People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

University of Saida - Dr Moulay Tahar.

Faculty of Sciences.

Department of Mathematics.

This Thesis was Submitted in Partial Fulfillment of the Requirements for the degree of

Academic Master

Discipline : MATHEMATICS

Specialty: Statistical Stochastic Analysis of Processes and Applications

by

Amra **H**alima **M**oulai[1]

Under the supervision of

Dr. Chafiâa AYHAR

Theme:

Nonparametric estimation of transition function in continuous time semi-Markov processes

Defended on 15/06/2026 in front of the committee

Dr. Rajaa Hazeb	University of Saida Dr. Moulay Tahar	President
Dr. Chafiâa Ayhar	University center of nour bachir el bayadh	Supervisor
Dr. Imane Bouazza	University of Saida Dr. Moulay Tahar	Examiner

June, 2025/2026

[1]e-mail: moulaihalima20@gmail.com

* Dedication *

All praise to ALLAH. We tuck the day's effort, the accomplishment represented between the covers of this modest work. Looking back, it was all worth it.

Regret does not come from failure. It comes from giving up.

— Mel Robbins

I dedicate this work to:

My parents who planted in me the spirit of perseverance and diligence to reach what I aspire to, who always accompanied me by their blessed prayers and sacrifices over the years in order to climb the ladders of success. May ALLAH bless them.

To my brothers and sisters, nephews and nieces.

To my teacher who taught me the Holy Quran and guided my path with its light.

To my dear students in the Quran circle, whom I cherish and wish the best.

To everyone who loved me and sincerely prayed for my success.

To all my friends, my colleagues and to those who share with me my success.

To my professors from elementary school through graduation.

* *Acknowledgments* *

In the Name of Allah, the Most Merciful, the Most Compassionate all praise be to Allah and prayers and peace be upon Mohammed, His servant and messenger.

First and foremost, I must acknowledge my limitless thanks to Allah, the Ever-Magnificent, the Ever-Thankful, for His help and blessings. I am totally sure that this work would have never become a reality ,without His guidance.

I would like to express my sincere thanks and appreciation to my supervisor **Dr. Chafiâa Ayhar**, who spared no effort in supporting this project throughout its preparation period. Her thoughtful guidance, valuable information and constructive criticism prompted me to raise my work to a higher level and come out with this best picture.

My thanks is also extended to the committee members:

Rajaa Hazeb and **Imane Bouazza** for their contributions to this work, for offering me their valuable time, and for providing a clear imprint through their instructive comments in order to improve my work.

I would like to thank all the members of the Laboratory of Stochastic Models, Statistic and Applications for their kind assistance and facilitation, which provided us with a comfortable space to seek knowledge.

A very special mention of gratitude is owed to my dear sister, **Fatma Moulai**, for her unconditional love, endless support, and for being my ultimate source of strength throughout this journey.

I would also like to express my sincere thanks to **Miss Chaima Hamiani**, my friends in **LMSSA**, and everyone who contributed, directly or indirectly, to the smooth running of my work.

Contents

Dedication 2

Acknowledgments 3

Notations 7

General introduction 13

1 Background on stochastic processes and renewal theory 16

1.1 Basic definitions and properties 16

1.2 Markov renewal process and semi-Markov process 20

2 Empirical estimators for continuous-time semi-Markov process 28

2.1 Construction of the estimators 28

2.2 Asymptotic properties of the estimators 29

2.2.1 Assumptions 29

2.2.2 Comments on the assumptions 29

2.2.3 Strong consistency 30

2.2.4 Asymptotic normality 34

3 Kernel estimators for continuous-time semi-Markov processes 41

3.1 Nonparametric kernel estimators 41

3.2 Asymptotic properties of the estimators 42

3.2.1 Assumptions 43

3.2.2 Comments on the assumptions 43

3.2.3 Strong consistency 43

3.2.4 Asymptotic normality 46

4 Simulation study 57

4.1 The evolution equation numerical solution 57

4.2 Numerical example 58

Conclusion **65**

Bibliography **65**

List of Figures

1.1 Sample path of a semi-Markov process. 21

4.1 A three state semi-Markov system 58

4.2 Estimated renewal matrix 2 to state 1, $\Psi_{21}(t,T)$ 60

4.3 Estimated renewal matrix 3 to state 1, $\Psi_{31}(t,T)$ 60

4.4 Estimated transition functions from state i to state 1, $P_{i1}(t,T)$ $(i \in E)$ 61

4.5 Estimated transition functions from state i to state 2, $P_{i2}(t,T)$ $(i \in E)$ 62

4.6 Estimated transition functions from state i to state 3, $P_{i3}(t,T)$ $(i \in E)$ 62

4.7 The true value and the estimator transition functions from state i to state 3, $P_{i3}(t,T)$ 64

Notation

$\mathbb{N}$	Set of positive natural numbers
$\mathbb{R}_+$	Set of non negative real numbers
$(\Omega, \mathscr{F}, \mathbb{P})$	Probability space
$\mathbb{E}$	Expectation with respect to $\mathbb{P}$
$E = \{1, ..., s\}$	Finite state space
$\mathscr{M}_E$	Set of real matrix on $\mathbb{E} \times \mathbb{E}$
$\mathscr{M}_E(\mathbb{N})$	Matrix-valued functions defined on $\mathbb{N}$, with values in $\mathscr{M}_E$
$Z := (Z_k)_{k\in\mathbb{N}}$	Semi-Markov chain
$Z := (Z_t)_{t\in\mathbb{R}^+}$	Semi-Markov process (SMP)
$(J,S) := (J_n, S_n)_{n\in\mathbb{N}}$	Markov renewal chain (MRC)
$J := (J_n)_{n\in\mathbb{N}}$	Visited states, embedded Markov chain (EMC)
$S := (S_n)_{n\in\mathbb{N}}$	Jump times
$X := (X_n)_{n\in\mathbb{N}}$	Sojourn times
T	Fixed censoring time
$N(T)$	Number of jumps of Z in the time interval $[1,T]$
$N_i(T)$	Number of visits to state i of the EMC, up to time T
$N_{ij}(T)$	Number of transitions from state i to state j of the EMC, up to time T
$N_{ij}(k,T)$	Number of transitions from state i to state j of the EMC, up to time T, with sojourn time in state i inferior or equal to t

$\mathbf{p} := (p_{ij})_{i,j\in E}$	Transition matrix of the EMC J
$\mathbf{q} := (q_{ij}(k))_{i,j\in E,k\in\mathbb{N}}$	Semi-Markov kernel
$\mathbf{Q} := (Q_{ij}(k))_{i,j\in E,k\in\mathbb{N}}$	Cumulated semi-Markov kernel
$\mathbf{Q} := (Q_{ij}(t))_{i,j\in E,t\in\mathbb{R}^+}$	First derivative of semi Markov kernel
$\mathbf{f} := (f_{ij}(k))_{i,j\in E,k\in\mathbb{N}}$	Conditional sojourn time in state i, before visiting state j
$\mathbf{F} := (F_{ij}(k))_{i,j\in E,k\in\mathbb{N}}$	Conditional cumulative sojourn time distribution in state i, before visiting state j
$\mathbf{F} := (F_{ij}(t))_{i,j\in E,t\in\mathbb{R}^+}$	Sojourn time distribution in state i, before visiting state j
$\mathbf{F} := (F_i(t))_{i\in E,t\in\mathbb{R}^+}$	Sojourn time distribution in state i
$\overline{\mathbf{F}} := (F_i(k))_{i\in E,k\in\mathbb{N}}$	Survival function in state i
$\mathbf{P} := (P_{ij})(k)_{i,j\in E,k\in\mathbb{N}}$	Transition function of the semi-Markov chain Z
$\mathbf{P} := (P_{ij})(t)_{i,j\in E,t\in\mathbb{R}^+}$	Transition function of the semi-Markov process Z
$\Psi(t) = (\Psi_{ij}(t))_{i,j\in E,t\in\mathbb{R}^+}$	Markov renewal matrix
μ_{ij}	Mean first passage time from state i to state j, for the semi-Markov process Z
μ^*_{ij}	Mean first passage time from state i to state j for the embedded Markov chain J
$\nu = (\nu(j))_{j\in E}$	Stationary distribution of the EMC J
$\alpha = (\alpha_i)_{i\in E}$	Initial distribution of semi-Markov process Z
$A * B$	Discrete-time matrix convolution product of A, B
$Q * \phi$	Stieltjes convolution of ϕ, Q
$A^{(n)}$	n-fold convolution of $A \in \mathscr{M}_{\mathbf{E}}(\mathbb{N})$
$\xrightarrow{\text{a.s}}$	Almost sure convergence (strong consistency)

$\xrightarrow{\mathscr{P}}$	Convergence in probability
$\xrightarrow{D}$	Convergence in distribution
δ_{ij}	Symbol of Kronecker
$\mathbf{1}_A$	Indicator function of A
$\mathscr{N}(0,\sigma^2)$	Standard normal random variable
$CTMP$	Continuous-time Markov Process
MRP	Markov Renewal Process
SMP	Semi-Markov Process
$CTSMP$	Continuous-time Semi-Markov Process
EMC	Embedded Markov Chain
MLE	Maximum-Likelihood Estimator
$SLLN$	Strong Law of Large Numbers
CLT	Central Limit Theorem

الملخص

يتناول هذا العمل تقدير نظام شبه ماركوفي ذي حالات متنهية باستخدام نهج لامعلمي. نحن نقدم مقدرات تعتمد على النواة (الكيرنل) للعديد من الخصائص الرئيسية ومقاييس الأداء للعملية شبه الماركوفية، ونثبت اتساقها القوي بالإضافة إلى طبيعتها التقاربية.

أولاً، نقدم مقدرات تجريبية للخصائص الرئيسية لعملية شبه ماركوفية في الوقت المستمر، وهي أوقات الإقامة المشروطة وغير المشروطة في حالة ما، والنواة شبه الماركوفية والمشتقات المرتبطة بها، بالإضافة إلى دالة الانتقال للعملية شبه الماركوفية ودالة التجديد الماركوفية. ثم ندرس خصائصها التقاربية، مع التركيز بشكل خاص على الاتساق القوي المنتظم والطبيعية التقاربية.

ثانياً، نقدم مقدرات الكيرنل للخصائص الرئيسية لعملية شبه ماركوفية في الوقت المستمر، بما في ذلك أوقات الإقامة المشروطة وغير المشروطة في الحالة، والنواة شبه الماركوفية، والمشتقات المقابلة لها، ودالة الانتقال للعملية شبه الماركوفية ودالة التجديد الماركوفية. الهدف الأساسي هو تقصي سلوكها التقاربي، وخاصة الاتساق القوي المنتظم والطبيعية التقاربية.

لإثبات فاعلية نتائجنا النظرية، يتم تكميل كل قسم بمثال عددي.

الكلمات المفتاحية:

عمليات شبه ماركوفية، مقدر الكيرنل، الطريقة التجريبية، مصفوفة التجديد الماركوفية، مصفوفة الانتقال شبه الماركوفية، الخصائص التقاربية.

Résumé

Ce travail aborde l'estimation d'un système semi-markovien (SMS) à états finis en utilisant une approche non paramétrique. Nous présentons des estimateurs à noyau pour plusieurs caractéristiques clés et mesures de performance du processus semi-markovien, et établissons leur forte consistance ainsi que leur normalité asymptotique.

Premièrement, nous introduisons des estimateurs empiriques pour les principales caractéristiques d'un processus semi-markovien à temps continu, à savoir les temps de séjour conditionnels et inconditionnels dans un état, le noyau semi-markovien et ses dérivées associées, ainsi que la fonction de transition du processus semi-markovien et la fonction de renouvellement markovien. Nous étudions ensuite leurs propriétés asymptotiques, avec un accent particulier sur la consistance forte uniforme et la normalité asymptotique.

Deuxièmement, nous présentons des estimateurs à noyau pour les principales caractéristiques d'un processus semi-markovien à temps continu, y compris les temps de séjour conditionnels et inconditionnels dans un état, le noyau semi-markovien et leurs dérivées correspondantes, la fonction de transition du processus semi-markovien et la fonction de renouvellement markovien. L'objectif principal est d'étudier leur comportement asymptotique, en particulier la consistance forte uniforme et la normalité asymptotique.

Pour démontrer l'efficacité de nos résultats théoriques, chaque section est complétée par un exemple numérique.

MOTS clés : Processus semi-markoviens, Estimateur à noyau, Méthode empirique, Matrice de renouvellement markovien, Matrice de transition semi-markovienne, Propriétés asymptotiques

Abstract

This work addresses the estimation of a finite-state semi-Markov system (SMS) using a nonparametric approach. We present kernel-based estimators for several key characteristics and performance measures of the semi-Markov process, and establish their strong consistency as well as their asymptotic normality.

First, we introduce empirical estimators for the main characteristics of a continuous-time semi-Markov process, namely the conditional and unconditional sojourn times in a state, the semi-Markov kernel and its associated derivatives, as well as the transition function of the semi-Markov process and the Markov renewal function. We then study their asymptotic properties, with particular emphasis on uniform strong consistency and asymptotic normality.

Secondly, we present kernel estimators for the principal characteristics of a continuous-time semi-Markov process, including conditional and unconditional state sojourn times, the semi-Markov kernel, and their corresponding derivatives, the transition function of the semi-Markov process and the Markov renewal function. The primary objective is to investigate their asymptotic behavior, particularly uniform strong consistency and asymptotic normality.

To demonstrate the effectiveness of our theoretical results, each section is complemented by a numerical example.

KEYWORDS : Semi-Markov processes, Kernel estimator, Empirical method, Markov renewal matrix, Semi-Markov transition matrix, Asymptotic properties

General introduction

Motivation and bibliographic context

Systems are becoming increasingly complex across various scientific disciplines, such as reliability and maintenance [9], survival analysis, performance evaluation, and biology [20]. Similar complexity is observed in DNA analysis, risk processes, earthquake modeling, and finance [17, 5, 53]. Consequently, recent years have seen a growing interest in evaluating system performance, where their evolution is modeled using continuous-time stochastic processes.
Among the models widely used as a standard tool for describing system evolution, Markov and semi-Markov models are prominent. Semi-Markov models in continuous time are appropriate for describing the evolution of a system. One reason for their widespread use is that they generalize Markov processes by relaxing the memoryless assumption. In semi-Markov processes, the future evolution may depend not only on the current state but also on the time already spent in that state. These processes are especially useful because they allow arbitrary distributions for sojourn times, such as exponential, Weibull, Gamma, or log-normal distributions.

Unlike standard Markov processes which are restricted to exponential or geometric distributions for the time spent in each state, semi-Markov processes (SMPs) offer the flexibility to use any distribution. This makes them a more realistic tool for modeling complex systems.
The semi-Markov process (SMP) is a generalization of the Markov process, independently introduced by Lévy [30], Smith [49] and Takács [50], who proposed essentially the same type of process. Further contributions were made by Pyke and Schaufele [46], Cinlar [14], Koroliuk and Turbin [26, 27], Takács [51], and many other researchers worldwide.

The statistical estimation of semi-Markov processes (SMPs) has been extensively studied over the past decades. The essential developments as well as inference problems for continuous-time semi-Markov processes were proposed by several authors. Moore and Pyke [37] studied empirical estimators for finite semi-Markov kernels. Lagakos, Sommer and Zelen [28] gave maximum likelihood estimators for nonergodic finite semi-Markov kernels. Akritas and Roussas [1] investigated the asymptotic local normality of estimators. Empirical estimators for the sojourn-time distributions, with proven consistency and asymptotic normality, were developed by Atuncar [3]. Significant contributions were also made by Limnios [31], who provided empirical estimators for finite nonlinear functionals of SMPs with desirable asymptotic properties. Window-censored nonparametric SMP estimation was proposed by Alvarez [2]. Ouhbi and Limnios [41] were the first to present nonparametric estimators of the Markov renewal matrix, and they later examined the semi-Markov transition matrix [40].

Nonetheless, the empirical distribution function has a key limitation: it is discontinuous. If the true distribution is known to be continuous, this discontinuity can lead to poor approximations. Kernel smoothing addresses this issue by producing a continuous estimate. We recall that the first kernel estimator for a probability density was proposed by Rosenblatt [47]. This estimator was generalized by Parzen [42] six years later. Since then, it has been formally known as the nonparametric Parzen-Rosenblatt kernel estimator. It represents one of the most fundamental tools in density estimation. Extending these approaches, kernel-based techniques have been applied to semi-Markov processes, with the key advances of this method proposed by Ayhar et al. [4] who studied kernel estimators of the main characteristics of a continuous-time semi-Markov process, Mokhtari et al. [36] who studied kernel estimators of Markov renewal function and transition Matrix of semi-Markov systems, Shamsuddinov [48] investigated the asymptotic unbiasedness and consistency of a kernel estimator of the density of the sojourn time distribution.

The shift from empirical to kernel methods is particularly important in semi-Markov processes, where sojourn time distributions, Markov renewal and transition function are often assumed to have continuous densities. Markov renewal processes and semi-Markov transition functions are essential for modeling systems with variable timing between events. Transition functions describe the probability of moving between states over time. Together, they enable calculation of survival probabilities, hazard rates, and reliability measures. This makes them crucial for applications in reliability, finance, and queuing systems.

Outline of the dissertation

This Master's thesis is structured into four chapters:

Chapter 1: In this chapter, we provide the necessary mathematical background on renewal processes and continuous-time semi-Markov processes. We introduce the fundamental concepts, properties, and theorems required to characterize these processes with in a discrete state space, laying the groundwork for the following chapters.

Chapter 2: In this chapter, we consider a continuous-time semi-Markov model with finite state space. We present its basic probabilistic properties and we present the empirical estimators for its main characteristics, such as the semi-Markov kernel, sojourn time distributions, the transition function of the semi-Markov process and the Markov renewal function. The final section is devoted to investigating the asymptotic properties of these estimators, specifically their strong consistency and asymptotic normality.

Chapter 3: In this chapter, we extend our statistical study to nonparametric kernel estimation for continuous-time semi-Markov processes. We present the kernel estimators for the transition function of the semi-Markov process and the Markov renewal function. The main objective is to overcome the discontinuity of empirical estimators by providing smoother and more accurate approximations. Finally, we establish the asymptotic properties of these kernel-based estimators.

Chapter 4: To support the theoretical results achieved in the field of nonparametric estimation for the transition function of a semi-Markov process, we present in this chapter a simulation study that highlight the performance of these estimators.

CHAPTER 1

Background on stochastic processes and renewal theory

In this chapter, we consider a semi-Markov process with a finite state space, observed over a fixed time interval [0, T]. The main objective of this chapter is to provide the fundamental definitions, properties, and key theorems related to continuous-time homogeneous semi-Markov processes with a discrete set of states.

1.1 Basic definitions and properties

Let us consider $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(\mathbb{E}, \varepsilon)$ be a measurable space with countably generated σ-algebra, and let I be a set called a parameter set. Generally, I is a subset of $\mathbb{R}$, usually $\mathbb{N}$ or $\mathbb{R}_+$.

Definition 1.1.1. (Stochastic processes)
A stochastic process is a family of random variables indexed by a set I, $\{X(t), t \in I\}$ defined on $(\Omega, \mathscr{F}_n, \mathbb{P})$ with values in E. For every $t \in I$, $X(t)$ is a random variable $X(t) : \Omega \to E$, whose value for the outcome $\omega \in \Omega$ is noted $X(t, \omega)$.

If instead of t we fix an $\omega \in \Omega$, we obtain the function $X(., \omega) : I \to E$ which is called a trajectory, a path-function, or a sample function of the process.

The set E is called the state space of the stochastic process $X = (X(t), t \in I)$. The stochastic process may be denoted by X_t instead of $X(t)$ (respectively, X_n if $I = \mathbb{N}$) .

Consider a finite set $E = \{1, \ldots, s\}$. We denote by $\mathscr{M}_E$ the set of real matrices on $E \times E$ and by $\mathscr{M}_E(\mathbb{N})$ the set of matrix valued functions defined on $\mathbb{N}$, with values in $\mathscr{M}_E$.

Theorem 1.1.1. (Strong law of large numbers) [33]

Let $(X_1, X_2, \dots)$ be an infinite sequence of i.i.d. Lebesgue integrable random variables with expected value $\mathbb{E}[X_1] = \mathbb{E}[X_2] = \dots$, then we have

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[X_1].$$

Definition 1.1.2. (Martingale)

Let $\mathbf{F} = (\mathscr{F}_n, n \geq 0)$ be a filtration of $\mathscr{F}$. A real-valued $\mathbf{F}$-adapted stochastic process X_n is ($\mathscr{F}_n$-measurable for $n \geq 0$) called martingale with respect to a filtration $\mathbf{F}$ if, for every $n = 0, 1, \dots$

1. $\mathbb{E}|X_n| < \infty$, and

2. $\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n$ (a.s.)

Theorem 1.1.2. (CLT for martingales)[10]

Let $(X_n)_{n\in\mathbb{N}^*}$ be a martingale with respect to the filtration $\mathscr{F} = (\mathscr{F}_n)_{n\in\mathbb{N}}$ and define the process $Y_n = X_n - X_{n-1}, n \in \mathbb{N}^*$ (with $Y_1 := X_1$),called a difference martingale. If

1. $\displaystyle \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[Y_k^2|\mathscr{F}_{k-1}] \xrightarrow[n\to\infty]{P} \sigma^2 > 0,$

2. $\displaystyle \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[Y_k^2 \mathbf{1}_{\{|Y_k|>\varepsilon\sqrt{n}\}}] \xrightarrow[n\to\infty]{} 0,$ For all $\varepsilon > 0$,

then

$$\frac{X_n}{n} \xrightarrow[n\to\infty]{a.s.} 0,$$

and

$$\frac{1}{\sqrt{n}}X_n = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} Y_k \xrightarrow[n\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma^2).$$

Theorem 1.1.3. (Anscombe's Theorem) [11]

Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables and $(N_n)_{n\in\mathbb{N}}$ a positive integer-valued stochastic process. Suppose that

$$\frac{1}{\sqrt{n}}\sum_{m=1}^{n} Y_m \xrightarrow[n\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma^2) \quad \text{and} \quad N_n/n \xrightarrow[n\to\infty]{P} \theta,$$

where θ is a constant, $0 < \theta < \infty$. Then,

$$\frac{1}{\sqrt{N_n}}\sum_{m=1}^{N_n} Y_m \xrightarrow[n\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma^2).$$

Theorem 1.1.4. (Glivenko-Cantelli theorem) [11]
Let $F_n(x) = \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\{X_k \le x\}}$ be the empirical distribution function of the i.i.d. random sample $X_1, \dots, X_n$. Denote by F the common distribution function of $X_i, i = 1, \dots, n$. Thus

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \xrightarrow[n\to\infty]{a.s.} 0.$$

Theorem 1.1.5. (Slutsky's theorem)[52]
Let X, X_n and $Y_n, n \in \mathbb{N}$ be random variables or vectors. If

$$X_n \xrightarrow[n\to\infty]{\mathscr{D}} X,$$

and

$$Y_n \xrightarrow[n\to\infty]{\mathscr{D}} c,$$

with c a constant, then

1. $X_n + Y_n \xrightarrow[n\to\infty]{\mathscr{D}} X + c$,
2. $X_n Y_n \xrightarrow[n\to\infty]{\mathscr{D}} Xc$,
3. $X_n Y_n^{-1} \xrightarrow[n\to\infty]{\mathscr{D}} Xc^{-1}$, for $c \neq 0$.

Limit Theorems for Markov renewal process

We present the strong law of large numbers and the central limit theorem for additive functional of MRPs. Pyke and Schaufele [46] gave these results. The notation used here comes from Moore and Pyke [37].

For a real measurable function f, defined on $E \times E \times \mathbb{R}$, define, for each $T > 0$, the functional $W_f(T)$ as

$$W_f(T) := \sum_{n=1}^{N(T)} f(J_{n-1}, J_n, X_n). \tag{1.1}$$

Set

$$A_{ij} := \int_0^\infty f(i,j,x)dQ_{ij}(x), \quad A_i := \sum_{j=1}^{s} A_{ij},$$

$$B_{ij} := \int_0^\infty (f(i,j,x))^2 dQ_{ij}(x), \quad B_i := \sum_{j=1}^{s} B_{ij}.$$

Let μ_{ij} and μ_{ij}^* denote the mean first passage times from state i to j in the MRP (J_n, S_n) and in the corresponding Markov chain $(J_n)_{n\in\mathbb{N}}$, respectively.

Write

$$r_i := \sum_{u=1}^{s} A_u \frac{\mu_{ii}^*}{\mu_{uu}^*},$$

$$\sigma_i^2 := -r_i^2 + \sum_{u=1}^{s} B_u \frac{\mu_{ii}^*}{\mu_{uu}^*} + 2\sum_{u=1}^{s} \sum_{l\neq i} \sum_{j\neq i} A_{ul} A_j \mu_{ii}^* \frac{\mu_{li}^* + \mu_{ij}^* - \mu_{ij}^*}{\mu_{uu}^* \mu_{jj}^*}. \tag{1.2}$$

Finally, put

$$m_f := \frac{r_i}{\mu_{ii}},$$

$$B_f := \frac{\sigma_i^2}{\mu_{ii}}.$$

Theorem 1.1.6. (strong law of large numbers)

For an aperiodic MRPs that satisfies Assumptions H.1 and H.2 (see Chapter 3) we have

$$\frac{W_f(T)}{T} \xrightarrow[T\to\infty]{a.s.} m_f.$$

Theorem 1.1.7. (Central Limit Theorem)

For an irreducible recurrent MRPs that satisfies Assumptions H.1 and H.2 (see Chapter 3) we have

$$T^{-1/2}[W_f(T) - T\cdot m_f] \xrightarrow[T\to\infty]{\mathscr{D}} \mathscr{N}(0, B_f).$$

Theorem 1.1.8. (Theorem of strong consistency)[39]

Suppose that $K(x)$ is a function of bounded variation, $f(x)$ is a uniformly continuous density function, and the series $\sum_{n=1}^{\infty} e^{-\gamma n h^2}$ converges for every positive value of γ. Then

$$\sup_{-\infty<x<\infty} |f_n(x) - f(x)| \longrightarrow 0,$$

with probability one as $n \to \infty$.

Classification of states

In order to understand the Markov chains better. It is necessary to provide some basic ways to describe the state or Markov chain.

Definition 1.1.3. (Accessible state)

We say that the state j is accessible from the state i, written as $i \to j$ if $p_{ij}^{(n)} > 0$. We assume every state is accessible from it self since $p_{ii}(0) = 1$.

Definition 1.1.4. (Communicate state)

Two states i and j are said to communicate, written as $i \leftrightarrow j$ if they are accessible from each other. In other words,

$$i \leftrightarrow j \text{means } i \to j \text{and } j \to i.$$

Remark 1.1.1. (Irreducible Markov chain)

A Markov chain is irreducible if there is a single communication link.

Definition 1.1.5. (Recurrent state)

A state is said to be recurrent if, any time that we leave that state, we will return to that state in the future with probability one. On the other hand, if the probability of returning is less than one, the state is called transient. Here, we provide a formal definition. For any state i, we define

$$G_{ii} = \mathbb{P}(J_n = i, \quad n \geq 1 | J_0 = i).$$

A state i is said to be recurrent if $G_{ii}(\infty) = 1$, and it is transient if $G_{ii}(\infty) < 1$.

Definition 1.1.6. (Periodic, aperiodic state)

A state $i \in E$ is said to be periodic of period $d > 1$, or d-periodic if d is equal to the greatest common divisor of all n such that $\mathbb{P}(J_{n+1} = i | J_1 = i) > 0$. If $d = 1$, then the state i is said to be aperiodic.

Remark 1.1.2. (Ergodic state)

An aperiodic recurrent state is called ergodic. An irreducible Markov chain with one state ergodic (and then all states ergodic) is called ergodic.

1.2 Markov renewal process and semi-Markov process

The semi-Markov process can be defined, by a two dimensional Markov renewal process, where one variable represents the states and the other the times of state changes (jump times). Thus, all properties of a semi-Markov process can be deduced from the properties of this Markov renewal process. The definition of a semi-Markov process in its usual form, essentially requires two points: A restrictive Markov condition which causes invariance of the renewal process under time shift (stationarity), and the homogeneity, i.e., the independence of the transition probabilities of the number of renewals.

Definition 1.2.1. (**Markov renewal process**)

Let E be the state space. A Markov renewal process is a bivariate stochastic process (J_n, S_n) where J_n are the values of the state space E in the Markov chain and S_n are the jump times. We define $X_{n+1} = S_{n+1} - S_n$ to be the sojourn time in the state J_n. The process has to satisfy the following equality

$$\begin{aligned} &\mathbb{P}\left(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_0, J_1, \ldots, J_n, S_0, S_1, \ldots, S_n\right) \\ =&\mathbb{P}\left(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_n\right), \end{aligned} \tag{1.3}$$

for all $j \in E$, all $t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$.

Definition 1.2.2. (**Continuous-time semi-Markov process**)

Consider a Markov-renewal process $(J_n, S_n)_{n\in\mathbb{N}}$ defined on a complete probability space and with state space E. The process $Z = \{Z_t; t \geq 0\}$ defined by

$$Z_t = J_{N(t)},$$

is called a semi-Markov process (SMP) where $N(t) = \sup\{n \geq 0 : S_n \leq t\}$ is the counting process of the semi-Markov process up to time t.

$$Z_t = J_n \quad \text{for } S_n \leq t < S_{n+1}.$$

The relationships between the semi-Markov process Z and the embedded Markov chain J are given by

$$Z_t = J_{N(t)} \Leftrightarrow J_n = Z_{S_n},$$

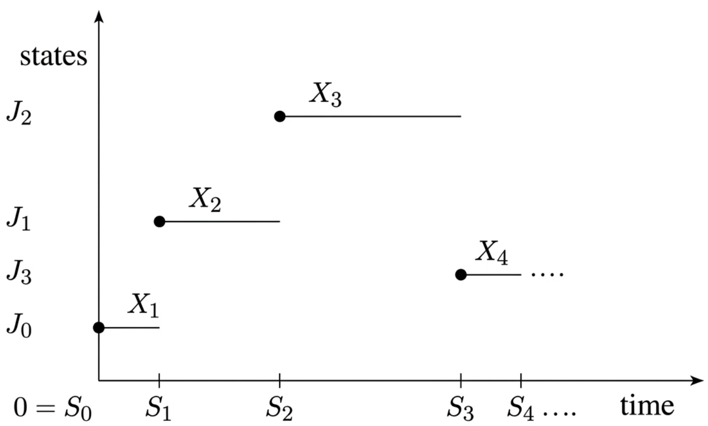


Figure 1.1: Sample path of a semi-Markov process.

Definition 1.2.3. (Homogeneous Markov chain)

Markov chain is homogeneous if for all $n \geq 0$, i and $j \in E$

$$\mathbb{P}(J_{n+1} = j | J_n = i) = \mathbb{P}(J_1 = j | J_0 = i).$$

This means that the probability $\mathbb{P}(J_{n+1} = j | J_n = i)$ does not depend on n.

Definition 1.2.4. (Renewal matrix, renewal kernel)

Let E be the state space and consider the Markov renewal process (J_n, S_n), we define $X_{n+1} = S_{n+1} - S_n$ to be the sojourn time in the state. The matrix defined as

$$\mathbf{Q}(t) = \{Q_{ij}(t) : i, j \in E\},$$

$$Q_{ij}(t) := \mathbb{P}(J_{n+1} = j, X_{n+1} \leq t | J_n = i), \tag{1.4}$$

is called a renewal matrix. We identify the renewal matrix $\mathbf{Q}$ as the renewal kernel(the semi-Markov kernel).

Proposition 1.2.1. [21]The Markov renewal matrix $\mathbf{Q}$ satisfies the following conditions:

(i) For all $t \geq 0$ and $i, j \in E$, it holds true that $Q_{ij}(t) \geq 0$.

(ii) The functions $Q_{ij}(t)$ are right-continuous and not decreasing.

(iii) For all $i, j \in E$, it holds true that $Q_{ij}(0) = 0$ and $Q_{ij}(t) \leq 1$ for all $t \geq 0$.

(iv) For all $i \in E$, it holds that $\lim\limits_{i \to \infty} \sum\limits_{j \in E} Q_{ij}(t) = 1$.

Definition 1.2.5. (Transition probability)

For all $i, j \in E$ and $n \geq 0$, we define the probability of transition from state i to state j between instants n and $n+1$ by the quantity:

$$p_{ij}(n) = \mathbb{P}(J_{n+1} = j | J_n = i)$$

We do not allow transitions to the same state, i.e. we set $p_{ii} = 0$ for any $i \in E$.

Definition 1.2.6. (Transition matrix)

The function $(i, j) \to p_{ij} := \mathbb{P}(J_{n+1} = j | J_n = i)$ is called transition function of the chain. For any $i, j \in E$ and $n \geq 0$, the transition function has the following properties:

1. $p_{ij} \geq 0$, for any $i, j \in E$,

2. $\sum_{j\in E} p_{ij} = 1$, for any $i \in E$.

Proposition 1.2.2. In order to define a Markov chain $(J_n)_{n\geq 0}$ we need:

1. Transition function (matrix) $\mathbf{p} = (p_{ij})_{i,j\in E}$.
2. $\alpha = (\alpha_1, \ldots, \alpha_s)$, the initial distribution of the chain, that is the distribution of J_0, $\alpha_i = \mathbb{P}(J_0 = i)$ for any state $i \in E$.

Definition 1.2.7. (**Distribution functions of sojourn time**) For all $i, j \in E, \forall t \in \mathbb{R}_+$.

1. $F_{ij}(.)$, the distribution function associated with the sojourn time in state i, before going to state j:
$$F_{ij}(t) := \mathbb{P}(X_{n+1} \leq t | J_n = i, J_{n+1} = j).$$
2. $F_i(.)$, the distribution function of the sojourn time, also called the waiting time, in state i:
$$F_i(t) := \mathbb{P}(X_{n+1} \leq t | J_n = i) = \sum_{j\in E} Q_{ij}(t).$$
3. $\bar{F}(t) = (\bar{F}_i(t); i \in E) = (1 - F_i(t); i \in E), \quad t \in \mathbb{R}^+$ the survival function of $F(t)$.

From the previous definition, we can deduce the following result.

Proposition 1.2.3. [21]For all $t \geq 0$ and $i, j \in E$, we have
$$F_{ij}(t) = \frac{Q_{ij}(t)}{p_{ij}}.$$

Proof.

From the definition of conditional probabilities, it follows that

$$\begin{aligned}
F_{ij}(t) &= \mathbb{P}(X_{n+1} \leq t | J_n = i, J_{n+1} = j) \\
&= \frac{\mathbb{P}(X_{n+1} \leq t, J_n = i, J_{n+1} = j)}{\mathbb{P}(J_n = i, J_{n+1} = j)} \\
&= \frac{\mathbb{P}(X_{n+1} \leq t, J_n = i, J_{n+1} = j)}{\mathbb{P}(J_n = i)} \frac{\mathbb{P}(J_n = i)}{\mathbb{P}(J_n = i, J_{n+1} = j)} \\
&= \frac{\mathbb{P}(J_{n+1} = j, X_{n+1} \leq t | J_n = i)}{\mathbb{P}(J_{n+1} = j | J_n = i)} \\
&= \frac{Q_{ij}(t)}{p_{ij}}.
\end{aligned}$$
□

Definition 1.2.8. (Mean sojourn times)

Let m_{ij} be the mean sojourn times of SMP Z in state i when the next state is j, defined by

$$m_{ij} = \frac{1}{p_{ij}} \int_0^\infty t q_{ij}(t) dt = \int_0^\infty t f_{ij}(t) dt. \tag{1.5}$$

The mean sojourn time in state i, m_i is defined by

$$m_i = \mathbb{E}[S_1 | J_0 = i] = \sum_{j \in E} p_{ij} m_{ij} = \int_0^\infty \overline{F}_i(t) dt. \tag{1.6}$$

The mean sojourn time m is defined by

$$m = \sum_{i \in E} \nu_i m_i = \sum_{i \in E} \sum_{j \in E} \nu_i p_{ij} m_{ij}.$$

where $\nu = (\nu_i,\ i \in E)$ the stationary distribution of the EMC.

Let $(S_n^i)_{n \geq 0}$ be the renewal process (eventually delayed) of the times of successive passages in state i. Then $N_i(t)$ is the counting process of renewals.

In the case where $Z_0 = i$, we have $S_0^i = 0$ and the renewal process S_n^i is an ordinary one; otherwise, if $Z_0 \neq i$, it is a delayed renewal process.

Denote by μ_{ii} the mean recurrence time of the state i of Z. This is the mean interarrival times of the (possibly delayed) renewal process $(S_n^i), n \geq 0$:

$$\mu_{ii} := \mathbb{E}[S_2^i - S_1^i].$$

Let us consider an irreducible positive recurrent Markov renewal process. For any $i \in E$ (see Limnios and Oprişan [34]) we have the following equality

$$\mu_{ii} = \frac{m}{\nu_i}. \tag{1.7}$$

Let $\mu^* := (\mu_{ii}^*)$ be the vector of mean recurrence times in the state i for the EMC J, defined by

$$\mu_{ii}^* := \mathbb{E}[T_i^* | J_0 = i],$$

where T_i^* is the first entry time in i defined by $T_i^* := \min\{n \geq 1 : J_n = i\}$, i.e. it is the minimal number of jumps to arrive in state i.

The relationship between the mean recurrence time and the stationary distribution (see Kemeny and Snell [25]) is given by

$$\mu_{ii}^* = \frac{1}{\nu_i}, \tag{1.8}$$

and, from relation (1.7), we have

$$\frac{\mu_{ii}}{\mu_{ii}^*} = m.$$

Definition 1.2.9. (Stieltjes convolution)

Let $\phi(i,t)$ for $t \geq 0$ and $i \in E$, be a real-valued measurable function and define the convolution of ϕ by Q as follows

$$Q * \phi(i,t) := \sum_{k \in E} \int_0^t Q_{ik}(ds)\phi(k,t-s).$$

The n-fold Stieltjes convolution of $Q_{ij}(t)$ by itself, for any $i,j \in E$ is defined by,

$$Q_{ij}^{(n)}(t) := \begin{cases} \sum_{k \in E} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t-s), & \text{if } n \geq 2, \\ Q_{ij}(t), & \text{if } n = 1, \\ \delta_{ij}\mathbf{1}_{\{t \geq 0\}}, & \text{if } n = 0, \end{cases}$$

where δ_{ij} is Kronecker's delta symbol. We have

$$Q_{ij}^{(n)}(t) = \mathbb{P}(J_n = j, S_n \leq t | J_0 = i).$$

Remark 1.2.1.

The renewal function $\psi(t) = (\psi(t), i, j \in E)$ is called a Markov renewal function of the semi-Markov process defined by

$$\psi_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t), \quad i,j \in E, \quad t \geq 0.$$

All along this work we consider that the MRP and the SMP are homogeneous in time, in the sense that equation (1.3) is independent of n.

Theorem 1.2.1. If some real numbers, say $\alpha > 0$ and $\beta > 0$, exist, such that $F_i(\alpha) < 1 - \beta$, for all $i \in E$, then the semi-Markov process is regular.

Let us write the Markov renewal function in matrix form

$$\Psi(t) = (\mathbf{I} - Q(t))^{(-1)} = \sum_{n=0}^{\infty} Q^{(n)}(t). \tag{1.9}$$

This can also be written as

$$\Psi(t) = \mathbf{I}(t) + Q * \Psi(t), \tag{1.10}$$

where $\mathbf{I} = \mathbf{I}(t)$ (the identity matrix), if $t \geq 0$ and $\mathbf{I}(t) = 0$, if $t < 0$. The upper index (-1) in the matrix $(\mathbf{I} - Q(t))$ means its inverse in the convolution sense.

Equation (1.10) is a special case of what is called a Markov Renewal Equation (MRE). A general MRE is as follows

$$\Theta(t) = \mathbf{L}(t) + Q * \Theta(t), \tag{1.11}$$

where $\Theta(t) = (\Theta_{i,j}(t))_{i,j\in E}$, $\mathbf{L}(t) = (L_{i,j}(t))_{i,j\in E}$ are matrix-valued measurable functions, with $\Theta_{i,j}(t) = L_{i,j}(t) = 0$ for $t < 0$. The function $\mathbf{L}(t)$ is a given matrix valued function whereas $\Theta(t)$ is an unknown matrix-valued function. We may also consider a vector version of Equation (1.11), i.e., consider corresponding columns of the matrices Θ and $\mathbf{L}$.
Let $\mathbb{A}$ be the space of all bounded on compact sets of $\mathbb{R}_+$ matrix-valued functions $\Theta(t)$, i.e., $\|\Theta(t)\| = \sup_{i,j} |\Theta_{i,j}(t)|$ is bounded on sets $[0,\eta]$ for all $\eta \in \mathbb{R}_+$.

We say that a matrix function $\Theta(t) = \Theta_{i,j}(t)$ belongs to $\mathbb{A}$, if for any fixed $j \in E$ the column vector function $\Theta_{\cdot,j}(\cdot)$ belongs to $\mathbb{A}$.

Equation (1.11) has a unique solution $\Theta = \Psi * \mathbf{L}(t)$ belonging to $\mathbb{A}$ when $\mathbf{L}(t)$ belongs to $\mathbb{A}$ (see Gámiz et al.[19]).

Transition matrix

Another important function is $P(\cdot) = (P_{ij}(\cdot),\ i,j \in E)$ called the transition matrix function of the semi-Markov process Z defined by

$$P_{ij}(t) = \mathbb{P}(Z_t = j | Z_0 = i), \quad i,j \in E,\ t \geq 0.$$

It is known, cf. Pyke [45], that

$$P_{ij}(t) = \mathbb{1}_{\{i=j\}} \left(1 - \sum_{k=1}^{s} Q_{ik}(t)\right) + \sum_{k\in E} \int_0^t P_{kj}(t-s) Q_{ik}(ds). \tag{1.12}$$

By solving the above Markov renewal equation, cf. Limnios [35], in a matrix notation, we obtain the solution,

$$P(t) = (I - Q(t))^{(-1)} * (I - diag(Q(t)\mathbf{1})), \tag{1.13}$$

where $diag(\cdot)$ is a diagonal matrix with i^{th} element $\sum_{j=1}^{s} Q_{ij}(t)$ and $\mathbf{1} = (1,1,\ldots,1)'$.

Definition 1.2.10. For a semi-Markov process Z, the stationary distribution $\pi = (\pi_i, i \in E)$ is defined, when it exists, by

$$\pi_i = \lim_{t\to\infty} P_{ij}(t), \quad \text{for every } i,j \in E. \tag{1.14}$$

Proposition 1.2.4. [32] If the MRP is positive recurrent, the limit distribution is given by

$$\pi_i = \frac{\nu_i m_i}{\sum_{k=1}^{s} \nu_k m_k}. \tag{1.15}$$

The transition function is illustrated by the following example .

Definition 1.2.11. (**Regularity of SMP**) The semi-Markov process Z is said to be regular if the corresponding counting process $\{N(t); t \geq 0\}$ has a finite number of jumps in a finite period with probability 1:

$$\forall t \in \mathbb{R}_+, \quad \mathbb{P}(N(t) < \infty) = 1. \tag{1.16}$$

The equality (1.16) is equivalent to the relation

$$\forall t \in \mathbb{R}_+, \quad \mathbb{P}(N(t) = \infty) = 0.$$

Definition 1.2.12. For all $i, j \in E$ and $t \leq T$, let us define the following counting processes:

(i) $N_i(T) = \sum_{l=1}^{N(T)} \mathbb{1}_{\{J_{l-1}=i\}}$: the number of visits to state i, up to time T;

(ii) $N_{ij}(T) = \sum_{l=1}^{N(T)} \mathbb{1}_{\{J_{l-1}=i, J_l=j\}}$: the number of transitions from i to j, up to time T;

(iii) $N_{ij}(t,T) = \sum_{l=1}^{N(T)} \mathbb{1}_{\{J_{l-1}=i, J_l=j, X_l \leq t\}}$: the number of transitions from i to j, up to time T, with sojourn time in state i less than or equal to t.

Lemma 1.1. [34] Under the previous notations, if the EMC $(J_n)_n$ is positive recurrent, then, for any $i, j \in E$ we have:

1. $\dfrac{N_i(T)}{N(T)} \xrightarrow[T\to\infty]{a.s.} \nu_i,$
2. $\dfrac{N_{ij}(T)}{N(T)} \xrightarrow[T\to\infty]{a.s.} \nu_i p_{ij},$
3. $\dfrac{N_i(T)}{T} \xrightarrow[T\to\infty]{a.s.} \dfrac{1}{\mu_{ii}},$
4. $\dfrac{N_{ij}(T)}{T} \xrightarrow[T\to\infty]{a.s.} \dfrac{p_{ij}}{\mu_{ii}},$
5. $\dfrac{N(T)}{T} \xrightarrow[T\to\infty]{a.s.} \dfrac{1}{\nu_i \mu_{ii}}.$

CHAPTER 2

Empirical estimators for continuous-time semi-Markov process

In this chapter, we investigate the empirical estimation of the main characteristics of continuous-time semi-Markov processes, including the semi-Markov kernel, the transition matrix of the embedded Markov chain, the conditional distributions of the sojourn times, and the semi-Markov transition function. We also study the asymptotic properties of the proposed estimators, namely their strong consistency and asymptotic normality.

2.1 Construction of the estimators

Let us define the following observation in the time interval $[0,T]$.

$$\mathscr{H}_T = (J_0, J_1, \dots, J_{N(T)}, X_0, X_1, \dots, X_{N(T)}, U_T),$$

where $U_T = T - S_{N(T)}$.

The empirical estimator $\widehat{Q}_{ij}(t,T)$ of the semi-Markov kernel $Q_{ij}(t)$, related to $\mathscr{H}_T$, is defined by

$$\widehat{Q}_{ij}(t,T) = \frac{1}{N_i(T)} \sum_{n=1}^{N(T)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n \le t\}}, \tag{2.1}$$

and the empirical estimator of the transition matrix of the embedded Markov chain p_{ij} is defined by

$$\widehat{p}_{ij} = \frac{N_{ij}(T)}{N_i(T)},$$

then from the Proposition(1.2.3) , we obtain

$$\widehat{F}_{ij}(t,T) = \frac{1}{N_{ij}(T)} \sum_{n=1}^{N(T)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n \le t\}},$$

where $\widehat{F}_{ij}(t,T)$ is the empirical estimators for the conditional transition functions.

Let us denote by $\hat{\Psi}(t)$ the estimators of $\Psi(t)$, defined by

$$\hat{\Psi}(t,T) = \sum_{n=0}^{\infty} \hat{\mathbf{Q}}^{(n)}(t). \tag{2.2}$$

Let $\hat{P}$ be the estimator of the transition function of the semi-Markov process, given by

$$\hat{P}(t,T) = (I - \hat{Q}(t,T))^{(-1)} * (I - \mathrm{diag}(\hat{Q}(t,T)\mathbf{1})). \tag{2.3}$$

2.2 Asymptotic properties of the estimators

In this part, we study the asymptotic properties (consistency and asymptotic normality) of the proposed estimators.

2.2.1 Assumptions

The following assumptions concerning the Markov renewal chain will be needed in the rest of this work.

(H.1) $p_{ii} = 0$ for all $i \in E$.

(H.2) $q_{ij}(0) = 0$ for all $i, j \in E$, such that q_{ij} is the derivative of Q_{ij}.

(H.3) The EMC $(J_n)_{n\in\mathbb{N}}$ is an ergodic irreducible Markov chain.

(H.4) The SMP is irreducible, aperiodic, with finite mean sojourn times.

(H.5) The SMP (or equivalently, the MRP) is regular, that is $\mathbb{P}_i(N(t) < \infty) = 1$ for all $t > 0, i \in E$, with $\mathbb{P}_i(\cdot)$ means $\mathbb{P}(\cdot|J_0 = i)$.

2.2.2 Comments on the assumptions

Notice that these conditions are usually assumed in this context. Assumption **H.1** means that transitions to the same state are not allowed, and it is equivalent to $q_{ii(k)} = 0$ for all $i \in E, k \in \mathbb{N}$, whereas Assumption **H.2** confirms that There are no instantaneous transitions, and it is equivalent to $f_{ij}(0) \equiv 0$ for all $i, j \in E$; remark that this implies that S is a strictly increasing sequence.

In Addition, Conditions **(H.3)**, **(H.4)** and **(H.5)** are clearly unrestrictive, since they are the same as those classically used for the semi-Markov processes setting, (see, for instance Dumitrescu et al. [18] and Barbu et al. [6]). Note also that under condition **(H.3)**, $S_n < S_{n+1}, n \in \mathbb{N}$, $S_n \xrightarrow[n\to\infty]{a.s.} \infty$, $N(t) \xrightarrow[t\to\infty]{a.s.} \infty$ (see Limnios [34]).

2.2.3 Strong consistency

Corollary 2.2.1. [5] For any $i, j \in E$, under **H3**, we have

$$\widehat{p}_{ij}(T) = \frac{N_{ij}(T)}{N_i(T)} \xrightarrow[T\to\infty]{a.s.} p_{ij}.$$

Theorem 2.2.1. [37] For all $i, j \in E$, the empirical estimator $\widehat{Q}_{ij}(t,T)$ of $Q_{ij}(t)$ is strongly consistent, i.e.

$$\max_{i,j\in E} \sup_{t\in[0,T]} |\widehat{Q}_{ij}(t,T) - Q_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0$$

Proof.

From the definition of $Q_{ij}(t)$ (Equation (1.4)) and that of $\widehat{Q}_{ij}(t)$ (Equation(2.1)) note that

$$Q_{ij}(t) = \mathbb{P}(J_{n+1} = j, X_{n+1} \le t | J_n = i) = F_{ij}(t)p_{ij},$$
$$\widehat{Q}_{ij}(t,T) = \widehat{F}_{ij}(t,T)\widehat{p}_{ij}(T).$$

Then it follows that

$$\begin{aligned}
\max_{i,j\in E} \sup_{t\in[0,T]} |\widehat{Q}_{ij}(t,T) - Q_{ij}(t)| &= \max_{i,j\in E} \sup_{t\in[0,T]} |\widehat{p}_{ij}(T)\widehat{F}_{ij}(t,T) - \widehat{p}_{ij}(T)F_{ij}(t,T) \\
&\quad + \widehat{p}_{ij}(T)F_{ij}(t,T) - p_{ij}F_{ij}(t)| \\
&\le \max_{i,j\in E} \widehat{p}_{ij}(T) \max_{i,j\in E} \sup_{t\in[0,T]} |\widehat{F}_{ij}(t,T) - F_{ij}(t,T)| \\
&\quad + \max_{i,j\in E} \sup_{t\in[0,T]} F_{ij}(t,T) \max_{i,j\in E} |\widehat{p}_{ij}(T) - p_{ij}| \\
&\le \max_{i,j\in E} |\widehat{p}_{ij}(T) - p_{ij}| + \max_{i,j\in E} \sup_{t\in[0,T]} |\widehat{F}_{ij}(t,T) - F_{ij}(t)|.
\end{aligned}$$

And from the consistency of $\widehat{p}_{ij}(T)$ and $\widehat{F}_{ij}(t,T)$ (Corollary(2.2.1) and Glivenko-Cantelli Theorem (1.1.4)), we obtain the desired result. □

Theorem 2.2.2. [41] The empirical estimator $\widehat{Q}^{(n)}_{ij}(t,T)$ of $Q^{(n)}_{ij}(t)$ for all $i, j \in E$ is strongly consistent, i.e. for any fixed $n \in \mathbb{N}$

$$\max_{i,j\in E} \sup_{t\in[0,T]} |\widehat{Q}^{(n)}_{ij}(t,T) - Q^{(n)}_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

Proof.

By induction. For the case $n=1$, the result follows from theorem(2.2.1). Assume that it holds true for $n=m$. So

$$\max_{i,j\in E}\sup_{t\in[0,T]}|\widehat{Q}^{(m)}_{ij}(t,T)-Q^{(m)}_{ij}(t)|\xrightarrow[T\to\infty]{a.s.}0.$$

Now, let $n=m+1$. It follows that

$$\begin{aligned}
\max_{i,j\in E}\sup_{t\in[0,T]}|\widehat{Q}^{(m+1)}_{ij}(t,T)-Q^{(m+1)}_{ij}(t)| &= \max_{i,j\in E}\sup_{t\in[0,T]}\left|\sum_{k\in E}\widehat{Q}_{ik}(t,T)\star\widehat{Q}^{(m)}_{kj}(t,T)-\sum_{k\in E}Q_{ik}(t)\star Q^{(m)}_{kj}(t)\right| \\
&= \max_{i,j\in E}\sup_{t\in[0,T]}\left|\sum_{k\in E}(\widehat{Q}_{ik}(t,T)\star\widehat{Q}^{(m)}_{kj}(t,T)-Q_{ik}(t)\star Q^{(m)}_{kj}(t))\right| \\
&= \max_{i,j\in E}\sup_{t\in[0,T]}\sum_{k\in E}|\widehat{Q}_{ik}(t,T)\star\widehat{Q}^{(m)}_{kj}(t,T)-Q_{ik}(t)\star Q^{(m)}_{kj}(t)| \\
&= \max_{i,j\in E}\sup_{t\in[0,T]}\sum_{k\in E}|\widehat{Q}_{ik}(t,T)\star\widehat{Q}^{(m)}_{kj}(t,T)-Q_{ik}(t)\star Q^{(m)}_{kj}(t,T) \\
&\quad+\widehat{Q}_{ik}(t)\star Q^{(m)}_{kj}(t,T)-\widehat{Q}_{ik}(t)\star Q^{(m)}_{kj}(t,T)| \\
&\le \max_{i,j\in E}\sup_{t\in[0,T]}\sum_{k\in E}|\widehat{Q}_{ik}(t,T)\star\widehat{Q}^{(m)}_{kj}(t,T)-\widehat{Q}_{ik}(t)\star Q^{(m)}_{kj}(t,T)| \\
&\quad+\max_{i,j\in E}\sup_{t\in[0,T]}\sum_{k\in E}|Q_{ik}(t)\star Q^{(m)}_{kj}(t,T)-Q_{ik}(t)\star Q^{(m)}_{kj}(t)| \\
&= \max_{i,j\in E}\sup_{t\in[0,T]}\sum_{k\in E}|[\widehat{Q}_{ik}(t,T)-Q_{ik}(t)]\star\widehat{Q}^{(m)}_{kj}(t,T)| \\
&\quad+\max_{i,j\in E}\sup_{t\in[0,T]}\sum_{k\in E}|Q_{ik}(t)\star[Q^{(m)}_{kj}(t,T)-Q^{(m)}_{kj}(t)]| \\
&\le \max_{i,k\in E}\sup_{t\in[0,T]}|\widehat{Q}_{ik}(t,T)-Q_{ik}(t)|\max_{k,j\in E}\sup_{t\in[0,T]}\sum_{k\in E}\widehat{Q}^{(m)}_{kj}(t,T) \\
&\quad+\max_{i,k\in E}\sup_{t\in[0,T]}|Q^{(m)}_{kj}(t,T)-Q^{(m)}_{kj}(t)|\max_{k,j\in E}\sup_{t\in[0,T]}\sum_{k\in E}Q_{ik}(t) \\
&\le s\max_{i,k\in E}\sup_{t\in[0,T]}|\widehat{Q}_{ik}(t,T)-Q_{ik}(t)| \\
&\quad+\max_{i,k\in E}\sup_{t\in[0,T]}|Q^{(m)}_{kj}(t,T)-Q^{(m)}_{kj}(t)|
\end{aligned}$$

The last step holds true, because $E=\{1,2,\ldots,s\}$. By theorem 2.2.1 the first converges to 0 (a.s). By the induction hypothesis, the second term converges to 0 (a.s) as well. The result follows from the principle of mathematical induction. □

Theorem 2.2.3. [40] The empirical estimator $\widehat{\psi}_{ij}(t,T)$ of the Markov renewal function $\psi_{ij}(t)$ is uniformly strongly consistent on compact $[0,L]$ for $L\in\mathbb{R}_+$, for all $i,j\in E$, i.e.

$$\max_{i,j\in E}\sup_{t\in[0,L]}|\widehat{\psi}_{ij}(t,T)-\psi_{ij}(t)|\xrightarrow[T\to\infty]{a.s.}0.$$

Proof.

Let i and j be two states and $t \in [0,L]$ and $\varepsilon > 0$. Since $S_n \longrightarrow \infty$, (because the MRP which we have considered is positive recurrent), there exists a constant $k_0 > 0$ such that $\max_i \sum_{j=1}^{s} Q_{ij}^{(k_0)}(t) < 1$.

Let Ω be a set of probability 1, where the convergence of theorem (2.2.2) is valid for all $l \geq 1$. Set $\varepsilon = 1 - \max_i \sum_{j=1}^{s} Q_{ij}^{(k_0)}(t)$. For all $\omega \in \Omega$, there exists $T_0(\omega)$, such that for all $T \geq T_0(\omega)$,

$$\max_i \sum_{j=1}^{s} \hat{Q}_{ij}^{(k_0)}(t,T) \leq \max_i \left| \sum_{j=1}^{s} [\hat{Q}_{ij}^{(k_0)}(t,T) - Q_{ij}^{(k_0)}(t)] \right| + \\ + \max_i \sum_{j=1}^{s} Q_{ij}^{(k_0)}(t) \leq 1 - \frac{\varepsilon}{2}.$$

Moreover, for all $m \geq k_0$, there exists $(q,r) \in \mathbb{N}^* \times \mathbb{N}$ such that $m = qk_0 + r$ where $0 \leq r < k_0$ and we see that,

$$\begin{aligned} \max_{i,j} \hat{Q}_{ij}^{(m)}(t,T) &= \max_{i,j} \sum_{n=1}^{s} \hat{Q}_{in}^{(r)} * \hat{Q}_{nj}^{(qk_0)}(t,T) \\ &\leq \max_{i,j} \sum_{n=1}^{s} \hat{Q}_{in}^{(r)}(t,T) \cdot \hat{Q}_{nj}^{(qk_0)}(t,T) \\ &\leq \max_{i,j} \hat{Q}_{ij}^{(qk_0)}(t,T). \end{aligned}$$

Let us now prove that, for all $q \in \mathbb{N}^*$,

$$\max_i \sum_{j=1}^{s} \hat{Q}_{ij}^{(qk_0)}(t,T) \leq \left(1 - \frac{\varepsilon}{2}\right)^q.$$

In fact, for $q = 1$, the result is true. Suppose that the result is valid until order q and prove it to order $q+1$.

$$\begin{aligned} \max_i \sum_{j=1}^{s} \hat{Q}_{ij}^{((q+1)k_0)}(t,T) &= \max_i \sum_{j=1}^{s} \sum_{n=1}^{s} \hat{Q}_{in}^{(qk_0)} * \hat{Q}_{nj}^{(k_0)}(t,T) \\ &\leq \max_i \sum_{n=1}^{s} \hat{Q}_{in}^{(qk_0)}(t,T) \cdot \max_i \sum_{j=1}^{s} \hat{Q}_{ij}^{(k_0)}(t,T) \\ &\leq \left(1 - \frac{\varepsilon}{2}\right)^q \cdot \left(1 - \frac{\varepsilon}{2}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned}\hat{\psi}_{ij}(t,T) &= \sum_{l=0}^{\infty} \hat{Q}_{ij}^{(l)}(t,T) \\ &= \sum_{l=0}^{k_0} \hat{Q}_{ij}^{(l)}(t,T) + \sum_{l=k_0+1}^{2k_0} \hat{Q}_{ij}^{(l)}(t,T) + \sum_{l=2k_0+1}^{3k_0} \hat{Q}_{ij}^{(l)}(t,T) + \ldots\end{aligned}$$

Let $\beta_{ij}^{m}(t)$ be a sequence of functions defined by

$$\beta_{ij}^{m}(t) = \begin{cases} \hat{Q}_{ij}^{(m)}(t,T) & \text{if } m < k_0 \\ k_0(1-\frac{\varepsilon}{2})^{[m/k_0]} & \text{otherwise.} \end{cases}$$

Where $[x]$ is the integer part of x. We have, $\hat{\psi}_{ij}(t,T) \leq \sum_{m=0}^{\infty} \beta_{ij}^{m}(t) < \infty$. Thus by theorem (2.2.2) and the Lebesgue's dominated convergence theorem, we get

$$\hat{\psi}_{ij}(t,T) \longrightarrow \psi_{ij}(t)\, a.s., \quad \text{as} \quad T \longrightarrow \infty.$$

To prove that the estimator of the Markov renewal matrix is uniformly strongly consistent on compact $[0,L]$, for $L \in \mathbb{R}_+$, observe that $\psi_{ij}(t)$ is monotone and continuous (since F is continuous), so, the convergence of $\hat{\psi}_{ij}(t,T)$ to $\psi_{ij}(t,T)$ is uniform for $t \in [0,L]$. □

Theorem 2.2.4. Under the assumption of Theorem (2.2.1), the estimator of the semi-Markov matrix is uniformly strongly consistent in $[0,L]$, for all $L \in \mathbb{R}_+$, in the sense that

$$\max_{i,j} \sup_{t\in[0,L]} |\hat{P}_{ij}(t,T) - P_{ij}(t)| \longrightarrow 0 \quad a.s. \quad \text{when} \quad T \longrightarrow \infty.$$

Proof.

Let us consider the matrices $B(t) = I - \operatorname{diag}(Q(t)\cdot\mathbf{1})$ and $\hat{B}(t,T) = I - \operatorname{diag}(\hat{Q}(t,T)\cdot\mathbf{1})$, for $(i,j) \in E\times E$ be fixed then,

$$\begin{aligned}&\sup_{t\in[0,L]} |\hat{P}_{ij}(t,T) - P_{ij}(t)| \\ &\quad = \sup_{t\in[0,L]} |(\hat{\psi} * \hat{B})_{ij}(t,T) - (\psi * B)_{ij}(t)| \\ &\quad \leqslant \sup_{t\in[0,L]} |(\hat{\psi}_{ij}(t,T) - \psi_{ij})(t)| + \\ &\qquad + \sup_{t\in[0,L]} |(\hat{\psi}_{ij}(t,T) - \psi_{ij})(t)| \cdot \operatorname{diag}(\hat{Q}(t,T)\cdot\mathbf{1}) + \\ &\qquad + \sup_{t\in[0,L]} |\operatorname{diag}((\hat{Q}-Q)\cdot\mathbf{1})_{jj}(t,T)|\psi_{ij}(L).\end{aligned}$$

Since s, the number of states, is finite, the process is normal and therefore $\psi_{ij}(t)$ is finite [46] and from Theorems (2.2.1) and (2.2.3) the uniformly strong consistency of the estimators of the semi-Markov kernel and of the Markov renewal function in $[0,L]$, we see that $\operatorname{diag}((\hat{Q}-Q)\cdot \mathbf{1})_{jj}(t,T)$ and $\Delta\psi_{ij}(t,T)$ on $[0,L]$ converges a.s. to zero as T tends to infinity. □

2.2.4 Asymptotic normality

It is assumed throughout that the MRP is irreducible, recurrent, and that $F_{ij} = F_i$ for $1 \leq j \leq s$. Consider the estimator defined by:

$$\widehat{Q}_{ij}(t,T) = \widehat{F}_i(t,T)\widehat{p}_{ij}(T) \tag{2.4}$$

$$\widehat{F}_i(t,T) = \frac{1}{N_i(T)}\sum_{k=1}^{N_i(T)} \mathbb{1}_{\{X_{ik}\leq t\}} \tag{2.5}$$

$\widehat{F}_i(t,T)$ is therefore the ordinary empirical distribution function determined from the sample, of random size $N_i(T)$.

Theorem 2.2.5. [37] For fixed i,j,t,

$$(T^{1/2}[\widehat{p}_{ij}(T)-p_{ij}], T^{1/2}[\widehat{F}_i(t,T)-F_i(t)]), \tag{2.6}$$

converges in law as $T\to\infty$ to a bivariate normal r.v. with means zero and covariance matrix (σ_{ij}) given by

$$(\sigma_{ij})_{i,j\in E} = \begin{pmatrix} \mu_{ii}p_{ij}(1-p_{ij}) & 0 \\ 0 & \mu_{ii}F_i(t)(1-F_i(t)) \end{pmatrix}.$$

Proof.
Let w_1 and w_2 be arbitrary constants. To prove the asymptotic joint normality it suffices to show that

$$w_1T^{1/2}[\widehat{p}_{ij}(T)-p_{ij}] + w_2T^{1/2}[\widehat{F}_i(t,T)-F_i(t)], \tag{2.7}$$

converges in law to a normal r.v. for all w_1 and w_2. We rewrite Equation (2.7) as the product of $[T/N_i(T)]T^{-1/2}$ and a sum of the form Equation (1.1) by using the function f defined by

$$f(r,l,y) = \{w_1[\delta_{lj}-p_{ij}] + w_2[\mathbf{1}_{\{y\leq t\}} - F_i(t)]\}\delta_{ri}. \tag{2.8}$$

For this function

$$A_r = w_1\delta_{ri}[p_{rj}-p_{ij}] + w_2\delta_{ri}[F_r(t)-F_i(t)] = 0,$$

and

$$B_r = \{w_1^2[p_{rj}(T) + p_{ij}^2 - 2p_{ij}p_{rj}] + w_2^2[F_r(t) + F_i^2(t) - 2F_r(t)F_i(t)]\}\delta_{ri},$$

for $1 \leq r \leq s$, hence $r_i = 0$ and the third sum in the equation (1.2) is zero. Then

$$\sigma_i^2 = \sum_{r=1}^{s} B_r \frac{\mu_{ii}^*}{\mu_{rr}^*} = w_1^2 p_{ij}[1 - p_{ij}] + w_2^2 F_i(t)[1 - F_i(t)].$$

The variance σ_i^2 is finite, so from Lemma 7.1 of [46], the limiting distribution of $T^{-1/2}w_f(T)$ for the f given in Equation (2.8) is normal with zero mean and variance σ_i^2/μ_{ii}. But $T/N_i(T) \to \mu_{ii}$, (a.s.) so the limiting distribution of Equation (2.7) is normal with zero mean and variance $\sigma_i^2\mu_{ii}$ as required. The zero correlation between $\widehat{p}_{ij}(T)$ and $\widehat{F}_i(t,T)$ yields the following results.

Corollary 2.2.2. [37] For fixed i, j, t, $\widehat{p}_{ij}(T)$ and $\widehat{F}_i(t,T)$ are asymptotically independent.

The asymptotic normality of Equation (2.7) can be used to obtain the limiting distribution of $\widehat{Q}_{ij}(t,T)$.

Theorem 2.2.6. [37] The empirical estimator $\widehat{Q}_{ij}(t,T)$ is asymptotically normal, i.e. for fixed $t > 0$

$$\sqrt{T}[\widehat{Q}_{ij}(t,T) - Q_{ij}(t)] \xrightarrow[T\to\infty]{\mathscr{D}} \mathscr{N}(0,\sigma^2).$$

With

$$\sigma^2 = \mu_{ii}F_i(t)p_{ij}[F_i(t) - 2F_i(t)p_{ij} + p_{ij}].$$

Proof.

We rewrite $\sqrt{T}[\widehat{Q}_{ij}(t,T) - Q_{ij}(t)]$ as

$$\begin{aligned}&\sqrt{T}[\widehat{F}_i(t,T)\widehat{p}_{ij}(T) - \widehat{F}_i(t,T)p_{ij} + \widehat{F}_i(t,T)p_{ij} - F_i(t)p_{ij}] = \\ &T^{1/2}\widehat{F}_i(t,T)[\widehat{p}_{ij}(T) - p_{ij}] + T^{1/2}p_{ij}[\widehat{F}_i(t,T) - F_i(t)]. \end{aligned} \tag{2.9}$$

By a well-known convergence Theorem [16] the limiting distribution of Equation (2.9) is the same as that of

$$T^{1/2}F_i(t)[\widehat{p}_{ij}(T) - p_{ij}] + T^{1/2}p_{ij}[\widehat{F}_i(t,T) - F_i(t)]. \tag{2.10}$$

With the particular choice $\omega_1 = F_i(t)$ and $\omega_2 = p_{ij}$, Equation (2.10) is just the same as Equation and the proof is complete. □

Lemma 2.1. For all $i, j, k, l \in E$, $\sqrt{T}|[\Delta Q_{ij} * \Delta Q_{kl}](t)|$ converges in probability to zero, when T tends to infinity.

Proof.

For all $i,j,k,l \in E$, $\sqrt{T}|[\Delta Q_{ij} * \Delta Q_{kl}](t)|$ has the same limit as $\sqrt{T} p_{ij} p_{kl} |[\Delta F_{ij} * \Delta F_{kl}](t)|$. Suppose that $(i,j) \neq (k,l)$. For all $n,m \in \mathbb{N}, n \neq m$, X_{ijn} and X_{klm} are independent random variables, then,

$$
\begin{aligned}
[\Delta F_{ij} * \Delta F_{kl}](t) &= [\hat{F}_{ij} * \hat{F}_{kl} - \hat{F}_{ij} * F_{kl} - F_{ij} * \hat{F}_{kl} + F_{ij} * F_{kl}](t) \\
&= \frac{\sum_{n=1}^{N_{ij}} \sum_{m=1}^{N_{kl}} 1_{\{X_{ijn}+X_{klm} \leqslant t\}}}{N_{ij} N_{kl}} - \\
&\quad - \frac{\sum_{n=1}^{N_{ij}} F_{kl}(t - X_{ijn})}{N_{ij}} \frac{\sum_{m=1}^{N_{kl}} F_{ij}(t - X_{klm})}{N_{kl}} + [F_{ij} * F_{kl}](t) \\
&= \frac{\sum_{n=1}^{N_{ij}} \sum_{m=1}^{N_{kl}} \{1_{\{X_{ijn}+X_{klm} \leqslant t\}} - F_{kl}(t - X_{ijn}) - F_{ij}(t - X_{klm}) + F_{ij} * F_{kl}(t)\}}{N_{ij} N_{kl}}.
\end{aligned}
\tag{2.11}
$$

When $(i,j) = (k,l)$, for all $n,m \in \mathbb{N}$, $n \neq m$, X_{ijn} and X_{ijm} are independent random variables, but since the number of times that $n = m$ is at most N_{ij}, we see that in the expression of $\sqrt{T} \Delta F_{ij} * \Delta F_{ij}$, the terms of X_{ijn} and X_{ijm} with $n = m$ will be negligible. So the asymptotic analysis of this case is similar to the case where $(i,j) \neq (k,l)$.

Let Z_{nm} denote the random variable $1_{\{X_{ijn}+X_{klm} \leqslant t\}} - F_{ij}(t - X_{klm}) - F_{kl}(t - X_{ijn}) + F_{ij} * F_{kl}$. The conditional expectation of Z_{nm} given X_{ijn}

$$
\begin{aligned}
E(Z_{nm}|X_{ijn}) &= E(1_{\{X_{ijn}+X_{klm} \leqslant t\}} - F_{ij}(t - X_{klm}) - F_{kl}(t - X_{ijn}) + \\
&\quad + F_{ij} * F_{kl}(t) \mid X_{ijn}) \text{ a.s.} \\
&= F_{kl}(t - X_{ijn}) + F_{ij} * F_{kl}(t) - F_{ij} * F_{kl}(t) - \\
&\quad - F_{kl}(t - X_{ijn}) \text{ a.s.} \\
&= 0 \text{ a.s.}
\end{aligned}
\tag{2.12}
$$

So, from (2.11) and (2.12) and the fact that $(X_{ijl})_{1 \leqslant l \leqslant N_{ij}}$ is a sequence of independent random variables, it is clear that in developing the expression of $(\Delta F_{ij} * \Delta F_{kl})^2$, only $N_{ij} N_{kl}$ squared terms have non-zero expectation. Since for all n and m, $|Z_{nm}| \leqslant 2$, by the Markov inequality, we see that

$$
\begin{aligned}
P(\sqrt{T}|[\Delta F_{ij} * \Delta F_{kl}](t)| \geqslant \varepsilon) &\leqslant \frac{T}{\varepsilon^2} E[(\{\Delta F_{ij} * \Delta F_{kl}\}(t))^2] \\
&\leqslant \frac{4T}{\varepsilon^2} E\left[\frac{N_{ij} N_{kl}}{(N_{ij} N_{kl})^2}\right] \\
&= \frac{4}{\varepsilon^2} E\left[\frac{T}{N_{ij} N_{kl}}\right].
\end{aligned}
\tag{2.13}
$$

From the strong law of large numbers [46] applied to semi-Markov positive recurrent processes, we see that for all $(i,j) \in E \times E$, T/N_{ij} converges a.s. and in L^1 towards μ_{ii}/p_{ij}. We conclude that, $\sqrt{T}[\Delta F_{ij} * \Delta F_{kl}](t)$ converges in probability to zero. $\square$

Lemma 2.2. For all $i,j \in \{1,\dots,s\}$, $t \in [0,T]$ and $l \in \mathbb{N}^*$, the random variate $\sqrt{T}\Delta Q_{ij}^{(l)}(t)$ has the same limit in law as $\sum_{r=1}^{l}\sum_{k=1}^{s}\sum_{n=1}^{s}\sqrt{T}[Q_{ik}^{(r-1)} * \Delta Q_{kn} * Q_{nj}^{(l-r)}](t)$ as T tends to infinity.

Proof.
To give the proof of this lemma, we proceed by induction. For $l = 1$, it is obvious.
For $l = 2$, we see that

$$\begin{aligned}
\hat{Q}_{ij}^{(2)} - Q_{ij}^{(2)} &= \sum_{k=1}^{s}(\hat{Q}_{ik} * \hat{Q}_{kj} - Q_{ik} * Q_{kj}) \\
&= \sum_{k=1}^{s}\{(\hat{Q}_{ik} - Q_{ik}) * \hat{Q}_{kj} + Q_{ik} * (\hat{Q}_{kj} - Q_{kj})\} \\
&= \sum_{k=1}^{s}\{(\hat{Q}_{ik} - Q_{ik}) * (\hat{Q}_{kj} - Q_{kj}) + Q_{ik} * (\hat{Q}_{kj} - Q_{kj}) + \\
&\quad + (\hat{Q}_{ik} - Q_{ik}) * Q_{kj}\} \\
&= \sum_{k=1}^{s}\{\Delta Q_{ik} * Q_{kj} + Q_{ik} * \Delta Q_{kj} + \Delta Q_{ik} * \Delta Q_{kj}\}.
\end{aligned}$$

So, from the previous lemma we see that for fixed t, $\sqrt{T}[\Delta Q_{ik} * \Delta Q_{kj}](t)$ converges in probability to zero as T tends to infinity, thus $\sqrt{T}\Delta Q_{ij}^{(2)}$ has the same limit in law as $\sqrt{T}\sum_{k=1}^{s}\{\Delta Q_{ik} * Q_{kj} + Q_{ik} * \Delta Q_{kj}\}$.

Suppose now that the induction hypothesis is true until l and we shall prove it for $l+1$.

$$\begin{aligned}
\sqrt{T}\Delta Q_{ij}^{(l+1)}(t) &= \sqrt{T}\sum_{k=1}^{s}(\hat{Q}_{ik}^{(l)} * \hat{Q}_{kj} - Q_{ik}^{(l)} * Q_{kj}) \\
&= \sqrt{T}\sum_{k=1}^{s}(\Delta Q_{ik}^{(l)} * Q_{kj} + Q_{ik}^{(l)} * \Delta Q_{kj} + \Delta Q_{ik}^{(l)} * \Delta Q_{kj}).
\end{aligned}$$

From the previous lemma, for fixed t, $\sqrt{T}\Delta Q_{ij}^{(l+1)}(t)$ converges weakly to the same limit, when T tends to infinity, as

$$\sqrt{T}\sum_{k=1}^{s}[\Delta Q_{ik}^{(l)} * Q_{kj} + Q_{ik}^{(l)} * \Delta Q_{kj}](t).$$

From the induction hypothesis, it can be written as

$$\sqrt{T}\sum_{k=1}^{s}\left\{\left[\sum_{r=1}^{l}\sum_{k=1}^{s}\sum_{n=1}^{s}\left(Q_{ik}^{(r-1)}*\Delta Q_{kn}*Q_{nj}^{(l-r)}\right)*Q_{kj}\right](t)+\right.$$
$$\left.+\left[Q_{ik}^{(l)}*\Delta Q_{kj}(t)\right]\right\}(t)$$
$$=\sqrt{T}\left\{\sum_{r=1}^{l}\sum_{k=1}^{s}\sum_{n=1}^{s}[Q_{ik}^{(r-1)}*\Delta Q_{kn}*Q_{nj}^{(l-r+1)}](t)+\right.$$
$$\left.+\sum_{k=1}^{s}[Q_{ik}^{(l)}*\Delta Q_{kj}](t)\right\}$$
$$=\sum_{r=1}^{l+1}\sum_{n=1}^{s}\sum_{m=1}^{s}\sqrt{T}[Q_{in}^{(r-1)}*\Delta Q_{nm}*Q_{mj}^{(l+1-r)}](t).$$

Which is the desired result. □

Theorem 2.2.7. For all $i,j\in E$ and $t\in[0,T]$, $\sqrt{T}\Delta\psi_{ij}(t)$ converges in law to a normal random variable with mean zero and variance

$$\sigma_{ij}^{2}(t)=\sum_{l=1}^{s}\sum_{k=1}^{s}\mu_{ll}\{[\psi_{il}*\psi_{kj}]^{2}*Q_{lk}-[\psi_{il}*\psi_{kj}*Q_{lk}]^{2}\}(t),$$

where μ_{ll} is the mean recurrence time of state l.

Proof.

By the Markov renewal equation we see that,

$$\begin{aligned}\sqrt{T}\Delta\psi_{ij}(t,T)&=\sqrt{T}(\hat{\psi}_{ij}(t,T)-\psi_{ij}(t))\\&=\sqrt{T}[\hat{\psi}_{ij}-(\hat{\psi}*\psi)_{ij}+(\hat{\psi}*\psi)_{ij}-\psi_{ij}](t)\\&=\sqrt{T}\{[\hat{\psi}*(I-\psi)]_{ij}+[(\hat{\psi}-I)*\psi]_{ij}\}(t)\\&=\sqrt{T}\{-(\hat{\psi}*Q*\psi)_{ij}+(\hat{\psi}*\hat{Q}*\psi)_{ij}\}(t)\\&=\sqrt{T}[\hat{\psi}*\Delta Q*\psi]_{ij}(t)\\&=\sqrt{T}[\hat{\psi}*\Delta Q*\psi*\Delta Q*\psi]_{ij}(t)+\\&\quad+\sqrt{T}[\psi*\Delta Q*\psi]_{ij}(t).\end{aligned}\tag{2.14}$$

Since for all $i,k,l,r,v,w\in E$,

$$\sup_{s\le t}[\hat{\psi}_{ik}(\cdot,T)*\psi_{lr}(\cdot)*\psi_{vw}(\cdot)](s)\le\left[\sum_{m=1}^{\infty}\beta_{ij}^{m}(t)\right]\cdot\psi_{lr}(t)\cdot\psi_{vw}(t)<+\infty,$$

we conclude by theorem (2.2.2), that the first term of (2.14) converges in probability to zero as T tends towards infinity. Apply central limit theorem related to semi-Markov processes [46] to the function

$$f(n,m,x) = \sum_{l=1}^{s}\sum_{k=1}^{s}[\psi_{il} * \psi_{kj} * 1_{\{n=l,m=k,x\leq t\}}],$$

we deduce that the second term of (2.14) converges in law to a normal random variable with mean zero and variance $\sigma_{ij}^2(t)$. □

Theorem 2.2.8. For all $i,j \in E$ and $t \in [0,T]$, $\sqrt{T}(\hat{P}_{ij}(t,T) - P_{ij}(t))$ converges in law to a zero mean normal random variable with variance

$$\sigma_{ij}^2(t) = \sum_{n=1}^{s}\sum_{k=1}^{s}\mu_{nn}\left[\left(\left(1-\sum_{l=1}^{s}Q_{jl}\right) * B_{inkj} - \psi_{ij}1_{\{n=j\}}\right)^2 * Q_{nk}(t)\right] - \\ -\left\{\left[\left(1-\sum_{l=1}^{s}Q_{jl}\right) * B_{inkj} - \psi_{ij}1_{\{n=j\}}\right] * Q_{nk}(t)\right\}^2.$$

Proof.

From (2.3), we see that,

$$\begin{aligned}\sqrt{T}\Delta P_{ij} &= \sqrt{T}[\hat{\psi}_{ij} * (I - \mathrm{diag}(\hat{Q}\mathbf{1}))_{jj} - \psi_{ij} * (I - \mathrm{diag}(Q\mathbf{1}))_{jj}] \\ &= \sqrt{T}[(\hat{\psi}_{ij} - \psi_{ij}) * (I - \mathrm{diag}(Q\mathbf{1}))_{jj} - \\ &\quad - \psi_{ij} * \mathrm{diag}([\hat{Q} - Q]\mathbf{1})_{jj} - \\ &\quad - (\hat{\psi}_{ij} - \psi_{ij}) * \mathrm{diag}([\hat{Q} - Q]\mathbf{1})_{jj}].\end{aligned}$$

From lemma (2.1), $\sqrt{T}(\hat{\psi}_{ij} - \psi_{ij}) * \mathrm{diag}([\hat{Q} - Q]\mathbf{1})_{jj}$ converges, in probability, when T tends to infinity, to zero. So, $\sqrt{T}\Delta P_{ij}(t,T)$ has the same limit as

$$\sqrt{T}[(\hat{\psi}_{ij} - \psi_{ij}) * (I - \mathrm{diag}(Q\mathbf{1}))_{jj} - \psi_{ij} * \mathrm{diag}([\hat{Q} - Q]\mathbf{1})_{jj}].$$

From Lemma (2.2) and Theorem (2.2.7), it has the same limit as,

$$\begin{aligned}&\sqrt{T}\left[\left(1-\sum_{l=1}^{s}Q_{jl}\right) * \left(\sum_{n=1}^{s}\sum_{k=1}^{s}B_{inkj} * \Delta Q_{nk}\right) - \psi_{ij} * \left(\sum_{k=1}^{s}\Delta Q_{jk}\right)\right] \\ &= \sum_{n=1}^{s}\sum_{k=1}^{s}\sqrt{T}\left[\left(1-\sum_{l=1}^{s}Q_{jl}\right) * B_{inkj} * \Delta Q_{nk}\right] - \sum_{k=1}^{s}\sqrt{T}\psi_{ij} * \Delta Q_{jk}.\end{aligned} \tag{2.15}$$

Let f be a real function defined on $E \times E \times \mathbb{R}_+$ by

$$\begin{aligned}f(r,m,x) = &\left[\left(1-\sum_{l=1}^{s}Q_{jl}\right) * B_{inkj} - \psi_{ij}1_{\{n=j\}}\right] \times \\ &\times 1_{\{r=n\}}(1_{\{m=k,x\leqslant t\}} - Q_{nk}).\end{aligned} \tag{2.16}$$

So, $\sqrt{T}\Delta P_{ij}(t,T) = \sqrt{T}W_f(t)$, where

$$W_f(t) = \sum_{n=1}^{s}\sum_{k=1}^{s}\frac{\sum_{l=1}^{N_n} f(J_{l-1}, J_l, X_l)}{N_n}, \tag{2.17}$$

and by Pyke and Schaufele's central limit theorem [46], we get the desired results. $\square$

CHAPTER 3

Kernel estimators for continuous-time semi-Markov processes

In this chapter, we study kernel-based estimators for the key characteristics of a continuous-time semi-Markov system, such as the semi-Markov kernel, the conditional distributions of sojourn times, and the semi-Markov transition function. The objective is to develop a smooth alternative to classical empirical methods, which often suffer from discontinuities. By using kernel techniques, we obtain more refined and accurate estimates. Furthermore, we analyze the asymptotic behavior of these estimators, focusing in particular on their strong consistency and asymptotic normality.

3.1 Nonparametric kernel estimators

Before stating the construction of the nonparametric kernel estimators, let us also introduce the following definition.

Definition 3.1.1. Let us consider a sample path of the Markov renewal processes $(J_l, S_l)_{l\in\mathbb{N}}$,

$$\mathscr{H}(T) := (J_0, X_1, \ldots, J_{N(T)-1}, X_{N(T)}, J_{N(T)}, u_T), \quad T \in \mathbb{R}_+,$$

where $u_T := T - S_{N(T)}$.

For fixed states i and j, let us denote by $X_{i;l}$ the corresponding sojourn time in state i during lth visit of this state and by $X_{ij;l}$ the corresponding sojourn time in state i before going to state j, during lth visit.

Taking a sample path $\mathscr{H}(T)$ of Markov renewal processes, for all $i, j \in E$ and $t \in \mathbb{R}_+$, $t \leq T$, we define the kernel estimators introduced by Ayhar et al.[4] of $F_i(t)$, $F_{ij}(t)$, $Q_{ij}(t)$ and of the derivatives $f_{ij}(t)$, $f_i(t)$, $q_{ij}(t)$ as follows:

$$\widehat{F}_i(t,T) = \frac{1}{N_i(T)} \sum_{l=1}^{N_i(T)} H\left(\frac{t-X_{i;l}}{h_{i,T}}\right) = \frac{1}{N_i(T)} \sum_{l=1}^{N(T)} H\left(\frac{t-X_l}{h_{i,T}}\right) \mathbb{1}_{\{J_{l-1}=i\}}; \tag{3.1}$$

$$\widehat{F}_{ij}(t,T) = \frac{1}{N_{ij}(T)} \sum_{l=1}^{N_{ij}(T)} H\left(\frac{t-X_{ij;l}}{h_{ij,T}}\right) = \frac{1}{N_{ij}(T)} \sum_{l=1}^{N(T)} H\left(\frac{t-X_l}{h_{ij,T}}\right) \mathbb{1}_{\{J_{l-1}=i,J_l=j\}}; \tag{3.2}$$

$$\widehat{f}_{ij}(t,T) = \frac{1}{N_{ij}(T)} \sum_{l=1}^{N_{ij}(T)} \frac{1}{h_{ij,T}} K\left(\frac{t-X_{ij;l}}{h_{ij,T}}\right); \tag{3.3}$$

$$\widehat{f}_i(t,T) = \frac{1}{N_i(T)} \sum_{l=1}^{N_i(T)} \frac{1}{h_{i,T}} K\left(\frac{t-X_{i;l}}{h_{i,T}}\right), \tag{3.4}$$

where $H(t) = \int_{-\infty}^{t} K(t)\,dt$. It should be noted that the smoothing parameter of the previous estimators depends on the sample size, so we should write $h_{i,T} = h_{i,N_i(T)}$ (resp. $h_{ij,T} = h_{ij,N_{ij}(T)}$), however we prefer to use a simpler notation.

Second, we can define an estimator of $Q_{ij}(T)$ defined by:

$$\widehat{Q}_{ij}(t,T) = \widehat{p}_{ij}(T)\widehat{F}_{ij}(t,T) = \frac{N_{ij}(T)}{N_i(T)}\widehat{F}_{ij}(t,T),$$

where $\widehat{p}_{ij}(T) := \frac{N_{ij}(T)}{N_i(T)}$ is the empirical estimator of p_{ij}. So we get the corresponding kernel estimators of $Q_{ij}(t)$ and $q_{ij}(t)$ given by

$$\widehat{Q}_{ij}(t,T) = \frac{1}{N_i(T)} \sum_{l=1}^{N_{ij}(T)} H\left(\frac{t-X_{ij;l}}{h_{ij,T}}\right), \tag{3.5}$$

$$\widehat{q}_{ij}(t,T) = \frac{1}{N_i(T)} \sum_{l=1}^{N_{ij}(T)} \frac{1}{h_{ij,T}} K\left(\frac{t-X_{ij;l}}{h_{ij,T}}\right). \tag{3.6}$$

Let us denote by $\hat{\Psi}(t)$ the estimators of $\Psi(t)$, defined by

$$\hat{\Psi}(t,T) = \sum_{n=0}^{\infty} \hat{\mathbf{Q}}^{(n)}(t).$$

Let $\hat{P}$ be the estimator of the transition function of the semi-Markov process, given by

$$\hat{P}(t,T) = (I-\hat{Q}(t,T))^{(-1)} * (I - \mathrm{diag}(\hat{Q}(t,T)\mathbf{1})). \tag{3.7}$$

3.2 Asymptotic properties of the estimators

Let us first focus on the assumptions we need to derive the asymptotic behaviour of our estimators.

3.2.1 Assumptions

The following assumptions will be needed in the rest of this work:

(H.6) (i) $Q_{ij}(t), F_i(t)$ and $F_{ij}(t)$ are absolutely continuous with respect to the Lebesgue measure, and let $q_{ij}(t), f_i(t)$ and $f_{ij}(t)$ be respectively the corresponding Radon-Nikodym derivative.

(ii) The first derivatives f_{ij}, f_i and q_{ij} are bounded.

(H.7) (i) The function H is a distribution function.

(ii) The kernel K is density function of bounded variation such that

$$\lim_{x\to\infty}|xK(x)| = 0 \quad \text{and} \quad \left|\int t^j K^k(t)dt\right| < \infty \text{ for } j = 0,1, \text{ and } k = 1,2,$$

where K is the derivative of H.

(H.8) The smoothing parameters $h_{i,T}$ and $h_{ij,T}$ are sequences of positive numbers satisfying:

$$\lim_{T\to\infty} h_{i,T} = 0, \quad \lim_{T\to\infty} h_{ij,T} = 0 \quad \text{and} \quad \lim_{T\to\infty} Th_{ij,T} = \infty.$$

(H.9) The series $\sum_{n=1}^{\infty} e^{-\gamma n h_{i,n}^2}$ and $\sum_{n=1}^{\infty} e^{-\gamma n h_{ij,n}^2}$ converge for every positive value of γ.

3.2.2 Comments on the assumptions

Assumption (H.6) imposed on $Q_{ij}(t), F_{ij}(t)$ and $F_i(t)$ is a regularity type hypothesis. Precisely, Hypothesis (H.6) (i) is a continuity-type constraint which will allow us to get strong consistency. Moreover, as soon as one wishes to state the asymptotic normality of our estimators, one has to introduce more restrictive constraints, which is the role played by the second derivative hypothesis (H.6) (ii). The technical conditions on the kernels are imposed for a sake of brevity of proofs. (H.8) and (H.9) are another technical constraints. Furthermore, (H.8) is also satisfied for $h_{ij,T}$ and $h_{i,T}$.

Our first results concern the uniform strong consistency of the proposed estimators.

3.2.3 Strong consistency

Theorem 3.2.1. (Gut [22]) Suppose that $Y_1, Y_2, \ldots$ are random variables such that

$$Y_n \xrightarrow[n\to\infty]{a.s.} Y,$$

and that $\{N(t), t \geq 0\}$ is a family of positive, integer valued random variables, such that

$$N(t) \xrightarrow[t\to\infty]{a.s.} \infty.$$

Then

$$Y_{N(t)} \xrightarrow[t\to\infty]{a.s.} Y.$$

We suppose in the sequel that $N_i(t) \xrightarrow[t\to\infty]{a.s.} \infty$ and $N_{ij}(t) \xrightarrow[t\to\infty]{a.s.} \infty$.

Theorem 3.2.2. For any fixed arbitrary states $i, j \in E$ and any fixed arbitrary positive $t \in \mathbb{R}_+$, $t \leq T$, under Assumptions (H.7)–(H.8), the following statement stands true.

The kernel estimator $\widehat{F_i}(t,T)$ introduced in (3.1) is uniformly strongly consistent, i.e.

$$\max_i \sup_{t\in[0,T]} |\widehat{F_i}(t,T) - F_i(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

Proof. Let

$$\widehat{F_i^*}(t) = \frac{1}{n}\sum_{l=1}^{n} H\left(\frac{t - X_{i,l}}{h_{i,n}}\right).$$

Under assumptions (H.7) and (H.8), applying Theorem 1 of Nadaraya [38] we have

$$\max_i \sup_{t\in[0,T]} |\widehat{F_i^*}(t) - F_i(t)| \xrightarrow[n\to\infty]{a.s.} 0.$$

Taking into account Theorem3.2.1, we obtain the desired result. □

According to the definition of the kernel estimators (3.2), (3.3), (3.4) and (3.5), we can establish their uniform strong consistency which proofs are a straightforward adaptation of the proof of Theorem 3.2.2.

Corollary 3.2.1. For any fixed arbitrary states $i, j \in E$ and any fixed arbitrary positive $t \in \mathbb{R}_+$, $t \leq T$, under Assumptions (H.7)–(H.8) and additional hypothesis (H.9) for (ii), (iii) and (iv), the following statements stand true.

(i) The kernel estimator $\widehat{F_{ij}}(t,T)$ introduced in (3.2) is uniformly strong consistent, i.e.

$$\max_{i,j} \sup_{t\in[0,T]} |\widehat{F_{ij}}(t,T) - F_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

(ii) The kernel estimator $\widehat{f_{ij}}(t,T)$ proposed in (3.3) is uniformly strong consistent, i.e.

$$\max_{i,j} \sup_{t\in[0,T]} |\widehat{f_{ij}}(t,T) - f_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

(iii) The kernel estimator $\widehat{f}_i(t,T)$ introduced in (3.4) is uniformly strong consistent, i.e.

$$\max_i \sup_{t\in[0,T]} |\widehat{f}_i(t,T) - f_i(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

(iv) The kernel estimator of the semi-Markov kernel density proposed in (3.6) is uniformly strongly consistent, i.e.

$$\max_{i,j} \sup_{t\in[0,T]} |\widehat{q}_{ij}(t,T) - q_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

(v) Since $\widehat{Q}_{ij}(t,T) = \widehat{p}_{ij}(T)\widehat{F}_{ij}(t,T)$, the uniform strong consistency of the estimators $\widehat{p}_{ij}(T)$ and $\widehat{F}_{ij}(t,T)$ allow us to deduce that:

$$\max_{i,j} \sup_{t\in[0,T]} |\widehat{Q}_{ij}(t,T) - Q_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

The following results concern the asymptotic normality of the proposed estimators.

Proof.
We give here only the proof of (iv). For the other points, according to the definition of the kernel estimators (3.2), (3.3), (3.4) and (3.5), we can establish their uniform strong consistency which proofs are a straightforward adaptation of the proof of Theorem 3.2.2.
(iv) We have for all $i,j\in E$: $\widehat{q}_{ij}(t,T) = \widehat{f}_{ij}(t,T)\widehat{p}_{ij}(T)$, $|\widehat{p}_{ij}(T)| \leq 1$, and $|f_{ij}(t,T)| \leq 1$. After some computation we obtain

$$\begin{aligned}
&\mathbb{P}\left(\max_{i,j} \sup_{t\in[0,T]} |\widehat{q}_{ij}(t,T) - q_{ij}(t)| > \varepsilon\right) \\
&\qquad \leq \mathbb{P}\left(\max_{i,j} |\widehat{p}_{ij}(T) - p_{ij}| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\max_{i,j} \sup_{t\in[0,T]} |\widehat{f}_{ij}(t,T) - f_{ij}(t)| > \frac{\varepsilon}{2}\right).
\end{aligned}$$

Hence the result is a consequence of (ii) in Corollary 3.2.1 under assumptions (H.8), (H.9) and of the strong consistency of $\widehat{p}_{ij}(T)$. □

Theorem 3.2.3. Under the assumptions **(H.7)**, **(H.8)**, we have, for all $i,j\in E$ and $n\in\mathbb{N}$,

$$\max_{i,j} \sup_{t\in[0,\infty]} |\hat{Q}_{ij}^{(n)}(t,T) - Q_{ij}^{(n)}(t)| \xrightarrow[T\to\infty]{a.s.} 0, \text{ as } T\to\infty.$$

Proof.
The proof of this Lemma is similar to the proof of that Ouhbi and Limnios [41]. For $n=1$, it is

the result of Ayhar et al. [4]. Suppose that this result is verified until the order $n-1$, for i and j two states, we have

$$\max_{i,j} \sup_{t\in[0,\infty]} |\hat{Q}_{ij}^{(n)}(t,T) - Q_{ij}^{(n)}(t)| \le \max_{i,j} \sup_{t\in[0,\infty]} |\hat{Q}_{ij}^{(n-1)}(t,T) - Q_{ij}^{(n-1)}(t)| + s \cdot \max_{i,j} \sup_{t\in[0,\infty]} |\hat{Q}_{ij}(t,T) - Q_{ij}(t)|.$$

This result converges to zero a.s., as $T \to \infty$ by the fact that $\sum_{j=1}^{s} |\hat{Q}_{ij}^{m}(t,T)| \le s$ for every $m \ge 0$ and the induction hypothesis. □

Theorem 3.2.4. For any fixed $t \in \mathbb{R}_+$ and $i, j \in E$, we have

(a) for any fixed $L > 0$ we have

$$\max_{i,j} \sup_{t\in[0,L]} |\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)| \xrightarrow{a.s.} 0, \text{ as } T \to \infty,$$

(b) for any fixed $L > 0$ we have

$$\max_{i,j} \sup_{t\in[0,L]} |\hat{P}_{ij}(t,T) - P_{ij}(t)| \xrightarrow{a.s.} 0, \text{ as } T \to \infty,$$

Proof.

(a)

The proof of this part is a straightforward adaptation of the proof of the uniform strong consistency established in Ouhbi and Limnios [41].

(b)

The proof of this part is a straightforward adaptation of the proof of the uniform strong consistency established in Ouhbi and Limnios [40], combining with Corollary 3.2.2(ii) and 3.2.4(a).

□

3.2.4 Asymptotic normality

Theorem 3.2.5. For any fixed arbitrary states $i, j \in E$ and any fixed arbitrary positive $t \in \mathbb{R}_+$, $t \le T$, under Assumptions (H.6), (H.7) and (H.8), the following statements stand true.

(i)

$$\sqrt{T}[\widehat{F}_i(t,T) - F_i(t)] \xrightarrow[T\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma_F^2(i,t)),$$

with the asymptotic variance

$$\sigma_F^2(i,t) = \mu_{ii} F_i(t)[1 - F_i(t)],$$

(ii) *Under the condition* $\lim_{T\to\infty} Th_{ij,T} = \infty$, *we have*

$$\sqrt{Th_{ij,T}}[\widehat{q}_{ij}(t,T) - q_{ij}(t)] \xrightarrow[T\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma_q^2(i,j,t)),$$

with the asymptotic variance

$$\sigma_q^2(i,j,t) = \mu_{ii} q_{ij}(t) \int_{-\infty}^{\infty} K^2(z)\,dz.$$

where μ_{ii} is the mean recurrent time of state i for the semi-Markov process $(Z_t)_{t\in\mathbb{R}_+}$.

Proof.

(i) Remark that we can write

$$\sqrt{T}[\widehat{F}_i(t,T) - F_i(t)] = \frac{T}{N_i(T)} \frac{1}{\sqrt{T}} \sum_{l=1}^{N(T)} \left[H\left(\frac{t-X_l}{h_{i,T}}\right) \mathbb{1}_{\{J_{l-1}=i\}} - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}} \right].$$

Taking into account Theorem 3.2.1, denote

$$U_l = \left(H\left(\frac{t-X_l}{h_{i,n}}\right) - F_i(t) \right) \mathbb{1}_{\{J_{l-1}=i\}} \tag{3.8}$$

If we denote by $\mathscr{F}_l$ the σ-algebra $\mathscr{F}_l := \sigma(J_n, X_n; n \le l)$, $l \ge 0$, thus U_l is $\mathscr{F}_l$-measurable and $\mathscr{F}_l \subseteq \mathscr{F}_{l+1}$, for all $l \in \mathbb{N}$. Moreover, we have

$$\begin{aligned}
\mathbb{E}(U_l|\mathscr{F}_{l-1}) &= \mathbb{E}\left(\left[H\left(\frac{t-X_l}{h_{i,n}}\right) \mathbb{1}_{\{J_{l-1}=i\}} - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}} \right] \middle| \mathscr{F}_{l-1} \right) \\
&= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} H\left(\frac{t-x}{h_{i,n}}\right) f_i(x)\,dx - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}},
\end{aligned}$$

where the last equation is obtained by the fact that $\mathbb{1}_{\{J_{l-1}=i\}}$ is $\mathscr{F}_{l-1}$-measurable. On the other hand, using a change of variable, an integration by parts followed by Taylor's expansion of $F_i(t - h_{i,n}z)$ in neighbourhood of t combining with assumptions (H.6), (H.7) and (H.8), we get

$$\begin{aligned}
\mathbb{E}(U_l|\mathscr{F}_{l-1}) &= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} H(z)\,dF_i(t - zh_{i,n}) - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
&= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} K(z) F_i(t - zh_{i,n})\,dz - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
&= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} K(z) \left(F_i(t - zh_{i,n}) - F_i(t)\right) dz \\
&= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} K(z) \left(-h_{i,n} z F_i'(t^*)\right) dz,
\end{aligned}$$

where t^* is between t and $t - h_{i,n}z$. It follows

$$\left| \int_{-\infty}^{\infty} K(z) F_i'(t^*) h_{i,n} z\,dz \right| \le C h_{i,n} \int_{-\infty}^{\infty} |z| K(z)\,dz = O(h_{i,n}).$$

This implies that

$$\mathbb{E}(U_l|\mathscr{F}_{l-1}) = 0, \quad \text{as } n \to \infty.$$

By using the CLT for martingales, we have

$$\frac{1}{\sqrt{n}} \sum_{l=1}^{n} U_l \xrightarrow[n\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma^2(i,t)).$$

To obtain the asymptotic variance $\sigma^2(i,t)$, we need to compute

$$\sigma^2(i,t) = \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}(U_l^2|\mathscr{F}_{l-1}) > 0.$$

Firstly,

$$\begin{aligned}
\mathbb{E}(U_l^2|\mathscr{F}_{l-1}) &= \mathbb{E}\left(H^2\left(\frac{t-X_l}{h_{i,n}}\right)\mathbb{1}_{\{J_{l-1}=i\}}\middle|\mathscr{F}_{l-1}\right) + \mathbb{E}\left(F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}}|\mathscr{F}_{l-1}\right) \\
&\quad - \mathbb{E}\left(2F_i(t)H\left(\frac{t-X_l}{h_{i,n}}\right)\mathbb{1}_{\{J_{l-1}=i\}}\middle|\mathscr{F}_{l-1}\right) \\
&= \mathbb{1}_{\{J_{l-1}=i\}}\left(\int_{-\infty}^{\infty} H^2\left(\frac{t-x}{h_{i,n}}\right) f_i(x)\,dx\right) + F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
&\quad - 2F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}\left(\int_{-\infty}^{\infty} H\left(\frac{t-x}{h_{i,n}}\right) f_i(x)\,dx\right),
\end{aligned}$$

Secondly, under Assumptions (H.6), (H.7), (H.8) and a change of variable, an integration by parts followed by Taylor's expansion, we obtain

$$\begin{aligned}
\mathbb{E}(U_l^2|\mathscr{F}_{l-1}) &= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} H^2(z)\,dF_i(t-zh_{i,n}) + F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
&\quad - 2F_i(t)1_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} H(z)\,dF_i(t-zh_{i,n}) \\
&= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} 2K(z)H(z)F_i(t-zh_{i,n})\,dz + F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
&\quad - 2F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}[F_i(t)+O(h_{i,n})] \\
&= \mathbb{1}_{\{J_{l-1}=i\}}F_i(t) \int_{-\infty}^{\infty} 2K(z)H(z)\,dz + O(h_{i,n}) - F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
&= \mathbb{1}_{\{J_{l-1}=i\}}F_i(t)\left[\int_{-\infty}^{\infty} (H^2)'(z)\,dz - F_i(t)\right] + O(h_{i,n}).
\end{aligned}$$

Thus,

$$\sigma^2(i,t) = v(i)F_i(t)[1-F_i(t)]. \tag{3.9}$$

Furthermore,

$$\sqrt{T}[\widehat{F}_i(t,T) - F_i(t)] = \frac{T}{N_i(T)}\sqrt{\frac{N(T)}{T}}\frac{1}{\sqrt{N(T)}}\sum_{l=1}^{N(T)}\left[H\left(\frac{t-X_l}{h_{i,T}}\right)\mathbb{1}_{\{J_{l-1}=i\}} - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}\right]$$

Combining the statements (3) and (5) of Lemma1.1 with Equation (3.9) and applying Anscombe's Theorem (see Billingsley [12]), we get

$$\begin{aligned}\sigma_F^2(i,t) &= \left(\mu_{ii}\sqrt{1/\mu_{ii}\nu(i)}\right)^2 \nu(i)F_i(t)[1-F_i(t)]\\ &= \mu_{ii}F_i(t)[1-F_i(t)].\end{aligned}$$

Now, it suffices to show that

$$\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(U_l^2\mathbb{1}_{\{|U_l|>\varepsilon\sqrt{n}\}}) \xrightarrow[n\to\infty]{} 0.$$

Indeed, using successively inequalities of Holder, Markov, Jensen and then Minkowski, we obtain for any $\varepsilon > 0$ and any p and q such that $\frac{1}{p}+\frac{1}{q}=1$:

$$\begin{aligned}\mathbb{E}(U_l^2\mathbb{1}_{\{|U_l|>\varepsilon\sqrt{n}\}}) &\leq (\mathbb{E}(U_l^{2q}))^{1/q}\mathbb{P}(\mid U_l\mid>\varepsilon\sqrt{n})^{1/p} \leq (\varepsilon\sqrt{n})^{-2q/p}\mathbb{E}(\mid U_l\mid^{2q})\\ &\leq (\varepsilon\sqrt{n})^{-2q/p}\mathbb{E}\left(\left|H\left(\frac{t-X_l}{h_{i,n}}\right)\mathbb{1}_{\{J_{l-1}=i\}}-F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}\right|^{2q}\right)\\ &\leq (\varepsilon\sqrt{n})^{-2q/p}\left(\mathbb{E}\left(H^{2q}\left(\frac{t-X_l}{h_{i,n}}\right)\mathbb{1}_{\{J_{l-1}=i\}}\right)+F_i^{2q}(t)\mathbb{1}_{\{J_{l-1}=i\}}\right)\\ &\leq (\varepsilon\sqrt{n})^{-2q/p}\left(\mathbb{1}_{\{J_{l-1}=i\}}\int_{-\infty}^{\infty}H^{2q}\left(\frac{t-x}{h_{i,n}}\right)f_i(x)\,dx+F_i^{2q}(t)\mathbb{1}_{\{J_{l-1}=i\}}\right)\\ &\leq (\varepsilon\sqrt{n})^{-2q/p}\left(h_{i,n}\int_{-\infty}^{\infty}H^{2q}(z)f_i(t-zh_{i,n})\,dz+F_i^{2q}(t)\right)\mathbb{1}_{\{J_{l-1}=i\}}\end{aligned}$$

Consequently, since f is a density function and by using Lemma 1.1, it follows that

$$\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(U_l^2\mathbb{1}_{\{|U_l|>\varepsilon\sqrt{n}\}}) \leq (\varepsilon\sqrt{n})^{-2q/p}\nu(i)(h_{i,n}\|H\|_{2q}^{2q}+F_i^{2q}(t)) \xrightarrow[n\to\infty]{} 0$$

From results above and the functional central limit theorem for martingale differences (see Hall and Heyde [23]; Billingsley [12]) we get the desired result.

(ii)

We start by writing

$$\begin{aligned}&\sqrt{Th_{ij,T}}[\widehat{q}_{ij}(t,T)-q_{ij}(t)]\\ &\quad= \frac{T}{N_i(T)}\frac{1}{\sqrt{T}}\sum_{l=1}^{N_{ij}(T)}\frac{1}{\sqrt{h_{ij,T}}}K\left(\frac{t-X_n}{h_{ij,T}}\right)-q_{ij}(t)\sqrt{h_{ij,T}}\\ &\quad= \frac{T}{N_i(T)}\frac{1}{\sqrt{T}}\sum_{l=1}^{N(T)}\left[\frac{1}{\sqrt{h_{ij,T}}}K\left(\frac{t-X_l}{h_{ij,T}}\right)\mathbb{1}_{\{J_{l-1}=i,J_l=j\}}-q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,T}}\right].\end{aligned}$$

According to Theorem 3.2.1, we denote

$$V_l = \frac{1}{\sqrt{h_{ij,n}}} K\left(\frac{t-X_l}{h_{ij,n}}\right) \mathbb{1}_{\{J_{l-1}=i, J_l=j\}} - q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,n}}. \tag{3.10}$$

Then, we use the same steps as in (i), so we give only the main details. Let us denote by $\mathscr{F}_l$ the σ-algebra $\mathscr{F}_l := \sigma(J_n, X_n; n \leq l)$, $l \geq 0$, for all $l \in \mathbb{N}$. Thus, we have

$$\begin{aligned}
\mathbb{E}(V_l|\mathscr{F}_{l-1}) &= \frac{1}{\sqrt{h_{ij,n}}}\mathbb{1}_{\{J_{l-1}=i\}} p_{ij} \int_{-\infty}^{\infty} K\left(\frac{t-x}{h_{ij,n}}\right) f_{ij}(x)\,dx - q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,n}} \\
&= \frac{1}{\sqrt{h_{ij,n}}}\mathbb{1}_{\{J_{l-1}=i\}} p_{ij} \int_{-\infty}^{\infty} K(z)\,dF_{ij}(t - zh_{ij,n}) - q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,n}} \\
&= \sqrt{h_{ij,n}}\mathbb{1}_{\{J_{l-1}=i\}} q_{ij}(t) \int_{-\infty}^{\infty} K(z)\,dz + O(h_{ij,n}) - q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,n}} \\
&= 0, \quad \text{as } n \to \infty,
\end{aligned}$$

the previous result is obtained by using the $\mathscr{F}_{l-1}$-measurability of $\mathbb{1}_{\{J_{l-1}=i\}}$, a change of variable, Taylor's expansions of order one and Assumptions (H.6), (H.7) and (H.8).
To get the asymptotic variance, we need first to compute $\mathbb{E}(V_l^2|\mathscr{F}_{l-1})$ and then to obtain

$$\sigma^2(i,j,t) = \lim_{n\to\infty} \frac{1}{n}\sum_{l=1}^{n} \mathbb{E}(V_l^2|\mathscr{F}_{l-1}) > 0.$$

Firstly,

$$\begin{aligned}
\mathbb{E}(V_l^2|\mathscr{F}_{l-1}) = {} & \frac{1}{h_{ij,n}}\mathbb{1}_{\{J_{l-1}=i\}} p_{ij} \left(\int_{-\infty}^{\infty} K^2\left(\frac{t-x}{h_{ij,n}}\right) f_{ij}(x)\,dx\right) + h_{ij,n} q_{ij}^2(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
& - 2q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}} p_{ij} \left(\int_{-\infty}^{\infty} K\left(\frac{t-x}{h_{ij,n}}\right) f_{ij}(x)\,dx\right).
\end{aligned}$$

Using the same steps as before, we obtain

$$\begin{aligned}
\mathbb{E}(V_l^2|\mathscr{F}_{l-1}) = {} & \mathbb{1}_{\{J_{l-1}=i\}} p_{ij} \int_{-\infty}^{\infty} K^2(z) f_{ij}(t - zh_{ij,n})\,dz + h_{ij,n} q_{ij}^2(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
& - 2q_{ij}(t) h_{ij,n}\mathbb{1}_{\{J_{l-1}=i\}} p_{ij} \int_{-\infty}^{\infty} K(z) f_{ij}(t - zh_{ij,n})\,dz \\
= {} & \mathbb{1}_{\{J_{l-1}=i\}} q_{ij}(t) \left[\int_{-\infty}^{\infty} K^2(z)\,dz - h_{ij,n} q_{ij}(t)\right] + O(h_{ij,n}).
\end{aligned}$$

Secondly,

$$\sigma^2(i,j,t) = v(i) q_{ij}(t) \int_{-\infty}^{\infty} K^2(z)\,dz. \tag{3.11}$$

Using (3.10) with Theorem 3.2.1 result, we can write

$$\begin{aligned}
& \sqrt{T h_{ij,T}}[\widehat{q}_{ij}(t,T) - q_{ij}(t)] \\
& \quad = \frac{T}{N_i(T)}\sqrt{\frac{N(T)}{T}} \frac{1}{\sqrt{N(T)}} \sum_{l=1}^{N(T)} \left[\frac{1}{\sqrt{h_{ij,T}}} K\left(\frac{t-X_l}{h_{ij,T}}\right) \mathbb{1}_{\{J_{l-1}=i, J_l=j\}} - q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,T}}\right].
\end{aligned}$$

By using the statements (3) and (5) of Lemma 1.1, with the application of Anscombe's Theorem, we deduce that

$$\sigma_q^2(i,j,t) = \left(\mu_{ii}\sqrt{1/\mu_{ii}\nu(i)}\right)^2 \nu(i)q_{ij}(t)\int_{-\infty}^{\infty} K^2(z)\,dz$$
$$= \mu_{ii}q_{ij}(t)\int_{-\infty}^{\infty} K^2(z)\,dz.$$

Now, it suffices to show that

$$\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(V_l^2 \mathbb{1}_{\{|V_l|>\varepsilon\sqrt{n}\}}) \xrightarrow[n\to\infty]{} 0$$

and we use the same steps as (i) to prove this result.
From the results above and the functional central limit theorem for martingale differences (see Hall and Heyde [23]; Billingsley [12]), we get the desired result. □

Similarly to Theorem 3.2.5, we establish the following asymptotic results.

Corollary 3.2.2. Under the same conditions of Theorem 3.2.5, the following statements stand true

(i)
$$\sqrt{T}[\widehat{F}_{ij}(t,T) - F_{ij}(t)] \xrightarrow[T\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma_F^2(i,j,t)),$$

with the asymptotic variance

$$\sigma_F^2(i,j,t) = \frac{\mu_{ii}}{p_{ij}}F_{ij}(t)[1 - F_{ij}(t)].$$

(ii)
$$\sqrt{T}[\widehat{Q}_{ij}(t,T) - Q_{ij}(t)] \xrightarrow[T\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma_Q^2(i,j,t)),$$

with the asymptotic variance

$$\sigma_Q^2(i,j,t) = \mu_{ii}Q_{ij}(t)[1 - Q_{ij}(t)].$$

(iii) *If* $\lim_{T\to\infty} Th_{ij,T} = \infty$ *holds, we have*

$$\sqrt{Th_{ij,T}}[\widehat{f}_{ij}(t,T) - f_{ij}(t)] \xrightarrow[T\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma_f^2(i,j,t)),$$

with the asymptotic variance

$$\sigma_f^2(i,j,t) = \frac{\mu_{ii}}{p_{ij}}f_{ij}(t)\int_{-\infty}^{\infty} K^2(z)\,dz.$$

(iv) *If* $\lim_{T\to\infty} Th_{i,T} = \infty$ *holds, we have*

$$\sqrt{Th_{i,T}}[\widehat{f}_i(t,T) - f_i(t)] \xrightarrow[T\to\infty]{\mathscr{D}} \mathscr{N}(0, \sigma_f^2(i,t)),$$

with the asymptotic variance

$$\sigma_f^2(i,t) = \mu_{ii} f_i(t) \int_{-\infty}^{\infty} K^2(z)\, dz.$$

Theorem 3.2.6. For any fixed $t \in \mathbb{R}_+$ and $i, j \in E$, we have

(a) under (H.6), (H.7) and (H.8) and we put $\min_{ij} h_{ij,T} = h_T$ we have

$$\sqrt{Th_T}[\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)] \xrightarrow[T\to\infty]{D} \mathscr{N}(0, \sigma_\Psi^2(i,j,t)),$$

where

$$\sigma_\Psi^2(i,j,t) \le \sum_{l=1}^{s} \mu_{ll} \sum_{r=1}^{s} (\Psi_{il} * \Psi_{rj})^2 * Q_{lr}(t) \int K^2(z)\, dz.$$

(b) under (H.6), (H.7) and (H.8) and we put $\min_{ij} h_{ij,T} = h_T$ we have

$$\sqrt{Th_T}[\hat{P}_{ij}(t,T) - P_{ij}(t)] \xrightarrow[T\to\infty]{D} \mathscr{N}(0, \sigma_P^2(i,j,t)),$$

where

$$\sigma_P^2(i,j,t) \le \sum_{l=1}^{s} \mu_{ll} \sum_{u=1}^{s} [\delta_{lj}\Psi_{ij} - (1-F_j) * \Psi_{il} * \Psi_{uj}]^2 * Q_{lu}(t) \int K^2(z)\, dz.$$

Proof.

(a)

we have,

$$\begin{aligned}\sqrt{Th_T}[\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)] &= \sqrt{Th_T}[\hat{\Psi}_{ij} - (\hat{\Psi} * \Psi)_{ij} + (\hat{\Psi} * \Psi)_{ij} - \Psi_{ij}](t) \\ &= \sqrt{Th_T}[(\hat{\Psi} - \Psi) * \Delta Q * \Psi]_{ij}(t) + \sqrt{Th_T}[\Psi * \Delta Q * \Psi]_{ij}(t) \quad (3.12)\end{aligned}$$

For every $t \ge 0, t \le T$, and for every $r, m, u, v \in E$, $\sqrt{T}(\hat{Q}_{rm}(t,T) - Q_{rm}(t))$ converges in distribution to a normal random variable (Corollary 3.2.2(ii)), as $T \to \infty$, and $\hat{\Psi}_{uv}(t,T) - \Psi_{uv}(t) \xrightarrow{P} 0$ as $T \to \infty$ (Theorem 3.2.4 (a)).
Thus, using Slutsky's Theorem we obtain that $\sqrt{Th_T}[(\hat{\Psi} - \Psi) * (\hat{Q} - Q) * \Psi]_{ij}(t) \xrightarrow{P} 0$ as $T \to \infty$.
Consequently, applying again Slutsky's Theorem we get that $\sqrt{Th_T}[\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)]$ has the

same limit in distribution as $\sqrt{Th_T}[\Psi*(\hat{Q}-Q)*\Psi]_{ij}(t)$.

The last term can be written as follows:

$$\begin{aligned}\sqrt{Th_T}(\Psi*(\hat{Q}-Q)*\Psi)_{ij}(t) &= \sqrt{Th_T}\sum_{m=1}^{s}\sum_{r=1}^{s}(\Psi_{im}*(\hat{Q}(\cdot,T)-Q)_{mr}*\Psi_{rj})(t)\\ &= \sqrt{\frac{h_T}{T}}\sum_{l=1}^{N(T)}\sum_{m=1}^{s}\frac{T}{N_m(T)}\sum_{r=1}^{s}[(\Psi_{im}*H\left(\frac{\cdot-X_l}{h_{mr,T}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=r\}}*\Psi_{rj})(t)\\ &-(\Psi_{im}*Q_{mr}\mathbb{1}_{\{J_{l-1}=m\}}*\Psi_{rj})(t)].\end{aligned}$$

Since $\frac{N_m(T)}{T}\xrightarrow{a.s.}\frac{1}{\mu_{mm}}$ (from (3) Lemme 2.1.), Let us consider the function

$$\begin{aligned}f(d,u,x) = \sum_{m=1}^{s}\mu_{mm}\sqrt{h_T}\sum_{r=1}^{s}&[(\Psi_{im}*H\left(\frac{\cdot-x}{h_{mr,T}}\right)\mathbb{1}_{\{d=m,u=r\}}*\Psi_{rj})(t)\\ &-(\Psi_{im}*Q_{mr}1_{\{d=m\}}*\Psi_{rj})(t)].\end{aligned}$$

Apply central limit theorem related to semi-Markov processes (see for instance Pyke and R. Schaufele [46]) to the function

$$\begin{aligned}W_f(T) &= \sum_{l=1}^{N(T)} f(J_{l-1},J_l,X_l)\\ &= \sum_{l=1}^{N(T)}\sum_{m=1}^{s}\mu_{mm}\sqrt{h_T}\sum_{r=1}^{s}[(\Psi_{im}*H\left(\frac{\cdot-X_l}{h_{mr,T}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=r\}}*\Psi_{rj})(t)\\ &-(\Psi_{im}*Q_{mr}\mathbb{1}_{\{J_{l-1}=m\}}*\Psi_{rj})(t)]\end{aligned}$$

For this function

$$A_{du} = \int_0^{\infty} f(d,u,x)dQ_{du}(x), \qquad A_d = \sum_{u=1}^{s} A_{du}$$

$$B_{du} = \int_0^{\infty}[f(d,u,x)]^2 dQ_{du}(x), \qquad B_d = \sum_{u=1}^{s} B_{du}$$

So using a change of variable, an integration by parts followed by Taylor's expansion under assumptions (H.6)-(H.8), we get

$$\begin{aligned}A_d \le \sum_{m=1}^{s}\mu_{mm}\sum_{r=1}^{s}&[(\Psi_{im}*\Psi_{rj})*\mathbb{1}_{\{d=m\}}\sqrt{h_T}(Q_{mr}\int_{-\infty}^{+\infty}K(z)\,dz+O(h_{mr,T}))(t)\\ &-(\Psi_{im}*\Psi_{rj})*Q_{mr}(t)\mathbb{1}_{\{d=m\}}\sqrt{h_T}].\end{aligned}$$

By using Jensen's inequality followed by the same steps as before, we get:

$$B_d \leq \sum_{m=1}^{s} \mu_{mm}^2 \mathbb{1}_{\{d=m\}} \sum_{r=1}^{s} (\Psi_{im} * \Psi_{rj})^2 * Q_{mr} \int_{-\infty}^{+\infty} K^2(z)\,dz(t); \quad \text{as } T \to \infty.$$

Write

$$r_m = \sum_{d=1}^{s} A_d \frac{\mu_{mm}^*}{\mu_{dd}^*} = 0, \quad \text{as } T \to \infty.$$

Then

$$\frac{\sigma_m^2}{\mu_m} = \frac{1}{\mu_{mm}} \sum_{l=1}^{s} B_l \frac{\mu_{mm}^*}{\mu_{ll}^*} \leq \sum_{l=1}^{s} \mu_{ll} \sum_{r=1}^{s} (\Psi_{il} * \Psi_{rj})^2 * Q_{lr} \int_{-\infty}^{+\infty} K^2(z)\,dz(t),$$

As a final step, we have to show that the variance σ_m^2 does not vanish. Assuming conditions (H.6)-(H.8), and let $0 < c \leq \frac{h_T}{h_{mr,T}}$, observe that:

$$\begin{aligned}
\sum_{d=1}^{s}\sum_{u=1}^{s} B_{du} \frac{\mu_{mm}^*}{\mu_{dd}^*} \geq{}& \sum_{d=1}^{s} \frac{\mu_{mm}^*}{\mu_{dd}^*} \sum_{u=1}^{s}\sum_{m=1}^{s} \mu_{mm}^2 [\beta \sum_{r=1}^{s} \int_0^t (\Psi_{im} * \Psi_{rj})(t-t_1) \frac{h_T}{h_{mr,T}^2} \int_0^{\infty} K\left(\frac{t_1 - x}{h_{mr,T}}\right) \\
& \mathbb{1}_{\{d=m,u=r\}} dQ_{du}(x)dt_1 \int_0^t (\Psi_{im} * \Psi_{rj})(t-t_1)dt_1 \\
& + \sum_{r_1,r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1) q_{mr_1}(t_1)\sqrt{h_T}dt_1 \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_1) q_{mr_2}(t_1)\sqrt{h_T}\mathbb{1}_{\{d=m\}} \\
& p_{mu}dt_1 - 2\sum_{r_1,r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1)\mathbb{1}_{\{d=m,u=r_1\}} \frac{\sqrt{h_T}}{h_{mr_1,T}} \int_0^{+\infty} K\left(\frac{t_1 - x}{h_{mr_1,T}}\right) dQ_{mr_1}(x)dt_1 \\
& \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_1) q_{mr_2}(t_1)\sqrt{h_T}dt_1],
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{d=1}^{s} \frac{\mu_{mm}^*}{\mu_{dd}^*} \sum_{m=1}^{s} \mu_{mm}^2 [c\beta \sum_{r=1}^{s} \int_0^t (\Psi_{im} * \Psi_{rj})(t-t_1)\mathbb{1}_{\{d=m\}} q_{mr}(t_1) \left(\int_{-\infty}^{\frac{t}{h_{mr,T}}} K(z)\,dz\right) dt_1 \\
&\int_0^t (\Psi_{im} * \Psi_{rj})(t-t_1)dt_1 - \left(\sum_{r=1}^{s} \mathbb{1}_{\{d=m\}} \sqrt{h_T} \Psi_{im} * \Psi_{rj} * Q_{mr}(t)\right)^2 + O(h_{mr,T})]
\end{aligned}$$

$$= c\beta \sum_{m=1}^{s} \mu_{mm}^2 [\sum_{r=1}^{s} (\Psi_{im} * \Psi_{rj}) * Q_{mr}(t) \int_0^t (\Psi_{im} * \Psi_{rj})(t-t_1)dt_1 > 0, \quad \text{as } T \to \infty.$$

(b)

From (3.7), we see that,

$$\begin{aligned}
\sqrt{Th_T}[\hat{P}_{ij}(t,T) - P_{ij}(t)] ={}& \sqrt{Th_T}\left[(\hat{\Psi}_{ij} - \Psi_{ij}) * (I - \operatorname{diag}(Q \cdot \mathbf{1}))_{jj} - \Psi_{ij} * (\operatorname{diag}(\hat{Q} - Q) \cdot \mathbf{1})_{jj}\right] \\
& - \sqrt{Th_T}\left[(\hat{\Psi}_{ij} - \Psi_{ij}) * (\operatorname{diag}(\hat{Q} - Q) \cdot \mathbf{1})_{jj}\right](t,T).
\end{aligned}$$

With $\text{diag}(\hat{Q}(t,T)\cdot\mathbf{1})_{jj} = \sum_{m=1}^{s}\hat{Q}_{jm}(t,T) = \hat{F}_j(t,T) \le 1$.

For every $t \ge 0$, $t \le T$, $\sqrt{Th_T}(\hat{\Psi}_{ij} - \Psi_{ij})(t,T)$ converges in distribution to a normal random variable (Theorem 3.2.6) and $\Delta Q_{jr}(t,T)$ converges in probability to zero (Corollary 3.2.2). Thus, using Slutsky's Theorem we obtain that $\sqrt{Th_T}\left[(\hat{\Psi}_{ij} - \Psi_{ij}) * (\text{diag}(\hat{Q} - Q)\cdot\mathbf{1})_{jj}\right](t,T)$ converges in probability to zero when T tends to infinity. Thus, $\sqrt{Th_T}[\hat{P}_{ij}(t,T) - P_{ij}(t)]$ has the same limit in distribution as

$$\sqrt{Th_T}\left[(\hat{\Psi}_{ij} - \Psi_{ij}) * (I - \text{diag}(Q\cdot\mathbf{1}))_{jj} - \Psi_{ij} * (\text{diag}(\hat{Q} - Q)\cdot\mathbf{1})_{jj}\right](t,T).$$

From the proof of Theorem 3.2.6 (a) we know that $\sqrt{Th_T}[\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)]$ has the same limit in distribution as $\sqrt{Th_T}[\Psi * \Delta Q * \Psi]_{ij}(t,T)$ when T tends to infinity. Writing

$$\Delta Q_{ij} = \frac{1}{N_i(T)}\sum_{l=1}^{N(T)}[H\left(\frac{t-X_l}{h_{ij,T}}\right)\mathbb{1}_{\{J_{l-1}=i,J_l=j\}} - Q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}],$$

and

$$\sum_{u=1}^{s}\Psi_{ij} * \Delta Q_{ju} = \sum_{u=1}^{s}\sum_{n=1}^{s}\delta_{nj}\Psi_{ij} * \Delta Q_{nu},$$

we obtain that $\sqrt{Th_T}[\hat{P}_{ij}(t,T) - P_{ij}(t)]$ has the same limit in distribution as

$$\begin{aligned}
&\sqrt{Th_T}\left\{\sum_{m=1}^{s}\sum_{u=1}^{s}[(1-F_j) * \Psi_{im} * \Psi_{uj} * \Delta Q_{mu}](t) - \sum_{u=1}^{s}\Psi_{ij} * \Delta Q_{ju}(t)\right\}\\
&= \frac{1}{\sqrt{T}}\sum_{l=1}^{N(T)}\sum_{m=1}^{s}\sum_{u=1}^{s}\frac{T}{N_m(T)}\sqrt{h_T}\\
&\times\left[-\delta_{mj}\Psi_{ij} * \left(H\left(\frac{\cdot - X_l}{h_{mu,T}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=u\}} - Q_{mu}(\cdot)\mathbb{1}_{\{J_{l-1}=m\}}\right)(t)\right.\\
&\left.+(1-F_j) * \Psi_{im} * \Psi_{uj} * \left(H\left(\frac{\cdot - X_l}{h_{mu,T}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=u\}} - Q_{mu}(\cdot)\mathbb{1}_{\{J_{l-1}=m\}}\right)(t)\right].
\end{aligned}$$

Since $\frac{N_m(T)}{T} \xrightarrow{a.s.} \frac{1}{\mu_{mm}}$ (from (3) Lemme 2.1.), let us consider the function

$$\begin{aligned}
f(d,r,x) = &\sum_{m=1}^{s}\sum_{u=1}^{s}\mu_{mm}\sqrt{h_T}\times\left[-\delta_{mj}\Psi_{ij} + (1-F_j) * \Psi_{im} * \Psi_{uj}\right] * \left(H\left(\frac{\cdot - x}{h_{mu,T}}\right)\mathbb{1}_{\{d=m,r=u\}}\right.\\
&\left.-Q_{mu}(\cdot)\mathbb{1}_{\{d=m\}}\right)(t).
\end{aligned}$$

Apply central limit theorem related to semi-Markov processes (see for instance Pyke and R. Schaufele [46]) to the function

$$
\begin{aligned}
W_f(T) &= \sum_{l=1}^{N(T)} f(J_{l-1}, J_l, X_l) \\
&= \sum_{l=1}^{N(T)} \sum_{m=1}^{s} \sum_{u=1}^{s} \mu_{mm} \sqrt{h_T} \times \left[-\delta_{mj}\Psi_{ij} + (1-F_j) * \Psi_{im} * \Psi_{uj}\right] \\
&* \left(H\left(\frac{\cdot - X_l}{h_{mu,T}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=u\}} - Q_{mu}(\cdot)\mathbb{1}_{\{J_{l-1}=m\}} \right)(t).
\end{aligned}
$$

So using the same notations of Theorem 3.2.6, a change of variable, an integration by parts followed by Taylor's expansion under assumptions (H.6)-(H.8), we get

$$
\begin{aligned}
A_d \leq & \sum_{m=1}^{s} \mu_{mm} \sum_{u=1}^{s} \left[-\delta_{mj}\Psi_{ij} + (1-F_j) * \Psi_{im} * \Psi_{uj}\right] * \mathbb{1}_{\{d=m\}} \sqrt{h_T} (Q_{mu} \int_{-\infty}^{+\infty} K(z)\, dz + O(h_{mu,T}))(t) \\
& - \left[-\delta_{mj}\Psi_{ij} + (1-F_j) * \Psi_{im} * \Psi_{uj}\right] * Q_{mu}(t) \mathbb{1}_{\{d=m\}} \sqrt{h_T}.
\end{aligned}
$$

By using Jensen's inequality followed by the same steps as before, we obtain:

$$
B_d \leq \sum_{m=1}^{s} \mu_{mm}^2 \mathbb{1}_{\{d=m\}} \sum_{u=1}^{s} [\delta_{mj}\Psi_{ij} - (1-F_j) * \Psi_{im} * \Psi_{uj}]^2 * Q_{mu} \int_{-\infty}^{+\infty} K^2(z)\, dz(t); \quad \text{as } T \to \infty.
$$

Write

$$
r_m = \sum_{d=1}^{s} A_d \frac{\mu_{mm}^*}{\mu_{dd}^*} = 0, \quad \text{as } T \to \infty.
$$

Then

$$
\frac{\sigma_m^2}{\mu_{mm}} = \frac{1}{\mu_{mm}} \sum_{l=1}^{s} B_l \frac{\mu_{mm}^*}{\mu_{ll}^*} \leq \sum_{l=1}^{s} \mu_{ll} \sum_{u=1}^{s} [\delta_{lj}\Psi_{ij} - (1-F_j) * \Psi_{il} * \Psi_{uj}]^2 * Q_{lu} \int_{-\infty}^{+\infty} K^2(z)\, dz(t).
$$

As a final step, we can show that the variance σ_m^2 does not vanish. □

CHAPTER 4

Simulation study

Although both semi-Markov chain (SMCs) and SMPs offer interesting and flexible frameworks to modeling real problems, it is well-known that they are difficult to be numerically solved especially in continuous-time of the time-dependent functions, i.e., transition function, etc. The emphasis is laid on solving the Markov renewal equation (MRE) which brings difficulty due to convolution products. Nevertheless, for solving the equation of the evolution of the process, we cite the paper of Janssen and Manca [24] where a numerical solution for the evolution equation of a continuous time non-homogeneous SMP is obtained using a quadrature method. In the same paper, the problem of obtaining the continuous time SMP from the discrete one is considered. The case of homogeneous SMP was considered by Corradi *et al.* [15], who have solved numerically the system of integral equations. In addition, they have proved that the solution of the discrete time SMP process always exists. Recently, Wu *et al.* [55] applied this method to a reliability problem by evaluating system availability. In this chapter, we illustrate the theoretical results of the nonparametric estimation for the transition function of a semi-Markov process with a numerical example.

4.1 The evolution equation numerical solution

In Corradi et all. [15] the authors have presented a novel method to solve the continuous-time MRE based on the algorithm from discrete-time case. They have studied the error bounds caused by discretization for transition function matrices of continuous-time SMP, which is an efficient way to decide the step size of discretization.
Mokhtari et al [36]. studied an alternative approach: instead of using the solution of the Markov renewal equation, as in the above-mentioned work (see Equation (1.13)), they approximated the solution of Equation (1.12) using a simple quadrature formula (see Corradi et al. [15] and

Janssen and Manca [24]).

Let $v > 0$ be the step size of discretization, $k \in \mathbb{N}$ and $kv \leq t < (k+1)v$, then we obtain the approximation of the transition function matrix $P(t)$ by using the following countable linear system:

$$P^{v}_{ij}(kv) = d_{ij}(kv) + v \sum_{l \in E} \sum_{\tau=1}^{k} P^{v}_{lj}(kv - \tau v) q_{il}(\tau v), \tag{4.1}$$

$$d_{ij}(kv) = \begin{cases} 1 - F_i(kv) & if \;\; i = j; \\ 0 & if \;\; i \neq j. \end{cases}$$

The main idea that we are concerned in the following section, is to combine between the previous theoretical results and the algorithm given by Corradi et al. [15] in order to highlight the efficiency of our estimators by a simulation study.

4.2 Numerical example

In this section we carry out a simulation study to evaluate the finite sample performance of the estimation procedure described in the previous sections. We will apply our results to a three-state semi-Markov processes. The transitions between states are given in Fig.4.1.

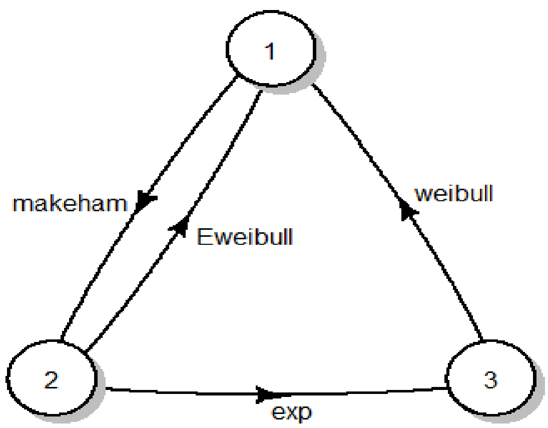


Figure 4.1: A three state semi-Markov system

Taking the initial distribution $\alpha = (1/3\ 1/3\ 1/3)$, the transition matrix of the embedded Markov chain $(J_n)_{n\in\mathbb{R}^+}$,

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0.95 & 0 & 0.05 \\ 1 & 0 & 0 \end{pmatrix}$$

and the conditional sojourn time distributions defined by:

- f_{12} is the Gompertz-Makeham distributed with parameter $\mu = 0.02$, $\nu = 0.1$ and $\gamma = 3$.
- f_{21} is the exponentially Weibull distributed, $W(\theta; k, \gamma)$ with $\theta = 0.8$, $k = 3$ and $\gamma = 2$.
- f_{23} is exponentially distributed, with parameter $\lambda = 0.3$.
- f_{31} is Weibull distributed with parameter $W(\alpha; \beta)$ with $\alpha = 2$ and $\beta = 0.5$.

Starting from this sample path, we can count the different transitions $N_{ij}(T)$, $N_i(T)$

$$\mathbf{N_{ij}}(\mathbf{T}) = \begin{pmatrix} 0 & 944 & 0 \\ 905 & 0 & 38 \\ 39 & 0 & 0 \end{pmatrix}$$

$$\mathbf{N_i}(\mathbf{T}) = \begin{pmatrix} 944 & 943 & 39 \end{pmatrix}$$

These quantities allow us to obtain the empirical estimators of the transition matrix of the embedded Markov chain $p_{ij.}$

The estimations of the transition probabilities from each state i to each state j are presented as elements of the matrix $\hat{P} = \hat{p}_{ij}$

$$\hat{\mathbf{P}} = \begin{pmatrix} 0.0000000 & 1 & 0.00000000 \\ 0.9597031 & 0 & 0.04029692 \\ 1.0000000 & 0 & 0.00000000 \end{pmatrix}$$

Afterward, kernel estimators for all the characteristics of the semi-Markov system $(Q, F, \Psi, P, etc.)$ are obtained. To construct the kernel estimators, the smoothed function $K(.)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{4}(1-u^2)$ for $|u| \leq 1$ and the cumulative distribution function $H(u) = \int_{-\infty}^{u} \frac{3}{4}(1-z^2)\mathbb{1}_{[-1,1]}(z)dz$. The bandwidth h_T has been obtained by the "PBbw" method, which computes the plug-in bandwidth of Polansky and Baker method [43]. We have considered that the observation period is the interval [0, T] with T = 10000.

The following figure corresponds to the Markov renewal matrix estimated (Figure 4.2)

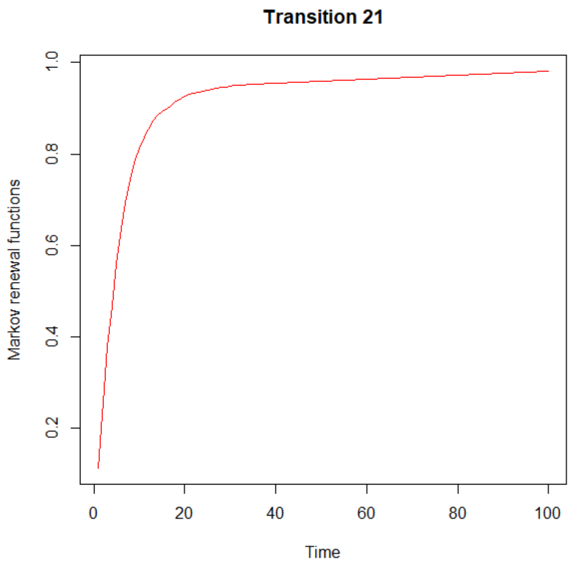


Figure 4.2: Estimated renewal matrix 2 to state 1, $\Psi_{21}(t,T)$

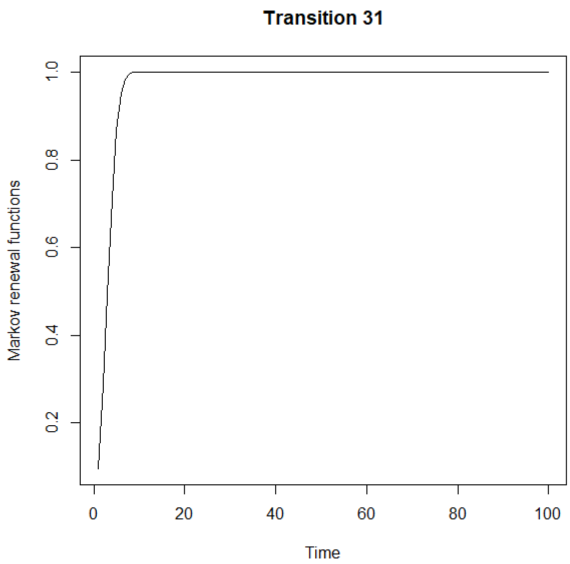


Figure 4.3: Estimated renewal matrix 3 to state 1, $\Psi_{31}(t,T)$

In Table 4.1 the kernel estimators of the quantities $\hat{\pi}_{ii}$ as well as the estimated mean recurrence times are exhibited.

State i	$\hat{v}_i$	$\hat{m}_i$	$\hat{\pi}_{ii}$
1	0.49013499	4.876430	0.45961911
2	0.48961578	5.592255	0.52652953
3	0.02024922	3.557164	0.01385136

Table 4.1: Estimated stationary distribution of the SMP

where $\hat{\pi}_i(T)$ is the estimator for the stationary distribution is determined by:

$$\hat{\pi}_i(T) = \frac{\hat{v}_i(T)\hat{m}_i(T)}{\sum_{k=1}^{s} \hat{v}_k(T)\hat{m}_k(T)}. \tag{4.2}$$

where $\hat{v}_i(T) = \dfrac{N_i(T)}{N(T)}$, for $i \in E$, is the empirical estimator of the stationary distribution of the EMC and $\hat{m}_i(T) = \int_0^{\infty}(1-\hat{F}_i)(t,T)dt$ is the estimated mean sojourn time in state i.

According to Figure 4.4 (resp. Figure 4.5, Figure 4.6), the transition probabilities estimator $P_{i1}(t)$ (resp. $P_{i2}(t)$, $P_{i3}(t)$) tends to the same limit, as t increases. In fact, this common limit represents the semi-Markov limit distribution $\hat{\pi}_1 = 0.45961911$ (resp. $\hat{\pi}_2 = 0.52652953$, $\hat{\pi}_3 = 0.01385136$).

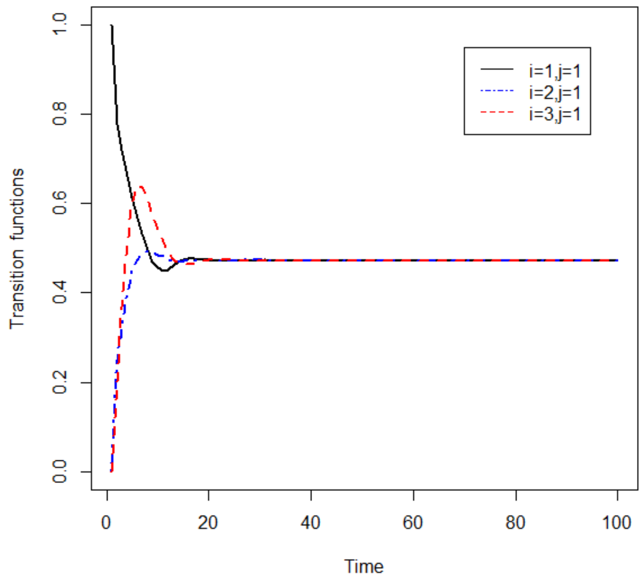


Figure 4.4: Estimated transition functions from state i to state 1, $P_{i1}(t,T)$ $(i \in E)$

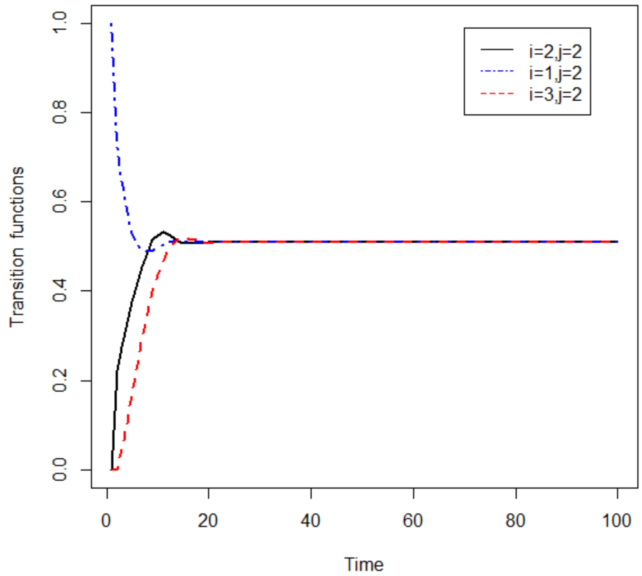


Figure 4.5: Estimated transition functions from state i to state 2, $P_{i2}(t,T)$ $(i \in E)$

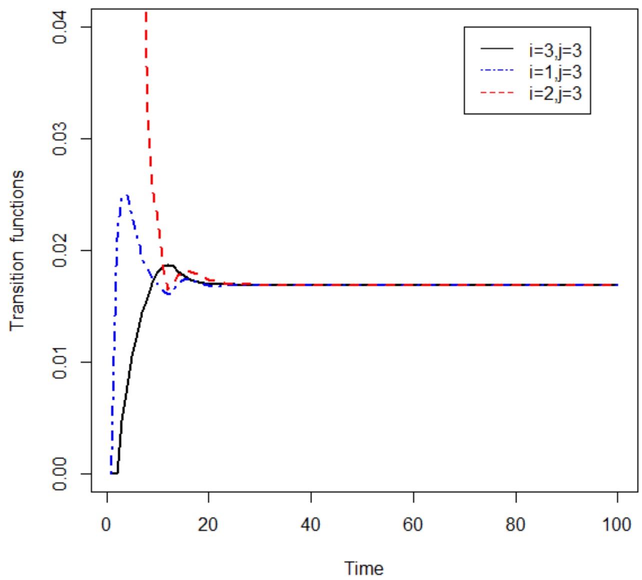
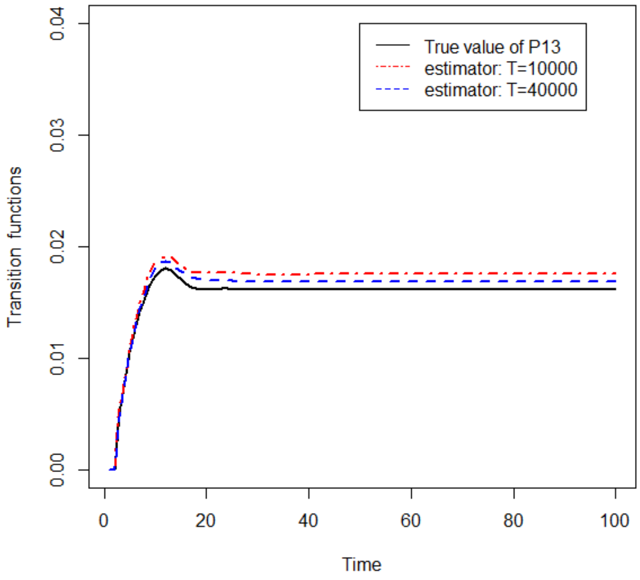
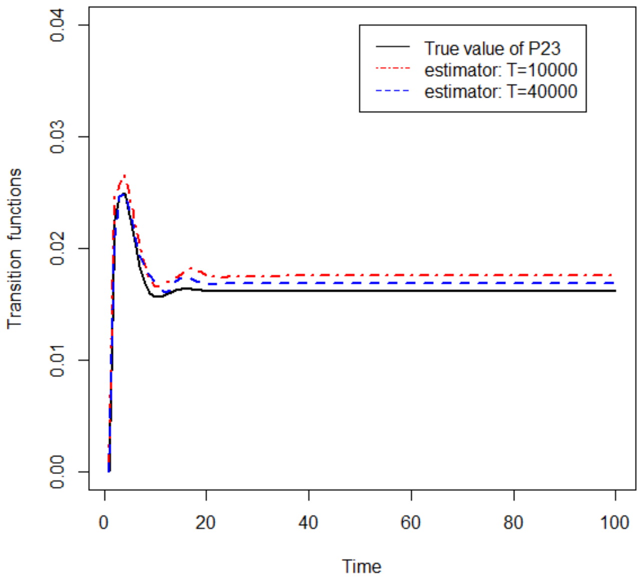


Figure 4.6: Estimated transition functions from state i to state 3, $P_{i3}(t,T)$ $(i \in E)$

Figure 4.7 below, gives a comparison between the true values of the elements $P_{13}(t)$,$P_{23}(t)$, $P_{33}(t)$ of the semi-Markov transition matrix $P(t)$, and their estimators


True value of P13
estimator: T=10000
estimator: T=40000
Transition functions
0.00
0.01
0.02
0.03
0.04
0
20
40
60
80
100
Time


True value of P23
estimator: T=10000
estimator: T=40000
Transition functions
0.00
0.01
0.02
0.03
0.04
0
20
40
60
80
100
Time

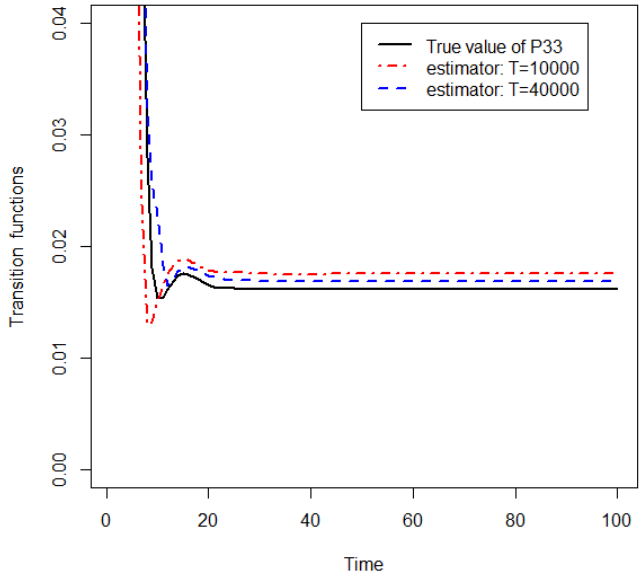


Figure 4.7: The true value and the estimator transition functions from state i to state 3, $P_{i3}(t,T)$ $(i \in E)$

Conclusion

Throughout this work, we were interested in presenting a study based on the nonparametric estimation of the transition functions in continuous time semi-Markov processes. The following points can be used to summarize the findings of this research:

- In the first part of this thesis, we have established the theoretical foundation for continuous-time homogeneous semi-Markov processes. We have presented the essential definitions and explored the fundamental properties and theorems governing these processes within a finite state space.
- In the second part of this dissertation, we have defined empirical estimators of semi-Markov kernel, sojourn time distributions, transition probabilities, the transition function of the semi-Markov process as well as the Markov renewal function. We then studied the asymptotic properties of these estimators, specifically their strong consistency and asymptotic normality. Then, we moved on to define the kernel estimation for the transition function of the semi-Markov process and Markov renewal function, and we have also studied their asymptotic properties.
- Finally, to validate our approach, we provided a numerical example that illustrates the performance of the nonparametric estimators for the transition function.

In conclusion, the problem with the empirical distribution function is that it is always a discontinuous function. For continuous distribution it is more suitable to use a continuous estimator, as the kernel estimator, instead of the empirical one.

Bibliography

[1] Akritas, M. G., & Roussas, G. G. (1980). Asymptotic inference in continuous time semi-Markov processes. *Scandinavian Journal of Statistics*, 73-79.

[2] Alvarez, E. E. (2005). Smoothed nonparametric estimation in window censored semi-Markov processes. *Journal of statistical planning and inference*, 131(2), 209-229.

[3] Atuncar, G. S., & Athreya, K. B. (2002). Kernel Estimators for Semi-Markov Processes. Brazilian *Journal of Probability and Statistics*, 69-85.

[4] Ayhar, C., Barbu, V. S., Mokhtari, F., & Rahmani, S. (2022). On the asymptotic properties of some kernel estimators for continuous-time semi-Markov processes. *Journal of Nonparametric Statistics*, 34(2), 299-318.

[5] Barbu, V. S., & Limnios, N. (2009). *Semi-Markov chains and hidden semi-Markov models toward applications: their use in reliability and DNA analysis*. Springer Science & Business Media, 191.

[6] Barbu, V. S., & Limnios, N. (2006). Empirical estimation for discrete-time semi-Markov processes with applications in reliability. *Nonparametric Statistics*, 18(7-8), 483-498.

[7] Barbu, V. S.,Bérard, C., Cellier, D., Sautreuil, M., & Vergne, N. (2018). SMM: An R package for estimation and simulation of discrete-time semi-Markov models.

[8] Barbu, V. S., Lecocq, F., Lothodé, C., & Vergne, N. (2023). smmR: A semi-Markov R package. *Journal of Open Source Software*, 8(85), 4365.

[9] Barlow, R. E., & Proschan, F. (1996). *Mathematical theory of reliability*. Society for Industrial and Applied Mathematics.

[10] Billingsley, P. (1961). The Lindeberg-Levy theorem for martingales. *Proceedings of the American Mathematical Society* , 12(5), 788-792.

[11] Billingsley, P. (1995). *Probability and Measure*, (WileyInterscience, New York).

[12] Billingsley, P. (1999). *Convergence of Probability Measures*. Wiley, New York, 2nd edition.

[13] Chryssaphinou, O., Karaliopoulou, M., & Limnios, N. (2008). On discrete time semi-Markov chains and applications in words occurrences. *Communications in Statistics-Theory and Methods*, 37(8), 1306-1322.

[14] Cinlar, E. (1969). Markov renewal theory. *Advances in Applied Probability*, 1(2), 123-187.

[15] Corradi, G., Janssen, J., & Manca, R. (2004). Numerical treatment of homogeneous semi-Markov processes in transient case–a straightforward approach. *Methodology and Computing in Applied Probability*, 6(2), 233-246.

[16] Cramér, H. (1999). *Mathematical methods of statistics*. Princeton university press, 26.

[17] D'amico, G., Manca, R., & Salvi, G. (2014). Bivariate semi-Markov process for counterparty credit risk. *Communications in Statistics-Theory and Methods*, 43(7), 1503-1522.

[18] Dumitrescu, M, Gámiz, M. L., & Limnios, N. (2016). Minimum divergence estimators for the Radon–Nikodym derivatives of the semi-Markov kernel. *Statistics*, 50(3), 486-504.

[19] Gámiz, M. L., Kulasekera, K. B., Limnios, N., & Lindqvist, B. H. (2011). *Applied nonparametric statistics in reliability*. Springer Science & Business Media.

[20] Gorissen, M., & Vanderzande, C. (2011). Semi-Markov models of mRNA-translation. *arXiv preprint arXiv:1104.0131*.

[21] Grabski, F. (2015). *Semi-Markov Processes: Applications in System Reliability and Maintenance*. Elsevier Amsterdam, The Netherlands, 599.

[22] Gut, A. (2005). *Probability: a graduate course* . New York: Springer.

[23] Hall, P., & Heyde, C. C. (1980). *Martingale Limit Theory and Its Application*, Probability and Mathematical Statistics. New York: Academic Press, Inc.

[24] Janssen, J., & Manca, R. (2001). Numerical solution of non-homogeneous semi-Markov processes in transient case. *Methodology and Computing in Applied Probability*, 3(3), 271-293.

[25] Kemeny, J. G., & Snell, J. L. (1976). *Finite markov chains*. Springer-verlag. New York .

[26] Korolyuk, V. S., & Turbin, A. F. (1993). Decompositions of Large Scale Systems.

[27] Korolyuk V.S., & Turbin, A.F. (1976). Semi-Markov Processes and Their Applications, *Naukova Dumka*, Kiev.

[28] Lagakos, S. W., Sommer, C. J., & Zelen, M. (1978). Semi-Markov models for partially censored data. *Biometrika*, 65(2), 311-317.

[29] Laksaci, A., & Yousfate, A. (2002). Estimation fonctionnelle de la densité de l'opérateur de transition d'un processus de Markov à temps discret. *Comptes rendus.Mathématique*, 334(11), 1035-1038.

[30] Levy, P. (1954). Processus semi-markoviens. In *Proc. Int. Congress. Math. III, Amsterdam*.

[31] Limnios, N. (2004). A functional central limit theorem for the empirical estimator of a semi-Markov kernel. *Nonparametric Statistics*, 16(1-2), 13-18.

[32] Limnios, N., Ouhbi, B., & Sadek, A. (2005). Empirical estimator of stationary distribution for semi-Markov processes. *Communications in Statistics-Theory and Methods*, 34(4), 987-995.

[33] Limnios, N., & Oprisan, G. (2012). *Semi-Markov processes and reliability*. Springer Science & Business Media.

[34] Limnios, N., & Oprişan, G. (2001). *Semi-Markov processes and reliability*. Birkhüser, Boston.

[35] Limnios, N., & Oprişan, G. (1997). *On an additive functional of a semi-Markov process with arbitrary state space and applications*. Research report, UTC/DMA.

[36] Mokhtari, F., Ayhar, C., Barbu, V. S., & Rahmani, S. (2025). Kernel estimators of Markov renewal and semi-Markov transition functions of semi-Markov systems. *Journal of Nonparametric Statistics*, 1–24.

[37] Moore, E. H., & Pyke, R. (1968). Estimation of the transition distributions of a Markov renewal process. *Annals of the Institute of Statistical Mathematics*, 20(1), 411-424.

[38] Nadaraya, E. A. (1964). Some new estimates for distribution functions. *Theory of Probability & Its Applications*, 9(3), 497-500.

[39] Nadaraya, E. A. (1965). On non-parametric estimates of density functions and regression curves. *Theory of Probability & Its Applications*, 10(1), 186-190.

[40] Ouhbi, B., & Limnios, N. (1999). Nonparametric estimation for semi-Markov processes based on its hazard rate functions. *Statistical Inference for Stochastic Processes*, 2(2), 151-173.

[41] Ouhbi, B., & Limnios, N. (1997). Comportement asymptotique de la matrice de renouvellement markovien. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 325(8), 921-924.

[42] Parzen, E. (1962). On estimation of a probability density function and mode. *The Annals of Mathematical Statistics*, 33(3), 1065–1076.

[43] Polansky, A. M., & Baker, E. R. (2000). Multistage plug–in bandwidth selection for kernel distribution function estimates. *Journal of Statistical Computation and Simulation*, 65(1-4), 63-80.

[44] Pyke, R. (1961). Markov renewal processes: definitions and preliminary properties. *The Annals of Mathematical Statistics*, 1231-1242.

[45] Pyke, R. (1961). Markov renewal processes with finitely many states. *The Annals of Mathematical Statistics*, 1243-1259.

[46] Pyke, R., & Schaufele, R. (1964). Limit theorems for Markov renewal processes. *The Annals of Mathematical Statistics*, 1746-1764.

[47] Rosenblatt, M. (1956). Remarks on some nonparametric estimates of a density function. *The Annals of Mathematical Statistics*, 27(3), 832–835.

[48] Shamsuddinov, B. R. (2001). On the consistency of a kernel estimator of the distribution density of the sojourn time in a fixed state for semi-Markov processes. *Journal of Mathematical Sciences*, 103(4), 487-492.

[49] Smith, W. L. (1955). Regenerative stochastic processes. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 232(1188), 6-31.

[50] Takács, L. (1954). Some investigations concerning recurrent stochastic processes of a certain type. *Maygyar Tud. Akad. Mad. Kutató int. Kozl*, 3, 115-128.

[51] Takács, L. (1959). On a sojourn time problem in the theory of stochastic processes. *Transactions of the American Mathematical Society*, 93(3), 531-540.

[52] Van der Vaart, A. W. (2000). *Asymptotic statistics*. Cambridge university press, 3.

[53] Votsi, I., Limnios, N., Tsaklidis, G., & Papadimitriou, E. (2012). Estimation of the expected number of earthquake occurrences based on semi-Markov models. *Methodology and Computing in Applied Probability*, 14(3), 685–703.

[54] Votsi, I., Limnios, N., Tsaklidis, G., & Papadimitriou, E. (2014). Hidden semi-Markov modeling for the estimation of earthquake occurrence rates. *Communications in Statistics-Theory and Methods*, 43(7), 1484-1502.

[55] Wu, B., Maya, B. I. G., & Limnios, N. (2021). Using semi-Markov chains to solve semi-Markov processes. *Methodology and computing in applied probability*, 23(4), 1419-1431.