i t)\, dt - F^{[x]}(y) \right] \|x - X_i\| \right\}$$

(iv) H is a square integrable function as

$$\theta^2_{\varepsilon_i}(X) = \operatorname{Var}\left[\frac{1}{h_i} H'\left(\frac{y - Y_i}{h_i}\right) \,\Big|\, X \right] \longrightarrow \theta^2_{\varepsilon}(X) = f^{[x]}(y) \int_{\mathbb{R}} (H'(t))^2\, dt; \text{ when } i \longrightarrow \infty.$$

(v) The function ϕ is derivable at 0.

$$\phi(\|x - X_i\|) = \mathbb{E}\left\{\left[\int_{\mathbb{R}} H'(t)\, f^{[x]}(y - h_i t)\, dt - f^{[x]}(y)\right] \|x - X_i\|\right\}.$$

(H7)

$$\exists \nu < \infty,\ \forall (t, y) \in \mathcal{N}_x \times \mathbb{R}, \quad f^{[x]}(y) \leq \nu;$$

(H8)

$$\exists \beta > 0,\ \forall (t, y) \in \mathcal{N}_x \times \mathbb{R}, \quad F^{[x]}(y) \leq 1 - \beta.$$

2.3 Almost sure convergence of the recursive kernel estimate of the conditional distribution function

In this section, we establish the almost sure convergence of the recursive estimators of the conditional distribution function $\widehat{F}_n^{[x,l]}(y)$.

The main result is stated in the following theorem.

Theorem 2.3.1. *[29]*
Assume that conditions (H1)–(H4), (H5)(i)–(ii), and (H6)(i)–(iii) hold. If

$$\lim_{n \to \infty} n h_n^2 = 0,$$

then

$$\limsup_{n \to \infty} \left(\frac{n F(h_n)}{\ln \ln n} \right)^{1/2} \left[\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y) \right] = \frac{\left[2 M_2\, \beta_{[1-2l]}\, F^{[x]}(y) \left(1 - F^{[x]}(y)\right) \right]^{1/2}}{M_1\, \beta_{[1-l]}} \quad a.s.$$

proof Let

$$F^{[x]}(y) = \frac{\varphi(x, y)}{f(x)}.$$

Then, the estimator can be written as

$$\widehat{F}_n^{[x,l]}(y) = \frac{\varphi_n^{[l]}(x, y)}{f_n^{[l]}(x)},$$

where $\varphi_n^{[l]}(x, y)$ and $f_n^{[l]}(x)$ are defined as above.

Then, let the following decomposition:

$$\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y) = \frac{\varphi_n^{[l]}(x, y) - F^{[x]}(y)\, f_n^{[l]}(x)}{f_n^{[l]}(x)}$$

The main idea is to show that $f_n^{[l]}(x)$ converges almost surely to $f^{[l]}(x)$ and that $\varphi_n^{[l]}(x, y) - F^{[x]}(y) f_n^{[l]}(x)$ converges almost surely to 0.

The numerator can be written

$$\varphi_n^{[l]}(x, y) - F^{[x]}(y)\, f_n^{[l]}(x) = \left\{ \varphi_n^{[l]}(x, y) - F^{[x]}(y)\, f_n^{[l]}(x) - \mathbb{E}\left[\varphi_n^{[l]}(x, y) - F^{[x]}(y)\, f_n^{[l]}(x) \right] \right\}$$
$$+ \left\{ \mathbb{E}\left[\varphi_n^{[l]}(x, y) - F^{[x]}(y)\, f_n^{[l]}(x) \right] \right\} = I_1 + I_2.$$

We begin by studying I_1. For this purpose,we set

$$W_i = \frac{1}{[F(h_i)]^l} K\left(\frac{\|x - X_i\|}{h_i}\right) \left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right]$$

$$Z_i = W_i - \mathbb{E}(W_i)$$

and

$$S_n = \sum_{i=1}^{n} Z_i.$$

Remark that

$$I_1 = \frac{S_n}{\sum_{i=1}^{n} [F(h_i)]^{1-l}}$$

Let $V_n = \sum_{i=1}^{n} Z_i^2$ we get

$$\begin{aligned} V_n = & \sum_{i=1}^{n} \operatorname{Var}(W_i) = \sum_{i=1}^{n} [F(h_i)]^{-2l} \left\{ \mathbb{E}\left(K^2\left(\frac{\|x - X_i\|}{h_i}\right) \left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right]^2 \right) \right\} \\ - & \sum_{i=1}^{n} [F(h_i)]^{-2l} \mathbb{E}^2 \left(K\left(\frac{\|x - X_i\|}{h_i}\right) \left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right] \right) \\ = & A_1 - A_2. \end{aligned}$$

A_1 is written as

$$A_1 = \sum_{i=1}^{n} [F(h_i)]^{-2l} \mathbb{E}\left\{ K^2\left(\frac{x - X_i}{h_i}\right) \mathbb{E}\left[\left(H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right)^2 \mid X_i \right] \right\}.$$

While

$$\begin{aligned} \mathbb{E}\left(\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right]^2 \mid X_i\right) &= \operatorname{Var}\left(\left[H\left(\frac{y - Y_i}{h_i}\right)\right] \mid X_i\right) \\ &+ \mathbb{E}^2\left(\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right] \mid X_i\right) \\ &= \sigma^2_{\varepsilon_i}(X) + \mathbb{E}^2\left(\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right] \mid X_i\right) \end{aligned}$$

Since

$$\begin{aligned} \mathbb{E}\left(\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right] \Big| X_i\right) &= \int_{\mathbb{R}} H'(t) \left[F^{[X_i]}(y - h_i t) - F^{[X_i]}(y)\right] dt \\ &+ \int_{\mathbb{R}} H'(t) \left[F^{[X_i]}(y) - F^{[x]}(y)\right] dt \\ &\leq O\left(h_i^{\beta_2}\right) + \left[F^{[X_i]}(y) - F^{[x]}(y)\right] \\ &\leq \|x - X_i\|^{\beta_1} \text{ as } i \to \infty. \end{aligned}$$

Let's note that the last inequality is ensured by (H5)(i),(ii) then we have

$$\mathbb{E}\left(\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right]^2 \Big| X_i\right) \leq \|x - X_i\|^{2\beta_1} + \sigma^2_{\varepsilon}(X)$$

In that case,

$$\mathbb{E}\left(K^2\left(\tfrac{\|x-X_i\|}{h_i}\right)\left[H\left(\tfrac{y-Y_i}{h_i}\right)-F^{[x]}(y)\right]^2\right)\leq\sigma_x^2(X)\mathbb{E}\left[K^2\left(\tfrac{\|x-X_i\|}{h_i}\right)\right]$$
$$+\mathbb{E}\left[\|x-X_i\|^{2\beta_1}K^2\left(\tfrac{\|x-X_i\|}{h_i}\right)\right]$$

But

$$\begin{aligned}\mathbb{E}\left[\|x-X_i\|^{2\beta_1}K^2\left(\frac{\|x-X_i\|}{h_j}\right)\right]&\leq\mathbb{E}\left[\sup_{X_i\in B(x,h_i)}\|x-X_i\|^{2\beta_1}K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right]\\&\leq h_i^{2\beta_1}\mathbb{E}\left[K^2\left(\frac{\|x-X_i\|}{h_j}\right)\right];\end{aligned}$$

where $B(x,h_i)$ is the closed ball with center x and radius h_i such that $B(x,h_i)=\{x'\in\mathcal{F}/\,\|x-x'\|\leq h_i\}$. Then we get

$$\begin{aligned}A_1&\leq\sum_{i=1}^{n}[F(h_i)]^{-2l}\left[\sigma_\varepsilon^2(X)+h_i^{2\beta_1}\right]\mathbb{E}\left[K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right]\\&=A_{11}+A_{12}\end{aligned}$$

By using equation (3.2) we get

$$A_{11}\leq\sigma_\varepsilon^2(X)\sum_{i=1}^{n}[F(h_i)]^{1-2l}\left[K^2(1)-\int_0^1\left(K^2(s)\right)'\tau_{h_1}(s)ds\right]$$

Now, using the hypothesis (H3) and applying Toeplitz' lemma, we obtain when n tends to infinity

$$\frac{A_{11}}{n\,[F(h_n)]^{1-2l}}\rightarrow\beta_{[1-2]]}\sigma_\varepsilon^2(X)M_2$$

When (H6)(i),(ii),(iii) is satisfied, we get When assumptions (H6)(i), (ii), and (iii) are satisfied, we obtain

$$\frac{A_{12}}{n[F(h_n)]^{1-2l}}\longrightarrow 0\quad\text{as } n\rightarrow\infty.$$

and

$$\frac{A_2}{n[F(h_n)]^{1-2l}}\longrightarrow 0\quad\text{as } n\rightarrow\infty.$$

Therefore, we can conclude that

$$V_n\sim n[F(h_n)]^{1-2l}\beta_{[1-2l]}\sigma_\varepsilon^2(X)M_2\quad\text{when } n\rightarrow\infty.$$

By assuming that $n[F(h_n)]\rightarrow\infty$ we obtain also

$$\frac{\ln(F(h_n))}{\ln n}\rightarrow 0\quad\text{when } n\rightarrow\infty.$$

It is clear then that

$$\mathbb{E}\big[\exp(\lambda|H|^\mu)\big]<\infty$$

for any λ and μ positive. This implies

$$\mathbb{E}\left(\max_{1\leq i\leq n}|U_i|^p\right)=O\left[(\ln n)^{\frac{p}{\mu}}\right],\quad\forall p\geq 1,\ n\geq 2.$$

where

$$U_i=H\left(\frac{y-Y_i}{h_i}\right).$$

By using the fact that

$$\lim_{n\to\infty}\frac{nF(h_n)(\ln n)^{-1-\frac{2}{\mu}}}{(\ln\ln n)^{2(\alpha+1)}}=\infty \quad \text{and} \quad \lim_{n\to\infty}(\ln n)^{\frac{2}{\mu}}F(h_n)=0.$$

α is a positive real.
We deduce that:

$$\lim_{n\to\infty}\frac{nF(h_n)(\ln n)^{-\frac{2}{\mu}}}{\ln\left[n(F(h_n))^{1-2l}\right]\left\{\ln\ln\left[n(F(h_n))^{1-2l}\right]\right\}^{2(\alpha+1)}}=\infty.$$

Let $b_n=(\delta lnn)^{\frac{1}{\mu}}$ with $\delta>0$. We will have the existence of $n_0\geq 1$ such that for all $i\geq n_0$

$$\frac{iF(h_i)(\ln i)^{-\frac{2}{\mu}}}{\ln\left[i(F(h_i))^{1-2l}\right]\left\{\ln\ln\left[i(F(h_i))\right]\right\}^{2(\alpha+1)}}>\frac{2\|K\|_\infty^2\max\left\{|F^x(y)|^2,(\delta lni)^{\frac{2}{\mu}}\right\}}{[F(h_i)]^{2l}}>Z_i^2$$

As the event

$$\left\{Z_i^2>\frac{i[F(h_i)]^{1-2l}}{\ln\left[i(F(h_i))^{1-2l}\right]\left\{lnln\left[i(F(h_i))\right]\right\}^{2(\alpha+1)}}\right\}$$

is impossible for $i\geq n_0$. From

$$V_n\sim n[F(h_n)]^{1-2l}\beta_{[1-2l]}\sigma_\varepsilon^2(X)M_2$$

we deduce that

$$\sum_{i=1}^{n}\frac{(\ln\ln V_i)^\alpha}{V_i}\mathbb{E}\left\{Z_i^2\mathbf{1}\left(\frac{V_i}{\ln(V_i)\left\{\ln\ln(V_i)\right\}^{2(\alpha+1)}}\right)\right\}<\infty.$$

Let S a random function defined on $[0,\infty[$, let

$$\text{for } t\in[V_n,V_{n+1}[,\quad S(t)=S_n.$$

Theorem A.2.1 in the Appendix of Amiri's thesis implies the existence of a Brownian motion ξ such that

$$\left|\frac{S(t)-\xi(t)}{(2t\ln\ln t)^{\frac{1}{2}}}\right|=o\left[(\ln\ln t)^{-\frac{\alpha}{2}}\right]\quad \forall t\in[V_n,V_{n+1}[.$$

But since, by Theorem A.3.1 in Amiri's thesis, the Brownian motion satisfies the law of the iterated logarithm, we obtain:

$$\limsup_{t\to\infty}\frac{S(t)}{(2t\ln\ln t)^{\frac{1}{2}}}=\limsup_{t\to\infty}\left[\frac{S(t)-\xi(t)}{(2t\ln\ln t)^{1/2}}+\frac{\xi(t)-S(t)}{(2t\ln\ln t)^{\frac{1}{2}}}\right]=1\quad a.s.$$

Then, we have

$$\frac{S_n}{(2V_n\ln\ln V_n)^{\frac{1}{2}}}\to 1\quad a.s.$$

By using the fact that

$$S_n=I_1\sum_{i=1}^{n}[F(h_i)]^{1-l}\quad \text{and}\quad \frac{V_{n+1}}{V_n}\to 1 \text{ when } n\to\infty,$$

We obtain

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n}[F(h_i)]^{1-l} I_1}{(2V_n \ln\ln V_n)^{\frac{1}{2}}} \frac{n(F(h_n))^{1-2l}\left(lnln[n[F(h_n))^{1-2l}]\right)^{\frac{1}{2}}}{n(F(h_n))^{1-2l}\left(lnln[n(F(h_n))^{1-2l}]\right)^{\frac{1}{2}}} = 1 \ a.s.$$

But

$$\sum_{i=1}^{n}[F(h_i)]^{1-l} = B_{n,(1-l)}\, n[F(h_n)]^{1-l}.$$

We have

$$\frac{\left(\ln\ln[n(F(h_n))^{1-2l}]\right)^{\frac{1}{2}} B_{n,(1-l)}}{(2V_n \ln\ln V_n)^{\frac{1}{2}}} \to \frac{\beta_{[1-l]}}{\left(2\beta_{[1-2l]}\sigma_\varepsilon^2(X)M_2\right)^{\frac{1}{2}}}, \quad \text{when } n\to\infty.$$

It comes, then:

$$\limsup_{n\to\infty} \left\{\frac{nF(h_n)}{\ln\ln[n(F(h_n))^{1-2l}]}\right\}^{\frac{1}{2}} I_1 = \sigma_l \ a.s.$$

with

$$\sigma_l = \frac{\left(2\beta_{[1-2l]}\sigma_\varepsilon^2(X)M_2\right)^{\frac{1}{2}}}{\beta_{[1-l]}}.$$

As

$$\ln\ln\left[n(F(h_n))^{1-2l}\right] = (\ln\ln n)[1+o(1)],$$

we conclude that

$$\limsup_{n\to\infty} \left(\frac{nF(h_n)}{\ln\ln n}\right)^{\frac{1}{2}} I_1 = \frac{\left(2\beta_{[1-2l]}\sigma_\varepsilon^2(X)M_2\right)^{1/2}}{\beta_{[1-l]}}.$$

Studying I_2 : We have to prove that

$$\limsup_{n\to\infty} \left\{\frac{nF(h_n)}{\ln\ln n}\right\}^{\frac{1}{2}} I_2 = 0.$$

We have Studying I_2 : We have to prove that

$$\overline{\lim_{n\to\infty}} \left\{\frac{nF(h_n)}{\ln\ln n}\right\}^{\frac{1}{2}} I_2 = 0.$$

We have

$$\begin{aligned}
I_2 &= \mathbb{E}\left[\varphi_n^{[l]}(x;y) - F^{[x]}(y) f_n^{[l]}(x)\right] \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{[F(h_i)]^{l}} \mathbb{E}\left\{\left[H\left(\frac{y-Y_i}{h_i}\right) - F^{[x]}(y)\right] K\left(\frac{\|x-X_i\|}{h_i}\right)\right\} \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{[F(h_i)]^{l}} \left\{h_i\varphi'(0)F(h_i)\left[K(1) - \int_0^1 (sK(s))'\tau_{h_i}(s)ds\right] + o(h_i)\right\}
\end{aligned}$$

The last equality above was obtained using equation (3.1). When n tends to infinity, based on hypothesis (H3), we have

$$I_2 \sim h_n\varphi'(0)\frac{\alpha_{[l]}}{\beta_{[1-l]}} M_0[1+o(1)].$$

Thus

$$\left\{\frac{nF(h_n)}{\ln\ln n}\right\}^{\frac{1}{2}} I_2 = \left\{\frac{nF(h_n)}{\ln\ln n}\right\}^{\frac{1}{2}} h_n\varphi'(0)\frac{\alpha_{[l]}}{\beta_{[1-l]}}M_0[1+o(1)] = o(1),$$

which is verified for $\lim_{n\to\infty} nh_n^2 = 0$

We conclude then

$$\limsup_{n\to\infty}\left\{\frac{nF(h_n)}{\ln\ln n}\right\}^{\frac{1}{2}} I_2 = 0.$$

Thus

$$\left\{\frac{nF(h_n)}{\ln\ln n}\right\}^{\frac{1}{2}}\left[\varphi_n^{[l]}(x,y) - F^{[x]}(y)f_n^{[l]}(x)\right] \to \frac{\left\{2\beta_{[1-2l]}\sigma_\varepsilon^2(X)M_2\right\}^{\frac{1}{2}}}{\beta_{[1-l]}}.$$

We now show the almost sure convergence of $f_n^{[l]}(x)$ to $f^{[l]}(x)$ in order to deduce that of $\widehat{F}_n^{[x,l]}(y)$ to $F^{[x]}(y)$.

In the same way, by letting $Z_i = W_i - \mathbb{E}(W_i)$ we can prove:

$$f_n^{[l]}(x) - \mathbb{E}f_n^{[l]}(x) = O\left(\sqrt{\frac{\ln\ln n}{nF(h_n)}}\right) \quad \text{a.s.}$$

As $\mathbb{E}\left[f_n^{[l]}(x)\right] = M_1[1+o(1)]$, $f_n^{[l]}(x)$ almost converge surely to M_1, since one can write

$$f_n^{[l]}(x) = \left[f_n^{[l]}(x) - \mathbb{E}f_n^{[l]}(x)\right] + \mathbb{E}\left[f_n^{[l]}(x)\right].$$

This completes the proof.

2.4 Mean square Convergence of the Recursive Kernel Estimator of the Conditional Distribution Function

We now present a theorem concerning the mean square convergence of the estimator of the conditional distribution function.

Theorem 2.4.1. *[29]*

Assume that conditions (H1)–(H4), (H5)(i)–(ii), and (H6)(i)–(iii) are satisfied. If there exists a constant $c > 0$ such that

$$nF(h_n)h_n^2 \longrightarrow c \quad \text{as } n \to \infty,$$

then we have

$$\lim_{n\to\infty} nF(h_n)\,\mathbb{E}\left[\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y)\right]^2 = \frac{\beta_{[1-2l]}}{\beta_{[1-l]}^2}\frac{M_2}{M_1^2}F^{[x]}(y)\big(1-F^{[x]}(y)\big) + c\,[\varphi'(0)]^2\frac{\alpha_{[l]}^2}{\beta_{[1-l]}^2}\frac{M_0^2}{M_1^2}.$$

Proof: It is known

$$\mathbb{E}\left[\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y)\right]^2 = \mathrm{Var}\left[\widehat{F}_n^{[x,l]}(y)\right] + \mathbb{E}^2\left[\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y)\right] = E_1 + E_2.$$

In this part, we will use the following decomposition for the calculation of E_2 :

$$\mathbb{E}\left[\widehat{F}_n^{[x,l]}(y)\right] = \frac{\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right]}{\mathbb{E}\left[f_n^{[l]}(x)\right]} - \frac{\mathbb{E}\left\{\left[f_n^{[l]}(x) - \mathbb{E}f_n^{[l]}(x)\right]\varphi_n^{[l]}(x,y)\right\}}{\left\{\mathbb{E}[f_n^{[l]}(x)]\right\}^2} + \frac{\mathbb{E}\left\{\left[f_n^{[l]}(x) - \mathbb{E}f_n^{[l]}(x)\right]^2\widehat{F}_n^{[x,l]}(y)\right\}}{\left\{\mathbb{E}[f_n^{[l]}(x)]\right\}^2}.$$

For the calculation of E_1 then we use the following decomposition of the variance that can be found in Collomb (1976):

$$\mathrm{Var}\Big[\widehat{F}_n^{[x,l]}(y)\Big] = \frac{\mathrm{Var}\big[\varphi_n^{[l]}(x,y)\big]}{\big\{\mathbb{E}[f_n^{[l]}(x)]\big\}^2} - 4\frac{\mathbb{E}[\varphi_n^{[l]}(x,y)]\mathrm{Cov}\big[f_n^{[l]}(x),\varphi_n^{[l]}(x,y)\big]}{\big\{\mathbb{E}[f_n^{[l]}(x)]\big\}^3}$$
$$+3\mathrm{Var}\big[f_n^{[l]}(x)\big]\frac{\big\{\mathbb{E}[\varphi_n^{[l]}(x,y)]\big\}^2}{\big\{\mathbb{E}[f_n^{[l]}(x)]\big\}^4} + o\bigg(\frac{1}{nF(h_n)}\bigg).$$

Studying the convergence of E_2:

Let us start by studying

$$\frac{\mathbb{E}[\varphi_n^{[l]}(x,y)]}{\mathbb{E}[f_n^{[l]}(x)]} - F^{[x]}(y):$$

One observes that

$$\frac{\mathbb{E}[\varphi_n^{[l]}(x,y)]}{\mathbb{E}[f_n^{[l]}(x)]} - F^{[x]}(y) = \frac{\displaystyle\sum_{i=1}^n \frac{1}{[F(h_i)]^l}\mathbb{E}\bigg\{\Big[H\Big(\frac{y-Y_i}{h_i}\Big) - F^{[x]}(y)\Big]K\Big(\frac{\|x-X_i\|}{h_i}\Big)\bigg\}}{\displaystyle\sum_{i=1}^n \frac{1}{[F(h_i)]^l}\mathbb{E}\bigg[K\Big(\frac{\|x-X_i\|}{h_i}\Big)\bigg]}.$$

Let $\varphi(t) = \mathbb{E}\Big\{\big[\int_{\mathbb{R}} H'(t)F^{[X]}(y-h_it)\,dt - F^{[x]}(y)\big] \,\Big|\, \|x-X\| = t\Big\}$.
Suppose that the function φ is derivable at point $t=0$.
By (H6)(iii), evidently

$$\mathbb{E}\bigg\{\Big[H\Big(\frac{y-Y_i}{h_i}\Big) - F^{[x]}(y)\Big]K\Big(\frac{\|x-X_i\|}{h_i}\Big)\bigg\} = \mathbb{E}\bigg[\varphi(\|x-X_i\|)K\Big(\frac{\|x-X_i\|}{h_i}\Big)\bigg]$$
$$= \int_0^1 \varphi(h_it)K(t)\,dP^{\|x-x_i\|/h_i}(t).$$

So using the Taylor expansion for φ around 0, one obtains

$$\mathbb{E}\bigg\{\Big[H\Big(\frac{y-Y_i}{h_i}\Big) - F^{[x]}(y)\Big]K\Big(\frac{\|x-X_i\|}{h_i}\Big)\bigg\} = h_i\varphi'(0)\int_0^1 tK(t)\,dP^{\|x-x_i\|/h_i}(t) + o[h_i]. \tag{3.1}$$

Based on the proof of Lemma 2 in Ferraty et al. (2007), assumption (H1), and Fubini's theorem, we obtain

$$\int_0^1 tK(t)\,dP^{\|x-x_i\|/h_i}(t) = F(h_i)\left[K(1) - \int_0^1 (sK(s))'\tau_{h_i}(s)\,ds\right]$$

and

$$\mathbb{E}\bigg[K\Big(\frac{\|x-X_i\|}{h_i}\Big)\bigg] = \int_0^{h_i} K\Big(\frac{t}{h_i}\Big)\,dP^{\|x-x_i\|}(t) = F(h_i)\left[K(1) - \int_0^1 (K(s))'\tau_{h_i}(s)\,ds\right].$$

Then, by (H1), we obtain

$$\frac{\mathbb{E}[\varphi_n^{[l]}(x,y)]}{\mathbb{E}[f_n^{[l]}(x)]} - F^{[x]}(y) = \frac{\displaystyle\sum_{i=1}^n h_i[F(h_i)]^{1-l}\left\{\varphi'(0)\left[K(1) - \int_0^1 (sK(s))'\tau_{h_i}(s)\,ds\right] + \gamma_i\right\}}{\displaystyle\sum_{i=1}^n [F(h_i)]^{1-l}\left[K(1) - \int_0^1 (K(s))'\tau_{h_i}(s)\,ds\right]} = \frac{D_1}{D_2}.$$

Finally, assumptions (H2) and (H3), together with Toeplitz's lemma (see Masry (1986)), allow us to obtain

$$\frac{D_1}{nh_n[F(h_n)]^{1-l}} = \alpha_{[l]}\varphi'(0)M_0[1+o(1)],$$

$$\frac{D_2}{n[F(h_n)]^{1-l}} = \beta_{[1-l]}M_1[1+o(1)],$$

and

$$\frac{\mathbb{E}[\varphi_n^{[l]}(x,y)]}{\mathbb{E}[f_n^{[l]}(x)]} - F_{[x]}(y) = h_n\varphi'(0)\frac{\alpha_{[l]}}{\beta_{[1-l]}}\frac{M_0}{M_1}[1+o(1)].$$

It is noted that the convergence of the other terms of the decomposition for calculating E_2 is a consequence of the terms of the variance.

Therefore, we establish the convergence of the variance. We have

$$\begin{aligned}
\mathbb{E}\left[f_n^{[l]}(x)\right] &= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}}\sum_{i=1}^{n}\frac{1}{[F(h_i)]^{l}}\mathbb{E}\left[K\left(\frac{\|x-X_i\|}{h_i}\right)\right] \\
&= \frac{\sum_{i=1}^{n}\frac{[F(h_i)]^{1-l}}{n[F(h_n)]^{1-l}}\left[K(1)-\int_0^1 (K(s))'\tau_{h_i}(s)ds\right]}{B_{n,(1-l)}} \\
&= M_1[1+o(1)]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right] &= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}}\sum_{i=1}^{n}\frac{1}{[F(h_i)]^{l}}\mathbb{E}\left[H\left(\frac{y-Y_i}{h_i}\right)K\left(\frac{\|x-X_i\|}{h_i}\right)\right] \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}}\sum_{i=1}^{n}\frac{\mathbb{E}\left\{\left[\int_{\mathbb{R}}H'(t)F^{[X]}(y-h_it)\,dt - F^{[X]}(y)+F^{[X]}(y)\right]K\left(\frac{\|x-X_i\|}{h_i}\right)\right\}}{[F(h_i)]^{l}} \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}}\sum_{i=1}^{n}\frac{1}{[F(h_i)]^{l}}\mathbb{E}\left\{\left[O\left(h_i^{\beta_2}\right)+F^{[X]}(y)\right]K\left(\frac{\|x-X_i\|}{h_i}\right)\right\} \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}}\sum_{i=1}^{n}\frac{1}{[F(h_i)]^{l}}F(h_i)M_1\left[F^{[X]}(y)+O\left(h_i^{\beta_2}\right)\right] \\
&= F^{[X]}(y)M_1[1+o(1)]
\end{aligned}$$

Concerning variances and covariance, we have

$$\mathbb{E}\left[K\left(\frac{\|x-X_i\|}{h_i}\right)\right] = \int_0^{h_i}K\left(\frac{t}{h_i}\right)dP^{\|x-x_i\|}(t) = F(h_i)\left[K(1)-\int_0^1 (K(s))'\tau_{h_i}(s)\,ds\right]$$

then

$$\mathbb{E}^2\left[K\left(\frac{\|x-X_i\|}{h_i}\right)\right] = O\big([F(h_i)]^2\big).$$

As far as

$$\mathbb{E}\left[K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right] = F(h_i)\left[K^2(1) - \int_0^1 (K^2(s))'\tau_{h_i}(s)\,ds\right] \tag{3.2}$$

thus

$$\mathrm{Var}\left[K\left(\frac{\|x-X_i\|}{h_i}\right)\right] = M_2 F(h_i)\,[1+\gamma_i]$$

with $\gamma_i = O(F(h_i))$.

Thus

$$\begin{aligned}
\mathrm{Var}[f_n^{[l]}(x)] &= \left[\frac{1}{\sum_{i=1}^n [F(h_i)]^{1-l}}\right]^2 \sum_{i=1}^n \left[\frac{1}{[F(h_i)]^l}\right]^2 M_2 F(h_i)\,[1+\gamma_i] \\
&= \frac{1}{\left(\sum_{i=1}^n [F(h_i)]^{1-l}\right)^2} \sum_{i=1}^n [F(h_i)]^{1-2l} M_2\,[1+\gamma_i] \\
&= \frac{\beta_{[1-2l]}}{\beta_{[1-l]}^2}\frac{1}{nF(h_n)} M_2[1+o(1)].
\end{aligned}$$

Then

$$\mathrm{Var}\big[\varphi_n^{[l]}(x,y)\big] = \left[\frac{1}{\sum_{i=1}^n [F(h_i)]^{1-l}}\right]^2 \sum_{i=1}^n \left[\frac{1}{[F(h_i)]^l}\right]^2 \mathrm{Var}\left[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\right]$$

with

$$\mathrm{Var}\left[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\right] = \mathbb{E}\left[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\right]^2 - \mathbb{E}^2\left[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\right].$$

As

$$\mathbb{E}^2\left[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\right] = O\big([F(h_i)]^2\big)$$

and

$$\mathbb{E}\left[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\right]^2 = \mathbb{E}\left\{K^2\left(\frac{\|x-X_i\|}{h_i}\right)\mathbb{E}^2\left[H\left(\frac{y-Y_i}{h_i}\right) \mid X\right]\right\} + \mathbb{E}\left\{\sigma_{\varepsilon_i}^2(X) K\left(\frac{\|x-X_i\|}{h_i}\right)\right\}$$

with

$$\mathbb{E}^2\left[H\left(\frac{y-Y_i}{h_i}\right) \mid X\right] = O(h_i^{\beta_2}) + [F^{[X]}(y)]^2$$

and

$$\sigma_{\varepsilon_i}^2(X) = \mathrm{Var}\left[H\left(\frac{y-Y_i}{h_i}\right) \mid X\right].$$

Thus we have, by (H6)(ii)

$$\operatorname{Var}\left[\varphi_n^{[l]}(x,y)\right] = \frac{\sum_{i=1}^{n}[F(h_i)]^{-2l}}{\left[\sum_{i=1}^{n}[F(h_i)]^{1-l}\right]^2} M_2 F(h_i)\left[\left(F^{[X]}(y)\right)^2+\sigma_\varepsilon^2(X)\right][1+\gamma_i]$$

$$= \frac{\beta_{[1-2l]}}{\beta_{[1-l]}^2}\left[\left(F^{[X]}(y)\right)^2+\sigma_\varepsilon^2(X)\right]\frac{1}{nF(h_n)}M_2[1+o(1)]$$

with $\gamma_i = o(h_i)$.

Finally

$$\operatorname{Cov}\left[f_n^{[l]}(x),\varphi_n^{[l]}(x,y)\right] = \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{H\left(\frac{y-Y_i}{h_i}\right)K\left(\frac{\|x-X_i\|}{h_i}\right)K\left(\frac{\|x-X_j\|}{h_j}\right)}{[F(h_i)]^l[F(h_i)]^l}\right]\right.$$

$$\left.-\sum_{i=1}^{n}\frac{\mathbb{E}\left[H\left(\frac{y-Y_i}{h_i}\right)K\left(\frac{\|x-X_i\|}{h_i}\right)\right]}{[F(h_i)]^l}\sum_{j=1}^{n}\frac{\mathbb{E}\left[K\left(\frac{\|x-X_j\|}{h_j}\right)\right]}{[F(h_j)]^l}\right\}$$

$$= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2}\sum_{i=1}^{n}\frac{\mathbb{E}\left[H\left(\frac{y-Y_i}{h_i}\right)K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right]}{[F(h_i)]^{2l}}$$

$$-\frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2}\sum_{i=1}^{n}\frac{\mathbb{E}\left[H\left(\frac{y-Y_i}{h_i}\right)K\left(\frac{\|x-X_i\|}{h_i}\right)\right]\mathbb{E}\left[K\left(\frac{\|x-X_i\|}{h_i}\right)\right]}{[F(h_i)]^{2l}}$$

$$= I - II$$

with

$$II = O\left(\frac{1}{n}(B_{n,1-l})^{-2}B_{n,2(1-l)}\right) = O\left(\frac{1}{nF(h_n)}\right)$$

and as

$$\mathbb{E}\left[H\left(\frac{y-Y_i}{h_i}\right)K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right] = F(h_i)M_2\left[F^{[X]}(y)+\gamma_i\right]$$

with $\gamma_i = o(h_i)$.

$$I = \frac{(B_{n,1-l})^{-2}}{nF(h_n)}\sum_{i=1}^{n}\frac{[F(h_i)]^{1-2l}}{n[F(h_n)]^{1-2l}}M_2F^{[X]}(y)[1+\gamma_i].$$

Then

$$\operatorname{Cov}\left[f_n^{[l]}(x),\varphi_n^{[l]}(x,y)\right] = \frac{\beta_{[1-2l]}}{\beta_{[1-l]}^2}F^{[X]}(y)M_2\frac{1}{nF(h_n)}[1+o(1)].$$

Finally, we have

$$\mathrm{Var}\Big[\widehat{F}_n^{[x,l]}(y)\Big] = \frac{\beta_{[1-2l]}}{\beta_{[1-l]}^2}\frac{M_2}{M_1^2}\sigma_\varepsilon^2(X)\frac{1}{nF(h_n)}[1+o(1)].$$

Given

$$\mathbb{E}\Big\{\big[f_n^{[l]}(x) - \mathbb{E}f_n^{[l]}(x)\big]\varphi_n^{[l]}(x,y)\Big\} = O\bigg(\frac{1}{nF(h_n)}\bigg)$$

and

$$\mathbb{E}\Big\{\big[f_n^{[l]}(x) - f(x)\big]^2\widehat{F}_n^{[x,l]}(y)\Big\} = O\bigg(\frac{1}{nF(h_n)}\bigg),$$

we get

$$\mathbb{E}\Big[\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y)\Big] = h_n\varphi'(0)\frac{\alpha_{[l]}}{\beta_{[1-l]}}\frac{M_0}{M_1}[1+o(1)] + O\bigg(\frac{1}{nF(h_n)}\bigg).$$

The proof takes end here. □

Chapter 3

Recursive kernel estimation of conditional distribution function under strong mixing conditions

In this chapter, we study the recursive estimation of the conditional distribution function and the conditional density function for dependent functional data. Contrary to the independent case, the observations are assumed to satisfy dependence conditions of the α-mixing type. We also investigate the asymptotic properties of the proposed estimators, including the almost sure convergence and the convergence in square mean under suitable assumptions.

3.1 model

Let $(X_1, Y_1), (X_2, Y_2), \dots (X_n, Y_n)$ n pairs of random variables with the observations (X_i), $i = 1, \dots, n$ are dependents of type strongly mixing, as (x, y) witch is a random pair valued in $\mathcal{F} \times \mathbb{R}$, where $(\mathcal{F}, d(;))$ is a semi-metric space and $d(x; X_i) = \|x - X_i\|$.

The conditional distribution function is defined by:

$$\widehat{F}^{[X]}(y) = \frac{\displaystyle\sum_{i=1}^{n} \frac{1}{[F(h_i)^l]} K\left(\frac{\|x - X_i\|}{h_i}\right) H\left(\frac{y - Y_i}{h_i}\right)}{\displaystyle\sum_{i=1}^{n} \frac{1}{[F(h_i)^l]} K\left(\frac{\|x - X_i\|}{h_i}\right)},$$

where K is a kernel, H is a distribution function h_n a sequence of positive reals and l is a parameter in $[0, 1]$, $F(h_i) = \mathbb{P}(\|x - X_i\| \leq h_i)$.

Our family of recursive estimators is defined by:

$$\widehat{F}_n^{[X]}(y) = \frac{\left[\displaystyle\sum_{i=1}^{n} F(h_i)\right]^{1-l} \varphi_n^l(y) + \left[\displaystyle\sum_{i=1}^{n+1} F(h_i)\right]^{1-l} H\left(\frac{y - Y_i}{h_i}\right) K_{n+1}^{[l]}(\|x - X_i\|)}{\left[\displaystyle\sum_{i=1}^{n} F(h_i)\right]^{1-l} G_n^l(y) + \left[\displaystyle\sum_{i=1}^{n+1} F(h_i)\right]^{1-l} K_{n+1}^{[l]}(\|x - X_i\|)},$$

with

$$\varphi_n^{[l]}(x, y) = \frac{\displaystyle\sum_{i=1}^{n} \frac{1}{[F(h_i)]^{1-l}} H\left(\frac{y - Y_i}{h_i}\right) K\left(\frac{\|x - X_i\|}{h_i}\right)}{\displaystyle\sum_{i=1}^{n} [F(h_i)]^{1-l}},$$

$$G_n^{[l]}(x) = \frac{\sum_{i=1}^{n} \frac{1}{[F(h_i)]^{1-l}} K\left(\frac{\|x - X_i\|}{h_i}\right)}{\sum_{i=1}^{n} [F(h_i)]^{1-l}},$$

and

$$K_i^{[l]}(\cdot) = \frac{1}{[F(h_i)]^l \sum_{j=1}^{i} [F(h_i)]^{1-l}} K\left(\frac{\cdot}{h_i}\right)$$

3.1.1 Hypothesis

we keep the same assumptions introduced in the independent case (i.i.d). However, in order to study the asymptotic properties of the estimator under dependence, we add some additional assumptions related to the α-mixing structure of the data. More precisely, all hypotheses previously stated remain valid, except that we supplement them with the conditions ensuring the strong mixing dependence framework.

(H_0)

(i) $\forall h_i > 0, \mathbb{P}(X \in B(x, h_i)) =: F(h_i)$ where $B(x, h_i) = \{x' \in \mathcal{F}/d(x, x' < h_i)\}$

(ii) $(X_i)_{i \in \mathbb{N}^*}$ is an $\alpha-$mixing sequence whose the coefficients of mixture verify:

$$\exists a > 0, \exists c > 0 : \forall n \in \mathbb{N}, \alpha(n) \leq cn^{-a}.$$

(iii) $0 < \sup_{i \neq j} \mathbb{P}((X_i, X_j) \in B(x, h_i) \times B(x, h_j)) = \partial\left(\frac{(F(h_i))^{\frac{a+1}{a}}}{n^{\frac{1}{a}}}\right).$

Note that $H_0(i)$ can be interpreted as a concentration hypothesis acting on the distribution of the f.r.v, X where as $H_0(iii)$ concerns the behavior of the joint distribution of the pairs (X_i, X_j). In the fact this hypothesis is equivalent to suppose that for n large enough

$$\sup_{i \neq j} \frac{\mathbb{P}((X_i, X_j) \in B(x, h_i) \times B(x, h_j))}{\mathbb{P}(X \in B(x, h))} \leq C\left(\frac{F(h_i)}{n}\right)^{\frac{1}{a}}.$$

3.2 Almost sure convergence of the recursive kernel estimate

Theorem 3.1. *[29] Under hypothesis $H_0(i)(ii)(iii)$; $H_1 - H_4$; $H_5(i)(ii)$ and $H_6(i)(ii)(iii)$ and if $\lim_{n \longrightarrow \infty} nh_n^2 = 0$, then*

$$\lim \sup_{n \longrightarrow \infty} \left[\frac{nF(h_n)}{\ln \ln n}\right]^{\frac{1}{2}} [\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y)] = \frac{[2M_2\beta_{1-2l}F^{[x]}(y)(1 - F^{[x]}(y))]^{\frac{1}{2}}}{M_1\beta_{1-l}}.$$

Proof: Let $F^{[x]}(y) = \frac{\phi(x, y)}{G(x)}$, this later can be written as

$$\widehat{F}_n^{[x,l]}(y) = \frac{\phi_n^{[l]}(x, y)}{G_n^{[l]}(x)},$$

let the following decomposition:

$$\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y) = \frac{\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x)}{G_n^{[l]}(x)}.$$

The idea is to show that $G_n^{[l]}(x)$ converges almost surely to $G^{[l]}(x)$ and that $\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x)$ converges almost surely to 0.

The numerator can be written

$$\begin{aligned}\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x) &= \{\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x) - \mathbb{E}[\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x)]\} \\ &+ \{\mathbb{E}[\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x)]\} \\ &= I_1 + I_2.\end{aligned}$$

We starting by studying I_1. For this purpose, we set:

$$W_i = \frac{1}{[F(h_i)^l]} K\left(\frac{\|x - X_i\|}{h_i}\right)\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right].$$

$$Z_i = W_i - \mathbb{E}(W_i)$$

and

$$S_n = \sum_{i=1}^{n} Z_i.$$

Remark that

$$I_1 = \frac{S_n}{\sum_{i=1}^{n}[F(h_i)]^{1-l}}.$$

Let $V_n = \sum_{i=1}^{n} \mathbb{E}(Z_i)^2$

$$\begin{aligned}V_n &= \sum_{i=1}^{n} Var(W_i) \\ &= \sum_{i=1}^{n}[F(h_i)]^{-2l}\left\{\mathbb{E}\left(K^2\left(\frac{\|x - X_i\|}{h_i}\right)\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right]^2\right)\right\} \\ &\quad -\sum_{i=1}^{n}[F(h_i)]^{-2l}\left\{\mathbb{E}^2\left(K\left(\frac{\|x - X_i\|}{h_i}\right)\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right]\right)\right\} \\ &\quad -\sum_{i \neq j}^{n} Cov\Big(W_i, W_j\Big) \\ &= \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3\end{aligned}$$

A_1 is written as

$$A_1 = \sum_{i=1}^{n}[F(h_i)]^{-2l}\left\{\mathbb{E}\left(K^2\left(\frac{\|x - X_i\|}{h_i}\right)\right)\mathbb{E}\left(\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right]^2 \mid X_i\right)\right\}.$$

As

$$\begin{aligned}\mathbb{E}\left(\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right]^2 \mid X_i\right)\Bigg\} &= Var\left(\left[H\left(\frac{y - Y_i}{h_i}\right)\right] \mid X_i\right) \\ &+ \mathbb{E}^2\left(\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right] \mid X_i\right) \\ &= \sigma_{\varepsilon_i}(X) + \mathbb{E}^2\left(\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right] \mid X_i\right).\end{aligned}$$

Since

$$\begin{aligned}\mathbb{E}\Big(\Big[H\Big(\frac{y-Y_i}{h_i}\Big)-F^{[x]}(y)\Big]\mid X_i\Big) &= \int_{\mathbb{R}} H'(t)\Big[F^{[x_i]}(y-h_it)-F^{[x_i]}(y)\Big]dt\\ &+ \int_{\mathbb{R}} H'(t)\Big[F^{[x_i]}(y)-F^{[X]}(y)\Big]dt\\ &\leq O(h_i^{\beta_2})+F^{[x_i]}(y)-F^{[X]}(y)\\ &\leq \|x-X_i\|^{\beta_i} as\ i\longrightarrow\infty.\end{aligned}$$

Under l'hypothesis $5(i)(ii)$ we have

$$\mathbb{E}\Big(K^2\Big(\frac{\|x-X_i\|}{h_i}\Big)\Big[H\Big(\frac{y-Y_i}{h_i}\Big)-F^{[x]}(y)\Big]^2\Big)\leq \|x-X_i\|^{2\beta_1}+\sigma_{\varepsilon_i}.$$

In these case,

$$\begin{aligned}\mathbb{E}\Big(K^2\Big(\frac{\|x-X_i\|}{h_i}\Big)^2\Big[H\Big(\frac{y-Y_i}{h_i}\Big)-F^{[x]}(y)\Big]^2\Big) &\leq \sigma_\varepsilon(X)\mathbb{E}\Big[K^2\Big(\frac{\|x-X_i\|}{h_i}\Big)\Big]\\ &+\mathbb{E}\Big[\|x-X_i\|^{2\beta_1}K^2\Big(\frac{\|x-X_i\|}{h_i}\Big)\Big],\end{aligned}$$

where

$$\begin{aligned}\mathbb{E}\Big[\|x-X_i\|^{2\beta_1}K^2\Big(\frac{\|x-X_i\|}{h_i}\Big)\Big] &\leq \mathbb{E}\Big[\sup_{X_i\in B(x,h_i)}\|x-X_i\|^{2\beta_1}K^2\Big(\frac{\|x-X_i\|}{h_i}\Big)\Big]\\ &\leq h_i^{2\beta_1}\mathbb{E}\Big[K^2\Big(\frac{\|x-X_i\|}{h_i}\Big)\Big],\end{aligned}$$

with $B(x,h_i)$ is the closed ball with center x ad radius h_i such that
$B(x,h_i)=\{x'\in\mathcal{F}/\|x-x'\|\leq h_i\}$, then we get

$$\begin{aligned}A_1 &\leq \sum_{i=1}^n[F(h_i)]^{-2l}[\sigma_\varepsilon(X)+h_i^{2\beta_1}]\mathbb{E}\Big[K^2\Big(\frac{\|x-X_i\|}{h_i}\Big)\Big]\\ &= A_{11}+A_{12}.\end{aligned}$$

We get

$$A_{11}\leq\sigma_\varepsilon(X)\sum_{i=1}^n[F(h_i)]^{1-2l}\Big[K^2(1)-\int_0^1(K^2(s))'\tau_{h_i}(s)ds\Big].$$

Under the hypothesis H_3 and applying Toeplitz lemma we obtain

$$\frac{A_{11}}{n[F(h_n)]^{1-2l}}\longrightarrow\beta_{[1-2l]}\sigma_\varepsilon^2(X)M_2,$$

and under $H_6(i)(ii)(iii)$, we get

$$\frac{A_{12}}{n[F(h_n)]^{1-2l}}\longrightarrow 0$$

and

$$\frac{A_2}{n[F(h_n)]^{1-2l}}\longrightarrow 0.$$

Now we studying A_3 .

$$\begin{aligned}A_3 &= \sum_{i\neq j}^n Cov(W_i,W_j)\\ &= \sum_{i\neq j}^n F(h_i)^{-2l}Cov(N_i,N_j),\end{aligned}$$

with $N_i = K_i H_i$ where $K_i = (h_i^{-1} K(\|x - X_i\|))$; $H_i = (h_i^{-1} K(y - Y_i))$.

Because H is a commultative kernel we have $H_i \leq 1$. By using systematically this fact to bound the variables H_i we get

$$Cov(N_i, N_j) = Cov(\Delta_i, \Delta_j),$$

with $\Delta_i = K_i - \mathbb{E}(K_i)$.

On one hand, we have by the hypothesis $H_0(i), H_0(iii)$ and H_1

$$|Cov(\Delta_i, \Delta_j)| = O\left(\left(\frac{\phi_x(h_i)}{n}\right)^{\frac{1}{a}} \phi_x(h_i)\right),$$

these covariance can be controlled by means of the usual Davydov's covariance inequality for mixing processes (see Rio (2000)[46], formula 1.12a) to get her with $H_0(ii)$ this inequality leads to:

$$\forall i \neq j \quad |Cov(\Delta_i, \Delta_j)| \leq C|i - j|^{-a}.$$

By the fact

$$\sum_{K \geq C_n + 1} K^{-a} \leq \int_{C_n}^{\infty} t^{-a} dt = \frac{C_n^{-a+1}}{a-1},$$

thus by using the following classical technique (see Bosq (2000)[9]) we can write

$$S_n^{cov} = \sum_{0 < |i-j| < u_n} |Cov(\Delta_i, \Delta_j)| + \sum_{|i-j| > u_n} |Cov(\Delta_i, \Delta_j)|.$$

Thus

$$S_n^{cov} \leq C_n \left(\frac{\phi_x(h_i)}{n}\right)^{\frac{1}{a}} \phi_x(h_i) + \frac{C_n^{a+1}}{a-1},$$

choosing $C_n = \left(\frac{\phi_x(h_i)}{n}\right)^{\frac{-1}{a}}$ we can deduce

$$S_n^{cov} = O(nF(h_i)),$$

$$A_3 = \frac{S_n^{cov}}{nF(h_i)^{2l}} \longrightarrow 0 \text{ where } n \longrightarrow \infty.$$

Therefore we can conclude that
$V_n \sim n[F(h_n)]^{1-2l} \beta_{[1-2l]} \sigma_\varepsilon^2(x) M_2$ when $n \longrightarrow \infty$.
By assuming that $n[F(h_n)] \longrightarrow \infty$ we obtain $\frac{\ln F(h_n)}{\ln n} \longrightarrow 0$ when $n \longrightarrow \infty$ It is clear that

$$\mathbb{E}[\exp(\lambda |H|^{\mu})] < \infty,$$

for any λ and μ positive this implies

$$\mathbb{E}(\max_{1 \leq i \leq n |U_i|^p}) = O[(\ln n)^{\frac{p}{\mu}}], \quad p \geq 1, n \geq 2,$$

where $H_i = H\left(\frac{y - Y_i}{h_i}\right)$.

By using the fact that

$$\lim_{n \longrightarrow \infty} \frac{nF(h_n)(\ln n)^{-1-\frac{2}{\mu}}}{(\ln \ln n)^{2(\alpha+1)}} = \infty \quad \textit{and} \quad \lim_{n \longrightarrow \infty} (\ln n)^{\frac{2}{\mu}} F(h_n) = 0,$$

α is a positive real.

We deduce that:

$$\lim_{n\longrightarrow\infty}\frac{nF(h_n)(\ln n)^{-\frac{2}{\mu}}}{\ln[n(F(h_n))^{1-2l}]\{\ln\ln[nF(h_n))^{1-2l}]\}^{2(\alpha+1)}}=\infty.$$

Let $b_n=(\delta\ln n)^{\frac{1}{\mu}}$ with $\delta>0$. We will have the existence of $n_0\geq 1$ such that for all $i\geq n_0$

$$\frac{iF(h_i)(\ln i)^{-\frac{2}{\mu}}}{\ln[i(F(h_i))^{1-2l}]\{\ln\ln[iF(h_i))^{1-2l}]\}^{2(\alpha+1)}}>\frac{2\|K\|_\infty^2\max\{|F^{[x]}(y)|^2,(\delta\ln i)^{\frac{2}{\mu}}\}}{[F(h_i)]^{2l}}\geq Z_i^2.$$

As the event $Z_i^2>\dfrac{i[F(h_i)]^{1-2l}}{\ln[i(F(h_i))^{1-2l}]\{\ln\ln[iF(h_i))^{1-2l}]\}^{2(\alpha+1)}}$ is impossible, for $i\geq n_0$. From $V_n\sim n[F(h_n)]^{1-2l}\beta_{[1-2l]}\sigma_\varepsilon^2(x)M_2$, we deduce that

$$\sum_{i=1}^{n}\frac{\ln\ln V_i^\alpha}{V_i}\mathbb{E}\left(Z_i^2\mathbf{1}_{\left\{\frac{V_i}{\ln[V_i]\{\ln\ln[V_i]\}^{2(\alpha+1)}}\right\}}\right)\leq\infty.$$

Let S a random function defined on $[0,\infty[$, let

$$for\quad t\in[V_n,V_{n+1}[,S(t)=S_n.$$

Theorem A.2.1 in the annexe (Amiri *and al.* (2014)[1]) imples that it exists a Brownian motion Ξ such that

$$\left|\frac{S(t)-\xi(t)}{(2t\ln\ln t)^{\frac{1}{2}}}\right|=O[(\ln\ln t)^{-\frac{a}{2}}]\forall t\in[V_n,V_{n+1}[.$$

But since, by the theorem of the Brownian motion verifies the law of iterated logarithm so:

$$\overline{\lim_{t\to\infty}}\frac{S(t)}{(2t\ln\ln t)^{\frac{1}{2}}}=\overline{\lim_{t\to\infty}}\left[\frac{S(t)-\xi(t)}{(2t\ln\ln t)^{\frac{1}{2}}}+\frac{\xi(t)-S(t)}{(2t\ln\ln t)^{\frac{1}{2}}}\right]=1\quad a.s.$$

Then, we have $\dfrac{S_n}{(2V_n\ln\ln V_n)^{\frac{1}{2}}}\longrightarrow 1\quad a.s.$

By using the fact that $S_n=I_1\sum_{i=1}^{n}[F(h_i)]^{1-l}$ and $\dfrac{V_{n+1}}{V_n}\longrightarrow 1$ when $n\longrightarrow\infty$, we obtain

$$\overline{\lim_{t\to\infty}}\frac{\sum_{i=1}^{n}[F(h_i)]^{1-l}I_1}{(2V_n\ln\ln V_n)^{\frac{1}{2}}}\frac{nF(h_n)^{1-2l}\{\ln\ln[nF(h_n)^{1-2l}]\}^{\frac{1}{2}}}{nF(h_n)^{1-2l}\{\ln\ln[nF(h_n)^{1-2l}]\}^{\frac{1}{2}}}=1\quad a.s$$

But $\sum_{i=1}^{n}[F(h_i)]^{1-l}=B_{n,(1-l)}n[F(h_n)]^{1-l}$.

We have

$$\frac{\{\ln\ln[nF(h_n)^{1-2l}]\}^{\frac{1}{2}}B_{n,(1-l)}B_{n,(1-l)}}{(2V_n\ln\ln V_n)^{\frac{1}{2}}}\longrightarrow\frac{\beta_{[1-l]}}{\{2\beta_{1-2l}\sigma_\varepsilon^2(X)M_2\}^{\frac{1}{2}}},$$

when $n\longrightarrow\infty$. It comes, then:

$$\overline{\lim_{t\to\infty}}\left\{\frac{nF(h_n)}{\ln\ln\left[n(F(h_n))^{1-2l}\right]}\right\}^{\frac{1}{2}}I_1=\sigma_l\text{ a.s}$$

with $\sigma_l=\dfrac{\left\{2\beta_{[1-2l]}\sigma_\varepsilon^2\left(X\right)M_2\right\}^{\frac{1}{2}}}{\beta_{[1-l]}}$.

As $\ln\ln\left[n\left(F\left(h_n\right)\right)^{1-2l}\right] = (\ln\ln n)\left[1+o\left(1\right)\right]$, we conclude that

$$\overline{\lim_{n\to\infty}}\left\{\frac{nF\left(h_n\right)}{\ln\ln n}\right\}^{\frac{1}{2}} I_1 = \frac{\left\{2\beta_{[1-2l]}\sigma_\varepsilon^2\left(X\right)M_2\right\}^{\frac{1}{2}}}{\beta_{[1-l]}}$$

Studying I_2:
We have to prove that

$$\overline{\lim_{t\to\infty}}\left\{\frac{nF(h_n)}{\ln\ln n}\right\}^{\frac{1}{2}} I_2 = 0.$$

We have

$$\begin{aligned} I_2 &= \mathbb{E}[\phi_n^{[l]}(x,y) - F^{[x]}(y)f_n^{[l]}(x)] \\ &= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}}\sum_{i=1}^{n}\frac{1}{[F(h_i)]^l}\mathbb{E}\left\{\left[H\left(\frac{y-Y_i}{h_i}\right)-F^{[x]}\right]K\left(\frac{\|x-X_i\|}{h_i}\right)\right\} \\ &= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}}\sum_{i=1}^{n}\frac{1}{[F(h_i)]^l}\left\{h_i\varphi'(0)F(h_i)\left[K(1)-\int_0^1(sK(s))'\tau_{h_i}(s)ds\right]+o(h_i)\right\}. \end{aligned}$$

The last equality above was obtained using the equation (3.1) when n tends to l'infinity (in the vicinity of infinity), based on the hypothesis H_3, we have

$$I_2 \simeq h_n\varphi(0)\frac{\alpha_{[l]}}{\beta_{1-l}}M_0[1+o(1)]$$

Thus

$$\left\{\frac{nF(h_n)}{\ln\ln n}\right\}\frac{1}{2}I_2 = \left\{\frac{nF(h_n)}{\ln\ln n}\right\}h_n\varphi(0)\frac{\alpha_{[l]}}{\beta_{1-l}}M_0[1+o(1)] = o(1).$$

In witch is verified for $\lim_{n\to\infty} nh_n^2 = 0$, we conclude then

$$\overline{\lim_{t\to\infty}}\left\{\frac{nF(h_n)}{\ln\ln n}\right\}^{\frac{1}{2}} I_2 = 0.$$

Thus

$$\left\{\frac{nF(h_n)}{\ln\ln n}\right\}\frac{1}{2}\left[\varphi_n^{[l]} - F^{[x]}G_n^{[l]}\right] \longrightarrow \frac{\left\{2\beta_{[1-2l]}\sigma_\varepsilon(X)M_2\right\}^{\frac{1}{2}}}{\beta_{[1-l]}}.$$

Now we show the almost sure convergence $G_n^{[l]}(x)$ to $G^{[l]}(x)$ to decide that of $F_n^{x,l}(y)$ to $F^{[x]}$.
In the same way, by letting $Z_i = W_i - \mathbb{E}(W_i)$ we can prove

$$G_n^{[l]}(x) - \mathbb{E}G_n^{[l]}(x) = O\left(\sqrt{\frac{\ln\ln n}{nF(h_n)}}\right) \quad as.$$

As $\mathbb{E}\left[G_n^{[l]}(x)\right] = M_1\left[1+o(1)\right]$, $G_n^{[l]}(x)$ converge almost surely to M_1 because we can write that,

$$G_n^{[l]}(x) = \left[G_n^{[l]}(x) - \mathbb{E}G_n^{[l]}(x)\right] + \mathbb{E}\left[G_n^{[l]}(x)\right].$$

That makes the end of the proof.

3.3 Mean square convergence of the recursive Kernel estimate

We consider the next theorem:

Theorem 3.2. *[29] Suppose that $H_0(i)(ii)(iii)$; $H_1 - H_4$; $H_5(i)(ii)$ and $H_6(i)(ii)(iii)$ and satisfied, if these is a constant $c > 0$ such that $nf(h_n)h_n^2 \to c$ when $n \to \infty$ then*

$$\lim_{n\to\infty} \mathbb{E}[F_n^{[x,l]}(y) - F^{[x]}(y)]^2 = \frac{B_{[1-2l]}}{B_{1-l}^2}\frac{M_2}{M_1^2}F_{(y)}^{[x]}(1 - F_{(y)}^{[x]}) + c[\varphi'(0)]^2\frac{\alpha_{[l]}^2}{B_{[1-l]}^2}\frac{M_0^2}{M_1^2}$$

Proof. It is known

$$\mathbb{E}\left[F_n^{[x,l]}(y) - F^{[x]}(y)\right]^2 = Var\left[F_n^{[x,l]}\right] + \mathbb{E}^2\left[F_n^{[x,l]}(y) - F^{[x]}(y)\right] = E_1 + E_2,$$

we will use the following decomposition for the calculation of E_2

$$\mathbb{E}\left[F_n^{[x,l]}(y)\right] = \frac{\mathbb{E}\left\{\left[\varphi_n^{[l]}(x,y)\right]\right\}}{\mathbb{E}\left[f_n^{[l]}(x)\right]} - \frac{\mathbb{E}\left\{\left[G_n^{[l]}(x) - \mathbb{E}G_n^l(x)\right]\varphi_n^l(x,y)\right\}}{\left\{\mathbb{E}\left[G_n^{[l]}(x)\right]\right\}^2} + \frac{\mathbb{E}\left\{\left[G_n^{[l]}(x) - \mathbb{E}G_n^{[l]}(x)\right]^2 F_n^{x,l}(y)\right\}}{\left\{\mathbb{E}[G_n^{[l]}(x)]\right\}^2}.$$

For the calculation ofE_1we use the following decomposition of the variance that can be found in (Collombe (1976)[15].

$$Var\left[F_n^{[x,l]}(y)\right] = \frac{Var\left[\varphi_n^{[l]}(x,y)\right]}{\mathbb{E}\left[G_n^{[l]}(x)\right]^2} - 4\frac{\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right]Cov\left[G_n^{[l]}(x), \varphi_n^{[l]}(x,y)\right]}{\left\{\mathbb{E}\left[G_n^{[l]}(x)\right]\right\}^3} + 3Var\left[G_n^l(x)\right]\frac{\left\{\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right]\right\}^2}{\mathbb{E}\left\{G_n^l(x)\right\}^4} + O\left[\frac{1}{nF(h_n)}\right]$$

Studying the convergence of E_2:

We start by studying $\dfrac{\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right]}{\mathbb{E}\left[G_n^l(x)\right]} - F^{[x]}(y)$ we observe that:

$$\frac{\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right]}{\mathbb{E}\left[G_n^{[l]}(x)\right]} - F^{[x]}(y) = \frac{\sum_{i=1}^{n}\frac{1}{[F(h_i)]^{[l]}}\mathbb{E}\left\{\left[H\left(\frac{y-Y_i}{h_i}\right) - F^{[x]}(y)\right]K\left(\frac{\|x-X_i\|}{h_i}\right)\right\}}{\sum_{i=1}^{n}\frac{1}{[F(h_i)]^{[l]}}\mathbb{E}[K\left(\frac{\|x-X_i\|}{h_i}\right)]}$$

Let:

$$\varphi(t) = \mathbb{E}\Big[\int_{\mathbb{R}} H(t)F^{[x]}(y - h_i t)dt - F^{[x]}(y)\Big] \|x - X_i\| = t$$

Suppose that the function φ is derivable at point $t = 0$ by the hypothesis $H_6(ii)$ we have:

$$\begin{aligned}\mathbb{E}\Big[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\Big]K\Big(\frac{\|x - X_i\|}{h_i}\Big) &= \mathbb{E}\Big[\varphi\Big(\|x - X_i\|\Big)K\Big(\frac{\|x - X_i\|}{h_i}\Big)\Big] \\ &= \int_0^1 \varphi(h_i t)K(t)d\mathbb{P}^{\frac{\|x-X_i\|}{h_i}}(t)\end{aligned}$$

So using the Taylor expansion for φ around 0, we obtain,

$$\mathbb{E}\Big\{\Big[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\Big]K\Big(\frac{\|x - X_i\|}{h_i}\Big)\Big\} = h_i\varphi'(0)\int_0^1 tK(t)d\mathbb{P}^{\frac{\|x-X_i\|}{h_i}}(t) + O[h_i], \tag{3.1}$$

based on the proof of Lemma 2 in (Ferraty and al(2007)[23]), H_1 and fubini theorem

$$\int_0^1 tK(t)d\mathbb{P}^{\frac{\|x-X_i\|}{h_i}}(t) = F(h_i)[K(1) - \int_0^1 (sK(s)')\tau_{h_i}(s)ds]$$

and by(H_1) we obtain:

$$\begin{aligned}\frac{\mathbb{E}\Big[\varphi_n^l(x,y)\Big]}{\Big[G_n^l(x,y)\Big]} - F^{[x]}(y) &= \frac{\sum_{i=1}^n h_i[F(h_i)]^{1-l}\Big\{\varphi'(0)\Big[K(1) - \int_0^1 (sk(s)')\tau_{h_i}(s)ds\Big]\Big\} + \gamma_i}{\sum_{i=1}^n [F(h_i)]^{1-l}[K(1) - \int_0^1 (sK(s)')\tau_{h_i}(s)ds]} \\ &= \frac{D_1}{D_2}\end{aligned}$$

Finally(H_2)and(H_3) and Toeplitz lemma (see Masry (2005)[33]) permits us have:

$$\frac{D_1}{nh_n[F(h_n)]^{1-l}} = \alpha_{[e]}\varphi'(0)M_0[1 + 0(1)]\frac{D_2}{n[F(h_n)]^{1-l}} = B_{[1-l]}M_1[1 + 0(1)]$$

$$\frac{\mathbb{E}[\varphi_n^l(x,y)]}{\mathbb{E}[F_n^l(x)]} - F^{[x]}(y) = h_n\varphi'(0)\frac{\alpha_{[l]}}{\beta_{[1-l]}}\frac{M_0}{M_1}[1 + 0(1)],$$

the convergence of the other terms of the composition for calculating E_2 is a consequence of the terms of the variance there fore, we establish the convergence of the variance we have:

$$\begin{aligned}\mathbb{E}[G_n^{[l]}(x)] &= \frac{1}{\sum_{i=1}^n [F(h_i)]^{1-l}}\sum_{i=1}^n \frac{1}{[F(h_i)]^{1-l}}\mathbb{E}\Big[K\left(\frac{\|x - X_i\|}{h_i}\right)\Big] \\ &= \frac{\sum_{i=1}^n \frac{[F(h_i)]^{1-l}}{n[F(h_n)]^{1-l}}[K(1) - \int_0^1 (K(s)\tau_{h_i}(s)ds)]}{\beta_{n,[1-l]}} \\ &= M_1[1 + o(1)]\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\varphi_n^{[l]}(x,y)] &= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{[F(h_i)]^l} \mathbb{E}\Big[H\left(\frac{y-Y_i}{h_i}\right) K\left(\frac{\|x-X_i\|}{h_i}\right)\Big] \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \mathbb{E}\Big[\int_{\mathbb{R}} H'(t)F^x(y-h_i t)dt \\
&\quad -F^{[x]}(y)+F^{[x]}(y)\Big] K\left(\frac{\|x-X_i\|}{h_i}\right) \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{F(h_i)^l} \mathbb{E}[(h_i^{\beta_2})+F^{[x]}(y)] K\left(\frac{\|x-X_i\|}{h_i}\right) \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{F(h_i)^{1-l}} F(h_i) M_1 [F^{[x]}(y)+o(h_i^{\beta_2})] \\
&= F^{[x]}(y) M_1 [1+o(1)]
\end{aligned}$$

$$\begin{aligned}
Var\Big(\varphi_n^{[l)}(x,y)\Big) &= \left[\sum_{i=1}^{n}[F(h_i)]^{1-l}\right]^{-2} \sum_{i=1}^{n} \left[\frac{1}{F(h_i)^l}\right]^2 \Bigg[Var\Bigg(K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\Bigg) \\
&\quad -Cov(K_iH_i, KjH_j)\Bigg],
\end{aligned}$$

$$\begin{aligned}
Var\Bigg(K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\Bigg) &= \mathbb{E}\Bigg[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\Bigg]^2 \\
&\quad -\mathbb{E}^2\Bigg(K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\Bigg),
\end{aligned}$$

as

$$\mathbb{E}^2\Bigg[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\Bigg] = o[\{F(h_i)\}^2],$$

$$\begin{aligned}
\mathbb{E}\Bigg[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right)\Bigg]^2 &= \mathbb{E}\left\{K^2\left(\frac{\|x-X_i\|}{h_i}\right) \mathbb{E}^2\left(H\left(\frac{y-Y_i}{h_i}\right)|X\right)\right\} \\
&\quad +\mathbb{E}\left\{\sigma_{\varepsilon_i}^2(x) K\left(\frac{\|x-X_i\|}{h_i}\right)\right\},
\end{aligned}$$

and

$$\mathbb{E}^2\Bigg[H\left(\frac{y-Y_i}{h_i}\right)|X\Bigg] = o(h_i) + [F^{[x]}(y)]^2.$$

There are $\sigma_{\varepsilon_i}^2(x) = Var\Bigg[H\left(\frac{y-Y_i}{h_i}\right)|X\Bigg]$ we have by $(h_6)(ii)$, because $H_i(y) \leq 1$ the distribution function

$$Cov(K_iH_i, K_jH_j) = Cov(\Delta_i, \Delta_j),$$

$$Cov(K_iH_i, K_jH_j) = o(nF(h_n)),$$

$$\begin{aligned} Var(\varphi_n^l(x,y)) &= \left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^{-2}\sum_{i=1}^{n}\Big(F(h_i)^l\Big)^{-2}M_2F(h_i) \\ &\quad \Big(F^x(y)^2+\sigma^2_{\varepsilon_i}(x)\Big)(1+\gamma_i)-O(nF(h_n)), \end{aligned}$$

with $\gamma_i = O(h_i)$, we get

$$Var(\varphi_n^{[l]}(x,y)) = \frac{\beta_{[1-2l]}}{\beta^2_{[1-l]}}\Big[[F^{[x]}(y)]^2+\sigma^2_{\varepsilon}(x)\Big]\frac{1}{nF(h_n)}M_2[1+o(1)]-o\big(nF(h_n)\big).$$

$$\begin{aligned} Var(G_n^l) &= \left\{\left[\sum_{i=1}^{n}[F(h_i)]^{1-l}\right]^{-2}\sum_{i=1}^{n}\left[\frac{1}{[F(h_i)]^l}\right]^2\right\}\left\{Var\left[K\left(\frac{\|x-X_i\|}{h_i}\right)\right]-Cov(K_i;K_j)\right\} \\ &= \left[\sum_{i=1}^{n}[F(h_i)]^{1-l}\right]^{-2}\sum_{i=1}^{n}\left[\frac{1}{[F(h_i)]^l}\right]^2\Big(M_2F(h_i)[1+\gamma_i]-Cov(\Delta_i;\Delta_j)\Big) \\ &= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2}\sum_{i=1}^{n}[F(h_i)]^{1-2l}M_2[1+\gamma_i]-o(nF(h_n)) \\ &= \frac{\beta_{[1-2l]}}{\beta^2_{[1-l]}}M_2[1+o(1)]-o(nF(h_n)) \end{aligned}$$

$$\begin{aligned} Cov(\varphi_n^{[l]}, G_n^{[l]}) &= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2}Cov(K_iH_i;K_j) \\ &= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2}\left\{\sum_{i,j=1}^{n}\mathbb{E}\Big[K_iH_iK_j\Big]-\left[\sum_{i=1}^{n}\mathbb{E}\Big(K_iH_i\Big)\times\sum_{j=1}^{n}\mathbb{E}\Big(K_j\Big)\right]\right\}, \end{aligned}$$

because $H_i \leq 1$ is distribution cumulative function we get:

$$\begin{aligned} Cov(\varphi_n^{[l]}, G_n^{[l]}) &= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(K_i,K_j) \\ &= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(\Delta_i,\Delta_j) \\ &= O\left(\frac{1}{nF(h_n)}\right). \end{aligned}$$

Finally, we have

$$Var\Big[\widehat{F}^{x,[l]}(y)\Big] = \frac{\beta_{1-2l}}{\beta_{1-l}}\frac{M_2}{M_1^2}\sigma^2_{\varepsilon}\frac{1}{nF(h_n)}[1+o(1)].$$

Given

$$\mathbb{E}\left\{\Big[G_n^l(x)-\mathbb{E}G_n^l(x)\Big]\varphi_n^{[l]}(x,y)\right\} = O\left(\frac{1}{nF(h_n)}\right),$$

$$\mathbb{E}\left\{\left[G_n^l(x) - G(x)\right]^2 \widehat{F}_n^{[x,l]}(y)\right\} = O\left(\frac{1}{nF(h_n)}\right),$$

and

$$\mathbb{E}\left[\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y)\right] = h_n \varphi'(0) \frac{\alpha_{[l]}}{\beta_{[1-l]}} \frac{M_0}{M_1}[1 + o(1)] + O\left(\frac{1}{nF(h_n)}\right).$$

The proof takes and here.

□

General Conclusion

This dissertation was devoted to the study of recursive nonparametric estimation methods in the framework of functional data analysis. The main objective of this work was to construct recursive estimators for both the conditional cumulative distribution function and the conditional density function when the explanatory variable is functional and belongs to an infinite-dimensional space.

First, we introduced the fundamental concepts related to functional statistics, conditional models, and recursive methods, with particular emphasis on the asymptotic tools and probabilistic results required for the theoretical study of the proposed estimators under both independence and strong mixing conditions.

In the case of independent observations, recursive kernel estimators for the conditional distribution function and the conditional density function were developed, and their asymptotic properties were investigated. The obtained results established almost sure convergence and convergence in quadratic mean under suitable regularity assumptions, confirming the efficiency and consistency of the recursive estimation approach in the functional framework.

The study was then extended to dependent observations satisfying strong mixing conditions. In this context, recursive estimators adapted to dependent functional data were introduced, and their convergence properties were established, highlighting the importance of these methods in the study of statistical models involving dependent functional data.

The recursive methodology adopted throughout this work offers several important advantages, since it allows estimators to be updated sequentially as new observations become available without repeating all previous computations. Such procedures are particularly suitable for large datasets and dynamic data structures where computational efficiency and memory reduction are essential.

The results obtained in this dissertation contribute to the development of recursive estimation methods for functional data and emphasize the importance of nonparametric approaches in infinite-dimensional statistical models. They also provide a solid theoretical foundation that may be useful for further studies and for applications in several scientific fields, including climatology, medical sciences, environmental studies, econometrics, and reliability analysis.

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