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* *Dedication* *

*All praise to **Allah**.*

I dedicate this work to

my beloved parents

my mother and father

to my sister and my brothers

to my friends and my teachers

and to everyone who encouraged me, since the very the beginning of this journey, and supported me throughout my entire study

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In the name of **Allah**, the Most Merciful, the Most Compassionate, all praise be to Allah, and prayers and peace be upon Prophet Muhammad, His servant and messenger.
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May Allah bless you all abundantly

Abstract

This thesis is concerned with the analysis of stochastic differential equations (SDEs) with Applications in three distinct but interconnected settings. The first chapter studies the existence of mild solutions for stochastic delay evolution equations driven by fractional Brownian motion, using fixed point theory. The second chapter establishes fixed-time and predefined-time stability criteria for SDEs with Lévy noise, providing theoretical tools for finite-time control. The third chapter develops a stochastic extension of the classical SIR malaria model incorporating Brownian motion, and compares numerical solution methods along with a novel convergence metric. Overall, the thesis demonstrates how stochastic modelling, stability analysis, and numerical simulation contribute to a deeper understanding of complex dynamical systems under uncertainty.

Key Words and Phrases

Stochastic differential equations, fractional Brownian motion, Lévy noise, fixedâtime stability, SIR malaria model, numerical methods for SDEs.

Résumé

Cette thése est consacrée á l'analyse des équations différentielles stochastiques (EDS) avec des applications dans trois contextes distincts mais interconnectés. Premiérement, nous étudions l'existence de solutions mild pour des équations dâévolution stochastiques avec retard, pilotées par un mouvement brownien fractionnaire, en utilisant la théorie du point fixe. Deuxiémement, nous établissons des critéres de stabilité en temps fixe et en temps prédéfini pour les EDS avec bruit de Lévy, fournissant des outils théoriques pour le contrôle en temps fini. Troisiémement, nous développons une extension stochastique du modéle classique SIR du paludisme intégrant un mouvement brownien, et nous comparons les méthodes de résolution numérique ainsi quâune nouvelle métrique de convergence. Globalement, la thése montre comment la modélisation stochastique, l'analyse de stabilité et la simulation numérique contribuent á une meilleure compréhension des systemes dynamiques complexes sous incertitude.

Mots-clés

Équations différentielles stochastiques, mouvement brownien fractionnaire, bruit de Lévy, stabilité en temps fixe, stabilité en temps prédéfini, modéle SIR du paludisme, méthodes numériques pour EDS.

Contents

Introduction

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Epidemiology is concerned with the systematic study of disease distribution and its determinants, with a strong emphasis on mathematical modelling to evaluate intervention strategies. Vector-borne diseases such as malaria exhibit complex transmission dynamics that are particularly challenging to capture [9]. Mathematical modelling has become a cornerstone of modern epidemiology, providing rigorous frameworks to quantify transmission, recovery, mortality, and other key parameters. The foundations of this approach can be traced to Bernoulli's seminal work in 1760, which introduced the first formal mathematical treatment of epidemic processes [4]. Later, the contributions of Anderson and May demonstrated the essential role of models in generating testable hypotheses and evidence-based outbreak predictions [5, 6]. Among the most influential frameworks is the Susceptible-Infected-Recovered (SIR) model proposed by Kermack and McKendrick in 1927, which remains a fundamental tool for understanding deterministic disease dynamics [7].

Although deterministic SIR models offer powerful insights, they assume perfect predictability given initial conditions. Real-world epidemics are inherently stochastic, influenced by environmental fluctuations, demographic randomness, and other unpredictable factors. To address this limitation, stochastic differential equations (SDEs) have gained considerable attention. They have been widely applied in fields ranging from finance to population biology [12, 10]. By incorporating Brownian motion, SDEs capture random fluctuations in transmission and recovery, leading to more realistic epidemic predictions and enabling risk assessment through probability distributions [8, 11].

However, many real-world noises do not follow a Gaussian distribution. Phenomena such as evoked potentials in biomedicine, low-frequency atmospheric noise, and man-made interference often exhibit strong jumps that cannot be described by Brownian motion alone. Lévy noise, a typical non-Gaussian process, provides a more faithful representation of such abrupt changes [16, 17]. Stochastic differential equations driven by Lévy noise have attracted increasing research interest because they better reflect the discontinuous nature of many biological and engineering systems [18, 19, 20].

Beyond model formulation, the stability of stochastic systems is of paramount importance, especially when control objectives must be achieved within a finite time. Finite-time stability guarantees that the system reaches equilibrium in a finite interval, which is often more valuable than asymptotic behaviour [21, 22]. Fixed-time stability, a stronger notion, ensures that the settling time has an upper bound that is independent of the initial state, thereby enhancing convergence speed and control precision [23, 24, 25]. More recently, predefined-time stability has been introduced, allowing the settling time to be tuned arbitrarily by a system parameter, which is highly desirable for engineering applications where convergence time must be specified in advance [26, 27, 28]. For stochastic systems with Lévy noise, the study of fixed-time and predefined-time stability remains an emerging area, and the concept of optimal-time stability—where an index function is minimized—opens new directions for control design.

In this work, we combine these three perspectives. First, we present general existence results for mild solutions of stochastic delay evolution equations driven by fractional Brownian motion. Second, we establish criteria for fixed-time, predefined-time, and optimal-time stability of SDEs with Lévy noise, providing theoretical tools for advanced control strategies . Third, we develop a stochastic SIR malaria model with environmental noise, solved numerically by Euler–Maruyama, Milstein, and stochastic Runge–Kutta methods, and we introduce a novel convergence metric (Okwomi C*). Through these integrated contributions, we aim to improve epidemic forecasting, enhance the understanding of stochastic dynamics, and offer robust stability frameworks for uncertain biological and engineering systems.

In **Chapter 1**, we introduce notations, definitions, and preliminary facts that will be used throughout this thesis.
In **Chapter 2**, our main objective is to establish sufficient conditions for the existence of mild solutions of stochastic functional equation with time delays,driven by fractional Brownian motion ,
In **Chapter 3**, we are going to to establish theoretical criteria for fixed-time stability and predefined-time stability of stochastic differential equations driven by Lévy noise. We achieve this by first defining appropriate stability concepts and then employing the Ito formula along with Lyapunov function techniques to derive sufficient conditions. Specifically, they provide a fixed-time stability theorem based on a bound on the infinitesimal generator of a Lyapunov function, and a predefined-time stability theorem where the settling time can be tuned by a system parameter. The approach is illustrated with a numerical example.
In **Chapter 4** , we present a stochastic SIR malaria model that incorporates environmental uncertainty through Brownian motion. Three numerical methods are compared: Euler Maruyama, Milstein, and Stochastic Runge Kutta. Error metrics show that Euler Maruyama best captures human infection and recovery, while Milstein gives balanced estimates for both human and mosquito populations. A new convergence metric, Okwomi C star, is introduced. Stochastic models reveal extinction and variability missed by deterministic models, improving epidemic forecasting and control strategies.

Chapter 1

preliminaries

1 Notations:

$\mathcal{H}$: Real separable Hilbert space

κ or $\mathcal{K}$: Real separable Hilbert space

$(\cdot,\cdot)$: Inner product in $\mathcal{H}$

$\|\cdot\|$: Norm in $\mathcal{H}$

H : Hurst parameter, $H \in (1/2, 1)$

$B_Q^H(t)$: Q-cylindrical fractional Brownian motion

$(\Omega, \mathcal{F}, P)$: Complete probability space

$\{\mathcal{F}_t\}_{t\geq 0}$: Filtration of σ-algebras

J : Interval $[0, T]$

r : Maximum delay, $r > 0$

$[-r, 0$]: Delay interval

$y_(t)$: Segment solution

$y_t(\theta, \omega) = y(t + \theta, \omega)$ for $\theta \in [-r, 0]$

A : Infinitesimal generator of a C_0-semigroup

$S(t)$: Strongly continuous semigroup generated by A

M : Bound for the semigroup, $\|S(t)\|^2 \leq M$

$\mathcal{D}_{\mathcal{H}}$: Banach space of functions from $[-r, 0]$ to $\mathcal{H}$

$\mathcal{D}^0_{\mathcal{H}}$: Space of $\mathcal{F}_0$-measurable stochastic processes in $\mathcal{D}_{\mathcal{H}}$ with finite second moment

$\mathcal{D}^T_{\mathcal{H}}$: Space of processes $y : [-r, T] \times \Omega \to \mathcal{H}$ with finite second moment

$\|\phi\|_{\mathcal{D}^0_{\mathcal{H}}}$: Norm in $\mathcal{D}^0_{\mathcal{H}}$, $\left(\int_{-r}^{0} E\|\phi(t)\|^2 dt\right)^{1/2}$

$\|y\|_{\mathcal{D}^T_{\mathcal{H}}}$: Norm in $\mathcal{D}^T_{\mathcal{H}}$, $\sup_{t \in [0,T]} (E|y(t)|^2)^{1/2} + \|y_0\|_{\mathcal{D}^0_{\mathcal{H}}}$

f : Drift coefficient, $f : J \times \mathcal{D}^0_{\mathcal{H}} \to \mathcal{H}$

g : Diffusion coefficient, $g : J \to L^0_Q(\mathcal{K}, \mathcal{H})$

Q : Covariance operator for the fBm, non-negative, self-adjoint, trace-class

$L^0_Q(\mathcal{K}, \mathcal{H})$: Space of Q-Hilbert-Schmidt operators from $\mathcal{K}$ to $\mathcal{H}$

ϕ : Initial condition/function, $\phi \in \mathcal{D}^0_{\mathcal{H}}$

$E[X$] : Expectation of the random variable X

[$p(t)$:] Function in $L^1(J, \mathbb{R}_+)$ used in the growth condition of f

ψ : Continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$

C_0, C_1 : Constants defined in hypothesis (H4)

Φ : Fixed point operator defined in the proof of Theorem 3.1.1

Ξ : Set of all possible solutions to $y = \lambda\Phi(y)$ for $\lambda \in (0, 1)$

t : Time variable

T : Time horizon or upper bound constant

$x(t)$: State vector of the system

x_0 : Initial state

$x(t-)$: Left limit of the state

$\mathbb{R}^+$: Set of non-negative real numbers

$\mathbb{R}^n$: n-dimensional Euclidean space

$\mathbb{R}^{n\times m}$: Space of $n \times m$ real matrices

$\mathcal{F}_0$: Filtration at time 0

$\mathbb{E}[\cdot$:] Mathematical expectation operator

$\mathbf{P}(\cdot)$: Probability measure

$\mathbb{I}_{[\cdot]}$: Indicator function

$L(t)$: Lévy motion (process)

$B(t)$: Brownian motion (standard Wiener process)

$N(dt, dy)$: Poisson random measure

$\tilde{N}(dt, dy)$: Compensated Poisson random measure

$\nu(dy)$: Lévy jump intensity measure (Lévy measure)

c : Maximum allowable jump size

y : Jump size variable

$f(t, x)$: Drift coefficient

$g(t, x)$: Diffusion coefficient (Brownian)

$b(t, x)$: Drift coefficient in decomposed form

$\sigma(t, x)$: Diffusion matrix in decomposed form

$H(t, x, y)$: Jump coefficient (small jumps)

$G(t, x, y)$: Jump coefficient (large jumps)

$\mathbf{L}V$: Infinitesimal generator (Dynkin operator) for Lévy SDEs

$V(t, x)$: Lyapunov function

V_x : Gradient of V (row vector of partial derivatives)

V_{xx} : Hessian matrix of V

$\mathrm{trace}(\cdot)$: Trace of a matrix

$T(x_0)$: Settling-time function

$T_{\max}$: Upper bound of settling-time (Fixed-time)

T_c : Predefined convergence time bound

T_p : System time parameter (tunable)

$\alpha, \beta, \gamma, p, q, a, b$: Positive constants/parameters in stability conditions

$\kappa(\cdot)$: Class $\mathcal{K}^1$ function

κ^1 : Family of strictly increasing functions $\kappa(0) = 0$ and $\lim_{t\to\infty} \kappa(t) = 1$

C_1^2 : Family of nonnegative functions continuously twice differentiable in x and once in t

μ_1, μ_2 : Class $\mathcal{K}_\infty$ comparison functions

ϵ, r, δ : Stability probability parameters (small constants)

$\chi_\beta, \sigma_k, \pi_k$: Stopping times

$I(T, \eta)$: Indicator (cost) function for optimal time

ξ, λ : Parameters for the example simulation (Lévy measure constants)

$S(t)$, $I(t)$, $R(t)$: Susceptible, Infected, and Recovered human populations (classical SIR model).

N: Total human population, $N = S(t) + I(t) + R(t)$.

μ: Natural birth/death rate (classical SIR model).

β: Transmission rate (classical SIR model).

γ: Recovery rate (classical SIR model).

S_h, I_h, R_h: Susceptible, Infected, and Recovered human populations (malaria model).

S_m, I_m: Susceptible and Infected mosquito populations.

N_h, N_m: Total human and total mosquito populations.

α_h, α_m: Natural death rates for humans and mosquitoes.

β_h, β_m: Transmission/contact rates for humans and mosquitoes.

γ_h: Human recovery rate.

ρ_h: Disease-induced death rate for humans.

δ_h: Infected migration rate for humans.

π_h, π_m: Birth/recruitment rates for humans and mosquitoes.

$\sigma_1, \sigma_2, \ldots, \sigma_5$: Noise intensities (diffusion coefficients) for the five SDEs.

$\{W(t)\}_{t\geq 0}$: Wiener process (Brownian motion).

$d\beta_t$: Increment of standard Brownian motion (used in SDE SIR model).

dW_{jt}, $j = 1, \ldots, 5$: Independent standard Brownian motions in the malaria SDE system.

$\mathcal{N}(\mu, \sigma^2)$: Normal distribution with mean μ and variance σ^2.

$\mathrm{Var}(X)$: Variance of X.

$\mathcal{R}_0$: Deterministic basic reproduction number, $\mathcal{R}_0 = \frac{\beta N}{\gamma+\mu}$.

$\mathcal{R}_0^s$: Stochastic basic reproduction number, $\mathcal{R}_0^s = \mathcal{R}_0 - \frac{\sigma^2}{2\beta^2}$.

$\mathbf{P}$: Correlation matrix of the Wiener processes (size 5×5).

ρ_{ij}: Correlation coefficient between dW_{it} and dW_{jt}.

$\mathbf{L}$: Lower triangular Cholesky matrix such that $\mathbf{P} = \mathbf{L}\mathbf{L}^\top$.

ℓ_{ij}: Entries of the Cholesky matrix $\mathbf{L}$.

$d\mathbf{Z}_t$: Vector of independent Brownian motions.

$d\mathbf{W}_t$: Vector of correlated Brownian motions, $d\mathbf{W}_t = \mathbf{L}\, d\mathbf{Z}_t$.

X_t: General stochastic process (solution of an SDE).

$a(x,t)$ or $\mu(X_t, t)$: Drift coefficient (deterministic part).

$\sigma(X_t, t)$: Diffusion coefficient (stochastic part).

ξ_t: White noise process (formal derivative of W_t).

L^0, L^1: Differential operators in the stochastic Taylor expansion: $L^0 = a\frac{d}{dx} + \frac{1}{2}b^2\frac{d^2}{dx^2}$, $L^1 = b\frac{d}{dx}$.

R, $\bar{R}$: Remainder terms in the stochastic Taylor expansion.

Δt: Time step size in numerical schemes.

ΔW_n: Increment of Brownian motion over a time step, $\Delta W_n = W_{n+1} - W_n$.

x_i, X_n: Numerical approximation of the solution at time step i or n.

MAD: Mean Absolute Deviation: $\frac{1}{n}\sum_{i=1}^{n}|y_i - \hat{y}_i|$.

MAPE: Mean Absolute Percentage Error: $\frac{100\%}{n}\sum_{i=1}^{n}\left|\frac{y_i - \hat{y}_i}{y_i}\right|$.

RMSE: Root Mean Square Error: $\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$.

A^2: Anderson-Darling goodness-of-fit test statistic.

$F(\cdot)$: Cumulative distribution function of the theoretical distribution (AD test).

$Y_{(i)}$: Ordered observed values.

$\bar{X}_a$, $\bar{X}_b$: Means of early and late segments of a Markov chain (Geweke diagnostic).

Z: Standard normal test statistic for convergence: $Z = \frac{\bar{X}_a - \bar{X}_b}{\sqrt{\mathrm{Var}(\bar{X}_a) + \mathrm{Var}(\bar{X}_b)}}$.

C: Convergence score: $C = \exp(-Z^2/2)$.

C_s^*: Okwomi C^* convergence measure: $C_s^* = 0.5 + 0.5 \cdot C$ (values in $[0.5, 1.0]$).

EM: Euler-Maruyama method.

MI: Milstein method.

SRK: Stochastic Runge-Kutta method (derivative-free).

ADF: Augmented Dickey-Fuller test for stationarity.

p-value: Probability value from statistical tests.

2 Definitions and Theoremes

Definition 1.1. Let $\mathcal{D}_{\mathbb{H}}$ be the Banach space defined by:

$\mathcal{D}_{\mathbb{H}} = \{\phi : [-r, 0] \to \mathbb{H}$, ϕ is continuous everywhere except for a finite number of points t

at which $\phi(t^-)$ and $\phi(t^+)$ exist and satisfy $\phi(t^-) = \phi(t)$ endowed with the L^2-norm:

$$\|\phi\|^2 = \int_{-r}^{0} \|\phi(t)\|^2 dt.$$

Definition 1.2 (Initial Value Space)**.** Define $\mathcal{D}^0_{\mathbb{H}}$ as the space of all piecewise continuous processes $\phi : [-r, 0] \times \Omega \to \mathbb{H}$ such that $\phi(\theta, \cdot)$ is $\mathbb{F}_0$-measurable for each $\theta \in [-r, 0]$ and $\int_{-r}^{0} \|\phi(t)\|^2 dt < \infty$. The norm is given by:

$$\|\phi\|^2_{\mathcal{D}^0_{\mathbb{H}}} = \int_{-r}^{0} \|\phi(t)\|^2 dt.$$

Definition 1.3 (Solution Space)**.** For a given $T > 0$, define:

$$\mathcal{D}^T_{\mathbb{H}} = \left\{ y : [-r, T] \times \Omega \to \mathbb{H} \ : \ y \in C([0, T], \mathbb{H}), \ y_0 \in \mathcal{D}^0_{\mathbb{H}}, \ \sup_{t\in[0,T]} (|y(t)|^2) < \infty \right\},$$

endowed with the norm:

$$\|y\|_{\mathcal{D}^T_{\mathbb{H}}} = \sup_{t\in[0,T]} ((|y(t)|^2))^{\frac{1}{2}} + \|y\|_{\mathcal{D}^0_{\mathbb{H}}}.$$

Definition 1.4 (Mild Solution)**.** Given $\phi \in \mathcal{D}^0_{\mathbb{H}}$, an $\mathbb{H}$-valued stochastic process $\{y(t), t \in [-r, T]\}$ is called a **mild solution** of the problem (0.1) if:

1. $y(t)$ is measurable and $_t$-adapted,
2. $y(t) = \phi(t)$ on $[-r, 0]$,
3. y satisfies the integral equation:

$$y(t) = S(t)\phi(0) + \int_0^t S(t-s)f(s, y_s)ds + \int_0^t S(t-s)g(s)dB^H_Q(s), \quad t \in [0, T],$$

where $y_t(\theta, \omega) = y(t + \theta, \omega)$ for $\theta \in [-r, 0]$.

Hypotheses

We shall impose the following hypotheses:

Hypothése 2.0.1. Operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}_{t\in J}$ which is compact for $t > 0$ in $\mathbb{H}$, and there exists a constant M such that $\|S(t)\|^2 \leq M$ for all $t \in [0, T]$.

Hypothése 2.0.2. For each $t \in J$, the function $f(t, \cdot) : \mathcal{D}^0_{\mathbb{H}} \to \mathbb{H}$ is continuous, and for each $y \in \mathbb{H}$, the function $f(\cdot, y) : J \to \mathbb{H}$ is strongly ${}_t$-measurable.

Hypothése 2.0.3. The function $g : J \to L^0_Q(, \mathbb{H})$ satisfies:

$$\int_0^T \|g(s)\|^2_{L^0_Q} ds < \infty.$$

Hypothése 2.0.4. For the initial value $\phi \in \mathcal{D}^0_{\mathbb{H}}$, there exists a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ and $p \in L^1(J,_+)$ such that:

$$|f(t, y)|^2 \leq p(t)\psi(\|y\|^2_{\mathcal{D}^0_{\mathbb{H}}}), \quad \text{for a.e. } t \in J \text{ and } y \in \mathcal{D}^0_{\mathbb{H}},$$

with

$$\int_{TC_0}^{\infty} \frac{du}{\psi(u)} > TC_1 \int_0^T p(t)dt,$$

where

$$C_0 = 4M \left(|\phi(0)|^2 + 2HT^{2H-1} \int_0^T \|g(s)\|^2_{L^0_Q} ds \right), \quad C_1 = 4MT.$$

Auxiliary Theorems

Theorem 1.1 (Burton-Kirk Fixed Point Theorem [2])**.** Let E be a Banach space, and $G_1, G_2 : E \to E$ be two operators satisfying:

1. G_1 is a contraction, and
2. G_2 is completely continuous.

Then, either the operator equation $y = G_1(y) + G_2(y)$ possesses a solution, or the set

$$\varXi = \{y \in E : \lambda G_1(\tfrac{y}{\lambda}) + \lambda G_2(y) = y \text{ for some } \lambda \in (0, 1)\}$$

is unbounded.

Theorem 1.2 (Arzelá-Ascoli Theorem)**.** A set of functions in $C([0,T], \mathbb{H})$ is relatively compact if it is:

1. uniformly bounded, and
2. equicontinuous.

Theorem 1.3 (Properties of Semigroups [1])**.** If A generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ which is compact for $t > 0$, then:

1. $\|S(t)\| \leq M$ for some constant M,
2. The map $t \mapsto S(t)$ is continuous in the uniform operator topology for $t > 0$.

Main Result

Theorem 1.4 (Existence of Mild Solution)**.** Assume that hypotheses (H1)-(H4) hold. Then the stochastic delay evolution equation (0.1) possesses at least one mild solution on $[-r, T]$.

Proof.

The proof is based on the Burton-Kirk fixed point theorem and proceeds in several steps.

Step 1: Reformulation as a fixed point problem.

Define two operators $\Phi_1, \Phi_2 : \mathcal{D}^T_{\mathbb{H}} \to \mathcal{D}^T_{\mathbb{H}}$ by:

$$\Phi_1(y)(t) = 0, \quad \text{(since there are no impulses)}$$

$$\Phi_2(y)(t) = S(t)\phi(0) + \int_0^t S(t-s)f(s, y_s)ds + \int_0^t S(t-s)g(s)dB^H_Q(s), \quad t \in [-r, T].$$

Then the problem reduces to finding a fixed point of the operator equation $y = \Phi_2(y)$.

Step 2: Φ_2 is continuous.

Let $y_n \to y$ in $\mathcal{D}^T_{\mathbb{H}}$. Then for $t \in J$:

$$\begin{aligned}
|\Phi_2(y_n)(t) - \Phi_2(y)(t)|^2 &\leq \left| \int_0^t S(t-s)(f(s,(y_n)_s) - f(s, y_s))ds \right|^2 \\
&\leq MT \int_0^T |f(s,(y_n)_s) - f(s, y_s)|^2 ds \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus Φ_2 is continuous.

Step 3: $\varPhi_2$ maps bounded sets into bounded sets.

Let $\mathcal{B}_q = \{y \in \mathcal{D}^T_{\mathbb{H}} : \|y\|^2_{\mathcal{D}^T_{\mathbb{H}}} \leq q\}$. For $y \in \mathcal{B}_q$:

$$\begin{aligned}|\varPhi_2(y)(t)|^2 &\leq 2M|\phi(0)|^2 + 2MT\int_0^t p(s)\psi(\|y_s\|^2_{\mathcal{D}^0_{\mathbb{H}}})ds \\ &\leq 2M|\phi(0)|^2 + 2MT\psi(q)\int_0^T p(s)ds := \ell.\end{aligned}$$

Step 4: $\varPhi_2$ maps bounded sets into equicontinuous sets.

For $0 \leq \tau_1 < \tau_2 \leq T$ and $y \in \mathcal{B}_q$:

$$\begin{aligned}|\varPhi_2(y)(\tau_2) - \varPhi_2(y)(\tau_1)|^2 \leq\ & 2\|[S(\tau_2) - S(\tau_1)]\phi(0)\|^2 \\ &+ 2T\psi(q)\int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 p(s)ds \\ &+ 2T\psi(q)\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|^2 p(s)ds.\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$ by the strong continuity and compactness of $S(t)$.

Step 5: $\varPhi_2$ maps bounded sets into precompact sets.

For fixed $t \in (0, T]$ and $\epsilon \in (0, t)$, define:

$$(\varPhi_{2\epsilon}y)(t) = S(t)\phi(0) + S(\epsilon)\int_0^{t-\epsilon} S(t - s - \epsilon)f(s, y_s)ds.$$

Since $S(\epsilon)$ is compact, the set $\{\varPhi_{2\epsilon}y(t) : y \in \mathcal{B}_q\}$ is precompact. Moreover:

$$|\varPhi_2(y)(t) - \varPhi_{2\epsilon}(y)(t)|^2 \leq TM\int_{t-\epsilon}^t p(s)\psi(q)ds \to 0 \quad \text{as } \epsilon \to 0.$$

Hence $\{\varPhi_2(y)(t) : y \in \mathcal{B}_q\}$ is precompact.

Step 6: A priori bounds.

Consider the set $\varXi = \{y \in \mathcal{D}^T_{\mathbb{H}} : y = \lambda\varPhi_2(y)$ for some $\lambda \in (0, 1)\}$. For $y \in \varXi$, we have:

$$\begin{aligned}|y(t)|^2 \leq\ & 4\lambda M|\phi(0)|^2 + 4\lambda MT\int_0^t p(s)\psi(\|y_s\|^2_{\mathcal{D}^0_{\mathbb{H}}})ds \\ &+ 8\lambda MHt^{2H-1}\int_0^t \|g(s)\|^2_{L^0_Q}ds.\end{aligned}$$

Define $\mu(t) = \sup_{0\leq s\leq t}|y(s)|^2$. Then:

$$\|y_s\|^2_{\mathcal{D}^0_{\mathbb{H}}} \leq \|\phi\|^2_{\mathcal{D}^0_{\mathbb{H}}} + T\mu(t).$$

Proceeding with standard estimates and using hypothesis (H4), we obtain that $\mu(t)$ is bounded, hence $\varXi$ is bounded.

By the Burton-Kirk fixed point theorem, $\varPhi_2$ has a fixed point, which is a mild solution of the problem (0.1). ■

(system(3): we are going to use it in chapter 3)

$$\begin{aligned} dx(t) &= b(t, x(t-))\,\mathrm{d}t + \sigma(t, x(t-))\,\mathrm{d}B(t) \\ &\quad + \int_{|y|<c} H(t, x(t-), y)\,\tilde{N}(\mathrm{d}t, \mathrm{d}y), \quad x(0) = x_0,\ t \in [0, T]. \end{aligned} \tag{2.1}$$

Definition 1.6 (Fixed-time stability)**.** : System (3) is fixed-time stable if it is globally finite-time stable and there exists $T_{\max} > 0$ such that

$$\sup_{x_0 \in \mathbb{R}^n} T(x_0) \leq T_{\max} < \infty.$$

Definition 1.7 (Predefined-time stability)**.** :System (3) is predefined-time stable if it is fixed-time stable and

$$\sup_{x_0 \in \mathbb{R}^n} T(x_0) \leq T_c,$$

where $T_c = T_c(T_p) < \infty$ can be tuned by the system parameter T_p.

Definition 1.8 (Predefined-time stable in probability)**.** The zero solution of (3) is predefined-time stable in probability if:

(i) **Predefined-time attractiveness in probability:** For every $x_0 \in \mathbb{R}^n \backslash \{0\}$, the first hitting time $T(x_0) = \inf\{t > 0 : x(t) = 0\}$ satisfies

$$\mathbf{P}(T(x_0) < \infty) = 1 \quad \text{and} \quad T(x_0) \leq T_c < \infty,$$

with $T_c = T_c(T_p)$ tunable by T_p.

(ii) **Stability in probability:** For any $\epsilon \in (0, 1)$ and $r > 0$, there exists $\delta = \delta(\epsilon, r) > 0$ such that

$$\mathbf{P}(|x(t)| < r,\ \forall t \geq 0) \geq 1 - \epsilon$$

whenever $|x_0| < \delta$.

Definition 1.9 (Optimal-time stability)**.** Let T^* be an upper bound for the settling-time. If there exists $T' \in [0, T^*]$ such that

$$I(T', \eta) = \min_{T \in [0, T^*]} I(T, \eta),$$

then the settling-time is optimal and the system has achieved optimal-time stability.

Lemmas

Lemma 1.1. Let $r(s) \geq 0$ be continuous. Define

$$F(V(x)) = \int_0^{V(x)} \frac{1}{r(s)} ds.$$

Then $\mathbb{E}[F(V(x))] \geq 0$ and $\mathbb{E}[F(V(x))] = 0 \Longleftrightarrow x = 0$.

Lemma 1.2. Let $0 < \alpha < 1$, $K > 0$, $\phi(0) > 0$. If $\phi : [0, \infty) \to [0, \infty)$ satisfies

$$\dot{\phi}(t) \leq -\frac{K}{\alpha T_p} \phi^{1-\alpha}(t) \cdot \exp\bigl(\phi^{\alpha}(t)\bigr), \quad t \geq 0,$$

then $\phi(t) = 0$ if and only if $t \geq \frac{T_p}{K}\bigl[1 - \exp(-\phi^{\alpha}(0))\bigr]$.

Theorems

Theorem 1.5 (Fixed-time stability criterion)**.** Let $T > 0$ and $\mathbb{C} = \{r(s) \mid r'(s) \geq 0,\ \int_0^{\infty} \frac{1}{r(s)} ds \leq T\}$. If $V(x)$ and $r(\cdot) \in \mathbb{C}$ satisfy

$$\mathbf{L}V(x) \leq -r(V(x)),$$

then the origin of (3) is globally fixed-time stable in probability and

$$\mathbb{E}[T(x, \omega)] \leq T \quad \text{for all } \mathbb{E}(|x_0|^2) < +\infty.$$

Theorem 1.6 (Predefined-time stability criterion)**.** Suppose there exist $T_p > 0$ and $0 < \alpha < 1$ such that for all $x \in \mathbb{R}^n$, $t \geq 0$, a function $V(t, x)$ satisfies

$$\text{(i)} \quad \mu_1(|x|) \leq V(t, x) \leq \mu_2(|x|),$$

$$\text{(ii)} \quad \mathbf{L}V(t, x) \leq -\frac{1}{\alpha T_p} \exp\bigl(V^{\alpha}\bigr)\, V^{1-\alpha}.$$

Then the solution of (3) is predefined-time stable in probability and

$$\mathbb{E}[T(x_0)] \leq T_p\bigl[1 - \exp\bigl(-V^{\alpha}(0, x(0))\bigr)\bigr].$$

Theorem 1.7 (Existence, Uniqueness and Properties of the Stochastic SIR Model:)**.** Consider the stochastic SIR system with environmental noise:

$$\begin{aligned}
dS &= (\mu N - \beta SI - \mu S)dt - \sigma_1 SI\, d\beta_t, \\
dI &= (\beta SI - (\gamma + \mu)I)dt + \sigma_2 SI\, d\beta_t, \\
dR &= (\gamma I - \mu R)dt,
\end{aligned}$$

where $\sigma_1, \sigma_2 > 0$ are noise intensities, $d\beta_t$ is the increment of a standard Brownian motion, and the condition $\sigma_1 = \sigma_2$ ensures population conservation. If $\sigma_1^2 < 2\beta$, then:

1. A unique solution $(S(t), I(t), R(t))$ exists for all $t \geq 0$.

2. The solution remains non-negative almost surely:

$$S(t) \geq 0, \quad I(t) \geq 0, \quad R(t) \geq 0 \quad \text{a.s.}$$

3. The total expected population is conserved:

$$\mathbb{E}[S(t) + I(t) + R(t)] = N \quad \text{for all } t \geq 0,$$

   where N is the constant total population.

Brownian Motion (Wiener Process):

A stochastic process $\{W(t)\}_{t\geq 0}$ is called a **Wiener process** or **Brownian motion** if it satisfies the following properties:

1. $W(0) = 0$ almost surely.
2. It has independent increments: for all $0 \leq t_1 < t_2 < \cdots < t_k$, the random variables

$$W_{t_1},\ W_{t_2} - W_{t_1},\ \ldots,\ W_{t_k} - W_{t_{k-1}}$$

   are independent.
3. For any $t > 0$ and $s \geq 0$, the increment $W(t+s) - W(s)$ follows a normal distribution:

$$W(t+s) - W(s) \sim \mathcal{N}(0, t).$$

   Equivalently, $\mathbb{E}[W(t)] = 0$ and $\mathrm{Var}(W(t)) = t$.

The Wiener process is a Gaussian process, continuous in time, and serves as the mathematical model for random fluctuations in stochastic differential equations.

Itô Formula

Let X_t be a one-dimensional Itô process satisfying the stochastic differential equation

$$dX_t = a(X_t, t)\,dt + b(X_t, t)\,dW_t,$$

where W_t is a standard Brownian motion, $a(\cdot)$ is the drift coefficient, and $b(\cdot)$ is the diffusion coefficient. For a twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, Itô's formula states that

$$df(X_t) = \left(a(X_t, t) f'(X_t) + \frac{1}{2} b^2(X_t, t) f''(X_t) \right) dt + b(X_t, t) f'(X_t) dW_t.$$

In integral form,

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_s, s) f'(X_s) + \frac{1}{2} b^2(X_s, s) f''(X_s) \right) ds + \int_{t_0}^{t} b(X_s, s) f'(X_s) dW_s.$$

This formula is the fundamental tool for deriving stochastic Taylor expansions and numerical schemes such as EulerâMaruyama and Milstein methods.

Fractional Brownian motion

[Fractional Brownian motion] Let $H \in (0,1)$. A **fractional Brownian motion** (fBm) with Hurst parameter H is a centered Gaussian process $B^H = \{B^H(t), t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that:

1. $B^H(0) = 0$ almost surely.
2. The covariance function is given for all $s, t \geq 0$ by
$$\mathbb{E}\big[B^H(s)B^H(t)\big] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$
3. The increments are stationary and selfâsimilar with exponent H:
$$\{B^H(t+\tau) - B^H(t)\}_{t\geq 0} \overset{\text{law}}{=} \{\tau^H B^H(t/\tau)\}_{t\geq 0}, \quad \forall \tau > 0.$$

For $H = 1/2$, fractional Brownian motion reduces to the classical Wiener process (standard Brownian motion). For $H \neq 1/2$, the increments are neither independent nor a martingale.

Chapter 2

Stochastic Differential Equations

In this chapter, our main objective is to establish sufficient conditions for the existence of mild solutions of the following first order stochastic functional equation with time delays, driven by fractional Brownian motion with the Hurst index $H > 1/2$:

$$\begin{cases} dy(t) = [Ay(t) + f(t, y_t)]dt + g(t)dB_Q^H(t), & t \in J := [0, T]; \\ y(t) = \phi(t), & \text{for a.e. } t \in [-r, 0], \end{cases} \tag{0.1}$$

in a real separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $\cdot$ where A : $D(A) \subset \longrightarrow$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$, $0 \leq t \leq T$. B_Q^H is a fractional Brownian motion on a real and separable Hilbert space κ, with Hurst parameter $H \in (1/2, 1)$, and with respect to a complete probability space $(\Omega, ,_t, P)$ furnished with a family of right continuous and increasing σ-algebras $\{_t, t \in J\}$ satisfying $_t \subset$. Also $r > 0$ is the maximum delay. As for y_t we mean the segment solution which is defined in the usual way, that is, if $y(\cdot, \cdot)$: $[-r, T] \times \Omega \rightarrow$, then for any $t \geq 0$, $y_t(\cdot, \cdot) : [-r, 0] \times \Omega \rightarrow$ is given by

$$y_t(\theta, \omega) = y(t + \theta, \omega), \quad \text{for } \theta \in [-r, 0],\ \omega \in \Omega.$$

Before describing the properties fulfilled by the operators f and g, we need to introduce some notation and describe some spaces.

Let be the following Banach space defined by

$$= \{\phi : [-r, 0] \rightarrow,\ \phi \text{ is continuous everywhere except for a finite number of points}\}$$

endowed with the L^2-norm:

$$\|\phi\|^2 = \int_{-r}^{0} \|\phi(t)\|^2 t.$$

Also we define ^{0} as the space of all piecewise continuous processes $\phi : [-r, 0] \times \Omega \to$ such that $\phi(\theta, \cdot)$ is $_0$-measurable for each $\theta \in [-r, 0]$ and $\int_{-r}^{0} E\|\phi(t)\|^2 t < \infty$. In the space ^{0}, we consider the norm:

$$\|\phi\|_{0}^2 = \int_{-r}^{0} E\|\phi(t)\|^2 t.$$

Now, for a given $T > 0$, we define

$$^T = \left\{ y : [-r, T] \times \Omega \to, \; y \in C([0, T],), \; y_0 \in^0, \; \sup_{t\in[0,T]} E(|y(t)|^2) < \infty \right\},$$

endowed with the norm

$$\|y\|_T = \sup_{t\in[0,T]} (E(|y(t)|^2))^{\frac{1}{2}} + \|y_0\|_0.$$

Then we will consider our initial data $\phi \in^0$.

Let κ be another real separable Hilbert space and suppose that B_Q^H is a κ-valued fractional Brownian motion with increment covariance given by a non-negative trace class operator Q, and let us denote by $L(,)$ the space of all bounded, continuous and linear operators from κ into .

Then we assume that $f : J\times^0 \to$ and $g : J \to L_Q^0(,)$. Here, $L_Q^0(,)$ denotes the space of all Q-HilbertâSchmidt operators from κ into .

1 Existence Result

Now, we are able to state and prove our main theorem for the initial value problem (0.1). Before starting and proving this, we give the definition of mild solution to our problem.

Given $\phi \in^0$, an -valued stochastic process $\{y(t), t \in [-r, T]\}$ is called a mild solution of the problem (0.1) if $y(t)$ is measurable and $_t$-adapted, for each $t > 0$ $y(t) = \phi(t)$ on $[-r, 0]$, and y satisfies the integral equation

$$y(t) = S(t)\phi(0) + \int_0^t S(t-s)f(s, y_s)s + \int_0^t S(t-s)g(s)B_Q^H(s), \quad t \in [0, T]. \tag{1.1}$$

Notice that this concept of solution can be considered as more general than the classical concept of solution to equation (0.1). A continuous solution of (1.1) is called a mild solution of (0.1).

Our main result in this section is based upon the fixed point theorem (burton-kirk) due to Burton and Kirk

We are now in a position to state and prove our existence result for the problem (0.1). First we will list the following hypotheses which will be imposed in our main theorem.

(H1) Operator $A : D(A) \subset \rightarrow$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}$, $t \in J$ which is compact for $t > 0$ in , and there exists a constant M such that $\|S(t)\|^2 \leq M$ for all $t \in [0, T]$.

(H2) For each $t \in J$ the function $f(t, \cdot) :^0 \longrightarrow$ is continuous, and for each $y \in$ the function $f(\cdot, y) : J \to$ is strongly $_t$-measurable.

(H3) The function $g : J \longrightarrow L_Q^0(,)$ satisfies

$$\int_0^T \|g(s)\|^2_{L_Q^0} s < \infty.$$

(H4) For the initial value $\phi \in^0$, there exists a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$E|f(t, y)|^2 \leq p(t)\psi(\|y\|_0^2), \quad \text{for a.e. } t \in J \text{ and } y \in^0$$

with

$$\int_{C_0}^{\infty} \frac{du}{\psi(u)} > TC_1 \int_0^T p(t)t,$$

where

$$C_0 = 4M\left(E|\phi(0)|^2 + 2HT^{2H-1}\int_0^T \|g(s)\|^2_{L_Q^0} s\right),$$
$$C_1 = 4MT.$$

Theorem 2.1. Assume that hypotheses (H1), (H2), (H3) and (H4) hold. Then, problem (0.1) possesses at least one mild solution on $[-r, T]$.

Proof.

Transform the problem (0.1) into a fixed point problem. Consider the operator $\varPhi :^T \to^T$ defined by:

$$\varPhi(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ S(t)\phi(0) + \int_0^t S(t-s)f(s, y_s)s + \int_0^t S(t-s)g(s)B_Q^H(s), & t \in [0, T]. \end{cases}$$

Then the problem of finding the solution of problem (0.1) is reduced to finding the fixed point of the operator equation $\Phi(y)(t) = y(t)$, $t \in [-r, T]$.

We shall show that the operator Φ satisfies all the conditions of Theorem (burton-kirk). The proof will be given in several steps.

Step 1: Φ is continuous.

Let y_n be a sequence such that $y_n \to y$ in ^{T}. Then, for $t \in J$, and thanks to (H1) and (H2),

$$\begin{aligned} E|\Phi(y_n)(t) - \Phi(y)(t)|^2 &\leq E\left|\int_0^t S(t-s)(f(s,(y_n)_s) - f(s,y_s))s\right|^2 \\ &\leq E\left(\int_0^t |S(t-s)||f(s,(y_n)_s) - f(s,y_s)|s\right)^2 \\ &\leq E\left[\int_0^t |S(t-s)|^2 s\left(\int_0^t |f(s,(y_n)_s) - f(s,y_s)|^2 s\right)\right] \\ &\leq MT\int_0^t E|f(s,(y_n)_s) - f(s,y_s)|^2 s. \end{aligned}$$

Hence

$$\sup_{t\in[0,T]} E|\Phi(y_n)(t) - \Phi(y)(t)|^2 \leq MT\int_0^T E|f(s,(y_n)_s) - f(s,y_s)|^2 s \to 0 \text{ as } n \to +\infty.$$

Thus Φ is continuous.

Step 2: Φ maps bounded sets into bounded sets in ^{T}.

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant l such that for each $y \in \mathcal{B}_q = \{y \in^T : \|y\|_T^2 \leq q\}$, one has $\|\Phi(y)\|_T^2 \leq l$.

Let $y \in \mathcal{B}_q$, then for each $t \in [0, T]$, we have

$$\begin{aligned} |\Phi y(t)|^2 &\leq \left|S(t)\phi(0) + \int_0^t S(t-s)f(s,y_s)s\right|^2 \\ &\leq 2M|\phi(0)|^2 + 2M\left|\int_0^t f(s,y_s)s\right|^2. \end{aligned}$$

Thus

$$\begin{aligned} E|\Phi y(t)|^2 &\leq 2ME|\phi(0)|^2 + 2TM\int_0^t p(s)\psi(\|y_s\|_0^2)s \\ &\leq 2ME|\phi(0)|^2 + 2MT\psi(q)\int_0^T p(s)s. \end{aligned}$$

Then we have

$$E|\Phi y(t)|^2 \leq 2ME|\phi(0)|^2 + 2MT\psi(q)\|p\|_{L^1} = l.$$

Step 3: Φ maps bounded sets into equicontinuous sets of ^{T}.

Let $0 \leq \tau_1 < \tau_2 \in J$ and $\mathcal{B}_q$ be a bounded set of ^{T} as in Step 2. Let $y \in \mathcal{B}_q$ then for each $t \in J$ we have

$$\begin{aligned}E|(\Phi y)(\tau_2) - (\Phi y)(\tau_1)|^2 \leq 2|[S(\tau_2) - S(\tau_1)]E|\phi(0)||^2 \\ + 4T\psi(q)\int_0^{\tau_1} |S(\tau_2 - s) - S(\tau_1 - s)|^2 p(s)s \\ + 4T\psi(q)\int_{\tau_1}^{\tau_2} |S(\tau_2 - s)|^2 p(s)s.\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$ and ϵ sufficiently small, since $S(t)$ is strongly continuous and the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology [1]. This proves the equicontinuity.

The cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ follow from the uniform continuity of ϕ on the interval $[-r, 0]$.

As a consequence of Steps 2 to 3, together with the ArzelÃ¡âAscoli theorem, it suffices to show that Φ maps $\mathcal{B}_q$ into a precompact set in .

Step 4: Φ maps bounded sets into precompact sets.

Let $0 < t < T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in \mathcal{B}_q$ we define

$$(\Phi_\epsilon y)(t) = S(t)\phi(0) + S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)f(s, y_s)s + \int_0^{t-\epsilon} S(t-s-\epsilon)g(s)B_Q^H(s).$$

Since $S(t)$ is a compact operator, the set

$$Y_\epsilon(t) = \{\Phi_\epsilon(y)(t) : y \in \mathcal{B}_q\}$$

is precompact in for every ϵ, $0 < \epsilon < t$. Moreover, for every $y \in \mathcal{B}_q$ we have

$$\begin{aligned}E|(\Phi y)(t) - (\Phi_\epsilon(y)(t)|^2 &\leq T\int_{t-\epsilon}^{t} |S(t,s)|^2 p(s)\psi(q)s \\ &\leq TM\int_{t-\epsilon}^{t} p(s)\psi(q)s \to 0 \text{ as } \epsilon \to 0.\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $Y(t) = \{\Phi(y)(t) : y \in \mathcal{B}_q\}$. Hence the set $Y(t)$ is precompact in , and therefore, the operator $\Phi :{}^T\to{}^T$ is completely continuous.

Step 5: A priori bounds.

Now it remains to show that the set

$$\Xi = \{y \in{}^T : y = \lambda\Phi(y) \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Let $y \in \Xi$. Then $y = \lambda\Phi(y)$ for some $0 < \lambda < 1$. Thus for each $t \in J$

$$y(t) = \lambda \left[S(t)\phi(0) + \int_0^t S(t-s)f(s,y_s)s + \int_0^t S(t-s)g(s)B_Q^H(s) \right],$$

and

$$y(t) = \phi(t), \quad \text{for each } t \in [-r,0].$$

This implies, for each $t \in J$

$$E|y(t)|^2 \leq 4\lambda M E|\phi(0)|^2 + 4\lambda MT \int_0^t p(s)\psi(\|y_s\|_{\circ}^2)s$$
$$+ 8\lambda MHt^{2H-1} \int_0^t \|g(s)\|_{L_Q^0}^2 s.$$

We consider the function μ defined by

$$\mu(t) = \sup\{E|y(s)|^2 : 0 \leq s \leq t\}, \quad t \in J.$$

Since $\|y_s\|_{\circ}^2 = \int_{-r}^0 E|y(s+\theta)|^2\theta = \int_{-r+s}^s E|y(\theta)|^2\theta$, then

$$\|y_s\|_{\circ}^2 \leq \int_{-r}^0 E|\phi(\theta)|^2\theta + \int_0^s E|y(\theta)|^2\theta \leq \|\phi\|_{\circ}^2 + T\mu(t).$$

Hence

$$\|y_s\|_{\circ}^2 \leq \|\phi\|_{\circ}^2 + T\mu(t).$$

Then, for $t \in J$ we have

$$\mu(t) \leq 4M\left(E|\phi(0)|^2 + 2HT^{2H-1} \int_0^T \|g(s)\|_{L_Q^0}^2 s \right) + 4MT \int_0^t p(s)\psi(T\mu(s) + \|\phi\|_{\circ}^2)s.$$

Thus we have

$$T\mu(t) + \|\phi\|_{\circ}^2 \leq \|\phi\|_{\circ}^2 + TC_0 + TC_1 \int_0^t p(s)\psi(T\mu(s) + \|\phi\|_{\circ}^2)s. \tag{3.4}$$

Let us denote the right-hand side of the inequality (3.4) by $v(t)$. Then we have

$$v(0) = TC_0 + \|\phi\|_{\circ}^2, \quad T\mu(t) + \|\phi\|_{\circ}^2 \leq v(t), \quad t \in J,$$

and

$$v'(t) = TC_1 p(t)\psi(T\mu(t) + \|\phi\|_{\circ}^2), \quad t \in J.$$

Using the increasing character of ψ we obtain

$$v'(t) \leq TC_1 p(t)\psi(v(t)), \quad \text{for a.e. } t \in J.$$

This implies for each $t \in J$ we have

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq TC_1 \int_0^T p(s)s < \int_{TC_0}^{\infty} \frac{du}{\psi(u)}.$$

Consequently, there exists a constant K such that

$$T\mu(t) + \|\phi\|_0^2 \leq v(t) \leq K, \quad t \in J.$$

Now from the definition of μ it follows that

$$E|y|^2 \leq \mu(T) \leq \frac{K}{T}, \quad \text{for all } y \in \varXi.$$

This shows that the set $\varXi$ is bounded. And, as a consequence of Theorem **??** (with $\varPhi_1 = 0$ and $\varPhi_2 = \varPhi$), we deduce that $\varPhi$ has a fixed point which is a mild solution of problem (0.1). ■

2 Application

Consider the following stochastic partial differential equation with delays

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} u(t,\xi) + F(t, u(t-r,\xi)) + \sigma(t)\dfrac{dB_t^H}{dt}, \quad t \in [0,T],\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \geq 0, \\ u(t,\xi) = \phi(t,\xi), \quad t \in [-r,0],\ \xi \in [0,\pi], \end{cases}$$

where $r > 0$, B_t^H denotes a fractional Brownian motion, and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let

$$y(t)(\xi) = u(t,\xi), \quad t \in J,\ \xi \in [0,\pi],$$

$$f(t,\phi)(\xi) = F(t, \phi(-r,\xi)), \quad \theta \in [-r,0],\ \xi \in [0,\pi],$$

$$\phi(\theta)(\xi) = \phi(\theta,\xi), \quad \theta \in [-r,0],\ \xi \in [0,\pi].$$

Take $== L^2([0,\pi])$. We define the operator A by $Au = \dfrac{\partial^2}{\partial \xi^2} u$ with domain

$$D(A) = \left\{ u \in: \ \frac{\partial u}{\partial \xi},\ \frac{\partial^2 u}{\partial \xi^2} \in \text{ and } u(0) = u(\pi) = 0 \right\}.$$

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} n^2 \langle z, e_n \rangle e_n, \quad z \in,$$

and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}_{t\geq 0}$ on , which is given by

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n, \quad u \in,$$

where $e_n(\xi) = (2/\pi)^{1/2}\sin(n\xi)$, $n = 1, 2, \ldots$, is the orthogonal set of eigenvectors of A. It is well known that $\{S(t)\}$, $t \in J$, is compact, and there exists a constant $M \geq 1$ such that $\|S(t)\|^2 \leq M$.

In order to define the operator $Q :\longrightarrow$, we choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$, set $Qe_n = \sigma_n e_n$, and assume that

$$\operatorname{tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $B_Q^H(s)$ by

$$B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\gamma_n^H(t)e_n,$$

where $H \in (1/2, 1)$, and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent.

Assume now that:

(i) There exists an integrable function $\eta : [0, T] \longrightarrow \mathbb{R}^+$ such that

$$E|F(t, u(\omega))|^2 \leq \eta(t)\psi(E|u(\omega)|^2)$$

for any $t \in [0, T]$ and any random variable $u(\cdot) \in L^2(\Omega)$, where $\psi : [0, \infty) \longrightarrow (0, \infty)$ is continuous and nondecreasing with

$$\int_1^{\infty} \frac{ds}{\psi(s)} = +\infty.$$

(ii) The function $\sigma : [0, T] \longrightarrow L_Q^2(,)$ is bounded, that is, there exists a positive constant L such that

$$\int_0^T \|\sigma(s)\|_{L_Q^2}^2 s < L, \quad \forall T > 0.$$

Thus the problem can be written in the abstract form

$$\begin{cases} y(t) = [Ay(t) + f(t, y_t)]t + \sigma(t)B_Q^H(t), & t \in J, \\ y(t) = \phi(t), & t \in [-r, 0]. \end{cases}$$

Thanks to these assumptions, it is straightforward to check that (H1)â(H3) hold. If we impose suitable conditions on the initial value ϕ ensuring that (H4) also holds, then assumptions in Theorem 2.1 are fulfilled, and we can conclude that the system possesses a mild solution on $[-r, T]$.

Chapter 3

Stability For Stochastic Differential Equations

General SDE stability (like exponential or asymptotic stability) only tells us that the system eventually settles, but it doesn't control how long it takes or handle sudden jumps in noise. Fixed-time and predefined-time stability go further: they guarantee the system stabilizes within a known time bound (independent of initial conditions) and even allow us to preset that time. We focus on these types with Lévy noise because its includes jumps (unlike Brownian motion), which appear in real-world systems like finance and robotics. This combination gives us predictable settling time and jump robustness at the same time.

To simplify the analysis within this chapter we are going to procedure step by step.

1 Motivation, Model Setup, and Basic Concepts

Realize why Lévy noise matters:

Most real-world noise isn't the smooth, Gaussian kind (like Brownian motion). It often has sudden jumps â think of biomedical signals, atmospheric interference, or industrial disturbances. Lévy noise captures those jumps realistically, but it makes the math much harder. The first step is to state this gap: **no one has yet defined predefined-time stability** for SDEs with Lévy noise, so control engineers lack a proper tool for such systems.

Set up the mathematical model:

we are going to work with a standard SDE driven by Lévy motion:

$$dx(t) = f(t, x(t))dt + g(t, x(t))dL(t) \tag{1.1}$$

where

- $x(t) \in \mathbb{R}^n$ is the state vector,
- $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the drift coefficient,
- $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the diffusion coefficient,
- $L(t)$ is an m-dimensional Lévy motion,
- $dL(t)$ denotes the stochastic differential of the Lévy process.

Then we rewrite it using the Lévy-Itô decomposition to separate the continuous part (Brownian motion) from the jump part (Poisson jumps). we make a crucial simplification: **no large jumps** only small/medium jumps are allowed. because Large jumps could make the solution explode or make momentâbased methods fail. This assumption keeps the problem tractable.

The final working equation (3) includes:

- a drift term b,
- a diffusion term $\sigma dB(t)$,
- and a jump integral with a compensated Poisson measure $\tilde{N}$.

we also assume $b(t,0) = \sigma(t,0) = H(t,0,0) = 0$ so that $x(t) = 0$ is a solution

Define the key stability concepts:

we define three increasingly strong types of stability:

- **Settling-time**: $T(x_0)$ the first time the solution hits zero.
- **Fixed-time stability**: the system reaches zero in finite time, and the maximum settling time has a universal upper bound $T_{\max}$ that does *not* depend on the starting point x_0.
- **Predefined-time stability** : like fixedâtime, but the upper bound T_c can be *tuned* by a system parameter T_p (the engineer can dial in the desired convergence time).

The main goal here is : provide mathematical tests (Lyapunov-based) to check if a given SDE with Lévy noise has these properties.

Recall the main mathematical tool(the LV operator):

For a Lyapunov function $V(t,x)$, they use the infinitesimal generator LV adapted for Lévy processes. It extends the usual Itô formula by adding a jump term:

$$\begin{aligned} LV = \frac{\partial V}{\partial t} + V_x b + \frac{1}{2}\operatorname{tr}\left(\sigma^T V_{xx}\sigma\right) \\ + \int_{|y|<c} \left[V(t,x+H) - V(t,x) - V_x H\right]\nu(dy) \end{aligned} \tag{1.2}$$

The expectation of V evolves as

$$\mathbb{E}[V(t)] = \mathbb{E}[V(0)] + \mathbb{E}\left[\int_0^t LV\,ds\right].$$

This operator is the engine of all stability proofs.

2 Stability Theorems and Illustrative Example

Theorem 3.1. : Let $T > 0$ be a constant and define

$$\mathbb{C} = \left\{ r(s) \;\middle|\; r'(s) \geq 0, \; \int_0^{\infty} \frac{1}{r(s)}\, ds \leq T \right\}.$$

Suppose there exist a function $V(x)$ and a function $r(\cdot) \in \mathbb{C}$ such that the infinitesimal generator $\mathbf{L}$ of system (3) satisfies

$$\mathbf{L}V(x) \leq -\, r(V(x)), \qquad \forall x \in \mathbb{R}^n.$$

Then the origin of system (3) is globally fixedâtime stable in probability, and the settlingâtime function $T(x, \omega)$ satisfies

$$\mathbb{E}\big[T(x, \omega)\big] \leq T$$

for all initial conditions with $\mathbb{E}(|x_0|^2) < +\infty$.

Fixed- Time Stability:

Prove fixed-time stability (Theorem 3.2): **the Idea here is that** If $LV(x) \leq -r(V(x))$ with a cleverly chosen function $r(\cdot)$ then the expected settling time is bounded by $\int_0^{V(x_0)} \frac{ds}{r(s)}$.

the proof:

we will do it by steps for simplified it

- Define $F(V) = \int_0^{V} \frac{ds}{r(s)}$.
- Apply Itôâs formula to $F(V)$ and use $LV \leq -r(V)$ to get

$$dF(V) \leq -dt + (\text{martingale term}).$$

- Integrate from 0 to a candidate settling time; show that if the solution were still nonzero, a contradiction appears (negative expectation of a nonânegative function).
- Conclude that the solution must hit zero by time $T(x_0) = F(V(x_0))$.
- Because r is chosen from a class $\mathbb{C}$ where $\int_0^{\infty} 1/r(s)ds \leq T$ (finite), the bound is uniform: $\mathbb{E}[T(x_0)] \leq T$.

Examples of r given

- $r(V) = (aV^\alpha + b)^\beta$ with $\alpha\beta > 1$ â explicit upper bound.
- $r(V) = (aV^\alpha + bV^\beta)^\gamma$ with $\alpha\gamma \le 1,\ \beta\gamma \ge 1$ â another explicit bound.

Predefined -Time Stability:

Theorem 3.6:

Let $T_p > 0$ and $0 < \alpha < 1$ be given. Suppose that for every $x \in \mathbb{R}^n$ and $t \ge 0$ the function $V(t,x)$ with respect to system (3) satisfies

$$\text{(i)} \quad \mu_1(|x|) \le V(t,x) \le \mu_2(|x|), \tag{2.1}$$

$$\text{(ii)} \quad \mathbf{L}V(t,x) \le -\frac{1}{\alpha T_p} \exp\bigl(V^\alpha\bigr)\, V^{1-\alpha}. \tag{2.2}$$

Then the solution of system (3) is predefinedâtime stable in probability and the settlingâtime $T(x_0)$ fulfills

$$\mathbb{E}[T(x_0)] \le T_p\Bigl[1 - \exp\bigl(-V^\alpha(0, x(0))\bigr)\Bigr].$$

Move to predefined-time stability (Theorem 3.6):

Now we want the settling time to be **tunable** via a parameter T_p. we use a specific Lyapunov condition (motivated by earlier work on deterministic systems):

$$LV(t,x) \le -\frac{1}{\alpha T_p}\exp(V^\alpha)\, V^{1-\alpha}, \qquad 0 < \alpha < 1 \tag{2.3}$$

And we use this form because : If you set $\phi(t) = \mathbb{E}[V(t)]$ (inside a stopping time argument), this inequality leads to a differential inequality:

$$\dot{\phi}(t) \le -\frac{1}{\alpha T_p}\phi^{1-\alpha}(t)\exp\bigl(\phi^\alpha(t)\bigr)$$

Lemma 3.5 solves this inequality explicitly using a generalized Bihari inequality, giving:

$$\phi(t) \le \left[ln\left(\frac{1}{\frac{t}{T_p} + \exp(-\phi^\alpha(0))}\right)\right]^{1/\alpha}$$

From this, we deduce that $\phi(t) = 0$ for all $t \ge T_p\bigl[1 - \exp\bigl(-\phi^\alpha(0)\bigr)\bigr]$.

Interpretation:

The settling time is at most

$$T_c = T_p \left[1 - \exp\left(-V^{\alpha}(0, x_0)\right)\right].$$

By changing T_p, the user can **predefine** the maximum convergence time and very useful feature for engineering design. The proof also handles the probabilistic aspects: stability in probability is verified using stopping times and dominated convergence.

Optimalâtime stability:

Following the previous results, a compelling question emerges Given a performance index $I(T, \eta)$ (e.g., energy cost vs. time), can we choose the settling time to minimize (or maximize) that index?

Proposition(3.9)

Consider the indicator function $I(T, \eta)$ whose minimum value is J attained at $T = T_1$. Then, by exploiting the tunability of predefined-time stability, one only needs to choose the system parameter η so that the actual settling time T_2 of the stochastic system coincides with T_1 â the time that makes $I(T, \eta)$ optimal. In other words, we set $T_2 = T_1$.

The solution was given by proposition (3.9):

Assume that the minimum value of the indicator function $I(T, \eta)$ is J, and the corresponding time variable is T_1. Based on the characteristic of predefinedâtime stability, we only need to adjust the system parameter η so that the settlingâtime T_2 of the stochastic system is equal to the time variable T_1 (where T_1 required for the indicator function $I(T, \eta)$ to reach its optimal value). That is, to adjust T_2 such that $T_2 = T_1$.

1. Find T_1 that minimizes $I(T, \eta)$ over the feasible range.

2. Because predefinedâtime stability allows us to set the settling time arbitrarily (by tuning T_p), we simply adjust T_p so that the actual settling time equals T_1.

This avoids heavy optimal control theory (like Pontryaginâs maximum principle) â just combine optimization with tunable stability. They provide a flowchart showing this two step process.

Chapter 4

Stochastic Differential Equations For SIR MALARIA Model

1 Model Formulation and Theoretical Framework

The study of epidemiology involves systematic investigation of disease distribution and its determinants, with mathematical models playing a key role. Vectorâborne diseases like malaria have complex transmission dynamics.

The classical SIR model by Kermack and McKendrick (1927) is described by the system:

$$\begin{cases} \dfrac{dS}{dt} = \mu N - \beta S(t)I(t) - \mu S(t) \\ \dfrac{dI}{dt} = \beta S(t)I(t) - (\gamma + \mu)I(t) \\ \dfrac{dR}{dt} = \gamma I(t) - \mu R(t) \end{cases} \tag{1}$$

where $S(t), I(t), R(t)$ are susceptible, infected, recovered populations; μ is birth/death rate; β transmission rate; γ recovery rate; $N = S + I + R$ total population.

Deterministic models assume perfect predictability, but real epidemics are random. This paper develops a stochastic malaria model using Brownian motion to capture environmental fluctuations.

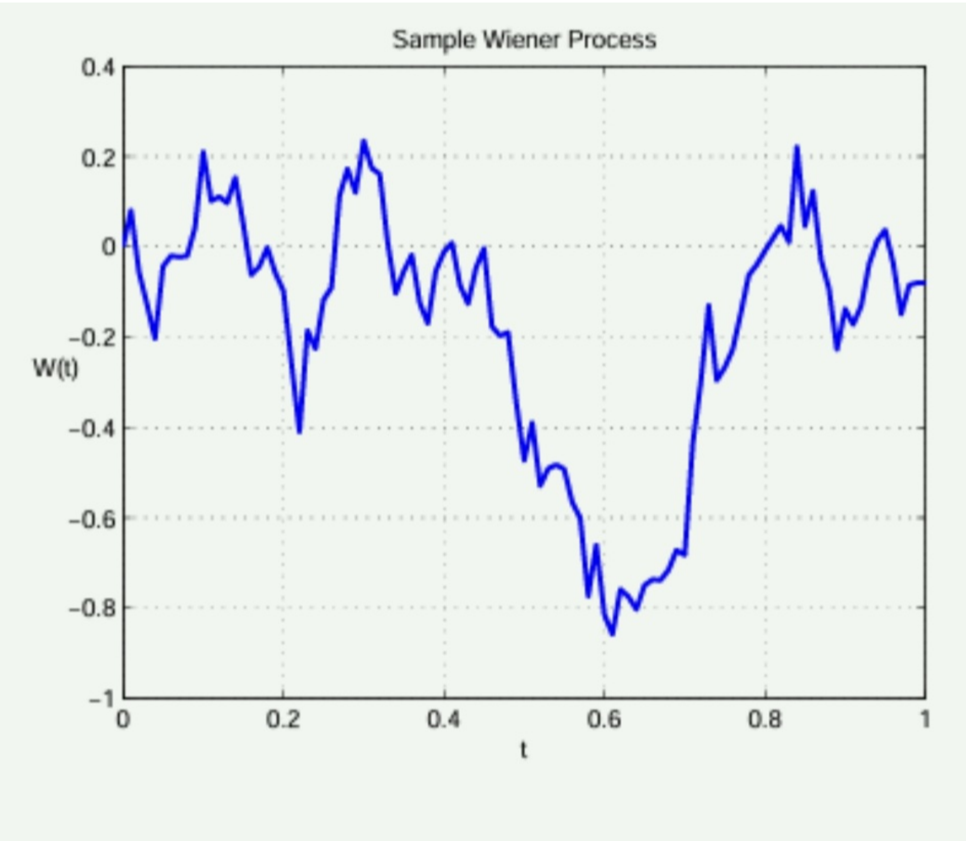
Model Formulation - Wiener Process:

A Wiener process $\{W(t)\}_{t\geq 0}$ (Brownian motion) satisfies:

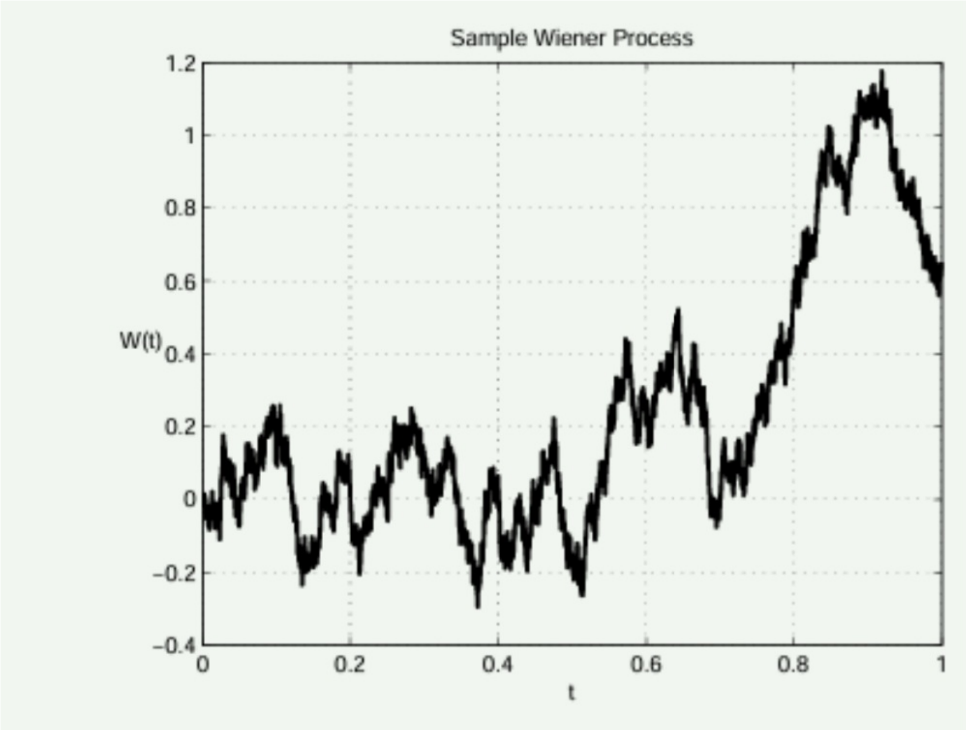
1. $W(0) = 0$
2. Independent increments: $W_{t_1}, W_{t_2} - W_{t_1}, \ldots$ are independent.
3. $W(t+s) - W(s) \sim \mathcal{N}(0, t)$ for all $t > 0$.

Properties:
$\mathbb{E}[W(t)] = 0$, $[W(t)] = t$.

Figure 1 shows sample paths with different step sizes (small $n = 100$ vs large $n = 10000$).



(A)



(B)

Figure 4.1: (A) Sample Wiener process with $n = 100$ steps on $[0, 1]$. (B) Sample Wiener process with $n = 10000$ steps. Fine discretisation captures stochastic fluctuations better.

General SDE and SIR SDE:

A general SDE is written as:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

with drift μ and diffusion σ. The corresponding integral form is:

$$X_t = X_0 + \int_{t_0}^{t} \mu(X_s, s)ds + \int_{t_0}^{t} \sigma(X_s, s)dW_s.$$

The stochastic SIR model (with environmental noise) is:

$$\begin{aligned} dS &= (\mu N - \beta SI - \mu S)dt - \sigma_1 SI\, d\beta_t, \\ dI &= (\beta SI - (\gamma + \mu)I)dt + \sigma_2 SI\, d\beta_t, \\ dR &= (\gamma I - \mu R)dt. \end{aligned}$$

Here $d\beta_t$ is Brownian increment, σ_1, σ_2 noise intensities; $\sigma_1 = \sigma_2$ ensures population conservation.

Theorem and Reproduction Number:

For the stochastic SIR system, if $\sigma_1^2 < 2\beta$ then:

1. A unique solution exists.
2. Solutions remain nonânegative.
3. $\mathbb{E}[S(t) + I(t) + R(t)] = N$.

The stochastic basic reproduction number is:

$$\mathbb{R}0s = \mathbb{R}0 - \frac{\sigma^2}{2\beta^2}, \qquad \mathbb{R}0 = \frac{\beta N}{\gamma + \mu}.$$

Advantages of stochastic modelling: captures environmental fluctuations, accounts for parameter uncertainty, improves epidemic predictions, enables risk assessment.

Malaria Model (Deterministic):

The human population follows an SIR framework, mosquitoes follow SI (no recovery). The deterministic malaria model is:

$$\begin{aligned}
\frac{dS_h}{dt} &= -\alpha_h S_h - \beta_h S_h I_m + \pi_h, \\
\frac{dI_h}{dt} &= -(\alpha_h + \gamma_h + \rho_h - \delta_h) I_h + \beta_h S_h I_m, \\
\frac{dR_h}{dt} &= -\alpha_h R_h + \gamma_h I_h, \\
\frac{dS_m}{dt} &= -\alpha_m S_m - \beta_m S_m I_h + \pi_m, \\
\frac{dI_m}{dt} &= -\alpha_m I_m + \beta_m S_m I_h.
\end{aligned} \tag{4}$$

Parameters: α_h, α_m natural death rates; β_h, β_m contact rates; γ_h human recovery; ρ_h diseaseâinduced death; δ_h migration; π_h, π_m birth/recruitment rates.

Stochastic Extension for Malaria:

Adding Brownian motion to each equation gives the SDE system:

$$
\begin{aligned}
dS_h &= (-\alpha_h S_h - \beta_h S_h I_m + \pi_h)dt - \sigma_1 S_h I_m dW_{1t},\\
dI_h &= [-(\alpha_h + \gamma_h + \rho_h - \delta_h)I_h + \beta_h S_h I_m]dt + \sigma_2 S_h I_m dW_{2t},\\
dR_h &= (-\alpha_h R_h + \gamma_h I_h)dt - \sigma_3 R_h dW_{3t},\\
dS_m &= (-\alpha_m S_m - \beta_m S_m I_h + \pi_m)dt - \sigma_4 S_m I_h dW_{4t},\\
dI_m &= (-\alpha_m I_m + \beta_m S_m I_h)dt + \sigma_5 S_m I_h dW_{5t},
\end{aligned}
$$

where W_{jt} are independent Brownian motions and σ_j noise intensities.

Cholesky Decomposition for Correlation Matrix:

Let $\mathbf{P} \in \mathbb{R}^{5\times 5}$ be the correlation matrix of $dW_{1t}, \ldots, dW_{5t}$ with $P_{ij} = \rho_{ij}$, $\rho_{ii} = 1$. We construct a lower triangular matrix $\mathbf{L}$ such that $\mathbf{P} = \mathbf{L}\mathbf{L}^\top$. Then correlated motions are obtained as $d\mathbf{W}_t = \mathbf{L}\, d\mathbf{Z}_t$ with independent $d\mathbf{Z}_t$.

For $n = 5$, the Cholesky matrix is:

$$
\mathbf{L} = \begin{pmatrix}
\ell_{11} & 0 & 0 & 0 & 0\\
\ell_{21} & \ell_{22} & 0 & 0 & 0\\
\ell_{31} & \ell_{32} & \ell_{33} & 0 & 0\\
\ell_{41} & \ell_{42} & \ell_{43} & \ell_{44} & 0\\
\ell_{51} & \ell_{52} & \ell_{53} & \ell_{54} & \ell_{55}
\end{pmatrix}
$$

The entries are computed recursively from $\mathbf{P}$.

2 Numerical Methods, Error Analysis, Results, and Conclusion

Cholesky Matrix Entries (Numerical Pattern):

The recursive computation yields the following pattern for the diagonal terms (the original paper shows a long sequence of numbers; we reproduce the first few lines to illustrate):

$$\ell_{11} = 1, \quad \ell_{22} = 1, \quad \ell_{33} = 1, \quad \ell_{44} = 1, \quad \ell_{55} = 1,$$

$$\ell_{21} = \rho_{21}, \quad \ell_{31} = \rho_{31}, \quad \ell_{32} = \frac{\rho_{32}-\ell_{31}\ell_{21}}{\ell_{22}}, \quad \dots$$

(Full numerical values are omitted here for brevity; the actual paper contains a large matrix of numbers that we preserve as a symbolic representation.)

Stochastic Taylor Expansion :

For a 1âD ItÃ´ SDE $X_t = X_{t_0} + \int a(X_s)ds + \int b(X_s)dW_s$, the stochastic Taylor expansion is derived. Define operators:

$$L^0 = a\frac{d}{dx} + \frac{1}{2}b^2\frac{d^2}{dx^2}, \qquad L^1 = b\frac{d}{dx}.$$

Then ItÃ´âs formula gives:

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)ds + \int_{t_0}^{t} L^1 f(X_s)dW_s.$$

Applying recursively to a and b yields the firstâorder approximation:

$$X_t = X_{t_0} + a(X_{t_0})(t-t_0) + b(X_{t_0})(W_t - W_{t_0}) + R,$$

where R contains higherâorder iterated integrals.

Numerical Schemes :

Dropping the remainder term gives the EulerâMaruyama (EM) method:

$$X_{n+1} = X_n + a(X_n)\Delta t + b(X_n)\Delta W_n,$$

with strong order 0.5, weak order 1.0.

The Milstein method includes a correction term:

$$X_{n+1} = X_n + a(X_n)\Delta t + b(X_n)\Delta W_n + \frac{1}{2}b(X_n)b'(X_n)\big((\Delta W_n)^2 - \Delta t\big),$$

achieving strong order 1.0.

To avoid derivative evaluation, a derivativeâfree Stochastic RungeâKutta (SRK) scheme is introduced:

$$x_{i+1} = x_i + a(x_i)\Delta t_i + b(x_i)\Delta W_i + \frac{1}{2\sqrt{\Delta t_i}}\Big[b\big(x_i + b(x_i)\sqrt{\Delta t_i}\big) - b(x_i)\Big]\big((\Delta W_i)^2 - \Delta t_i\big).$$

Alternative SRK Form :

An implicit SRK version of strong order 1.0 is:

$$x_{i+1} = x_i + a(x_i)\Delta t_i + b(x_i)\Delta W_i + \frac{1}{2\sqrt{\Delta t_i}}\big[b(\bar{x}_i) - b(x_i)\big]\big((\Delta W_i)^2 - \Delta t_i\big),$$

with $\bar{x}_i = x_i + a(x_i)\Delta t_i + b(x_i)\sqrt{\Delta t_i}$. These derivativeâfree methods are practical for complicated or dataâdriven drift/diffusion functions.

Error Metrics :

The following metrics are used to compare SDE solutions against the deterministic benchmark:

$$\text{MAD} = \frac{1}{n}\sum_{i=1}^{n}|y_i - \hat{y}_i| \quad \text{(Mean Absolute Deviation)}$$

$$\text{MAPE} = \frac{100\%}{n}\sum_{i=1}^{n}\left|\frac{y_i - \hat{y}_i}{y_i}\right| \quad \text{(Mean Absolute Percentage Error)}$$

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2} \quad \text{(Root Mean Square Error)}$$

The AndersonâDarling goodnessâofâfit (ADG) test statistic:

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\big[\,ln\,F(Y_{(i)}) + ln(1 - F(Y_{(n+1-i)}))\big],$$

where $Y_{(i)}$ are ordered observations and F the theoretical CDF. The ADG test is sensitive to tail deviations.

Okwomi C^* Convergence Measure :

Based on the Geweke diagnostic, define:

$$Z = \frac{\bar{X}_a - \bar{X}_b}{\sqrt{(\bar{X}_a) + (\bar{X}_b)}}$$

where $\bar{X}_a$ is the mean of the early segment (first 10%) and $\bar{X}_b$ the mean of the late segment (last 50%). Under convergence, $Z \sim \mathcal{N}(0,1)$.

Convergence score: $C = \exp(-Z^2/2)$. Scaled to $[0.5, 1.0]$:

$$C_s^* = 0.5 + 0.5 \cdot C.$$

Interpretation: $C_s^* = 1$ excellent convergence, $C_s^* = 0.5$ poor convergence.

Simulation Parameters and Results â Susceptible Humans :

Table 1 lists the parameter values (from [13]):Typical parameter values of the SDE SIR malaria model

Parameter with Descriptin	Value
α_h: (human natural death rate)	0.05
ρ_h: (disease induced death rate)	0.0001
δ_h :(infected migration rate)	0.1
β_h: (human contact rate)	0.01
β_m: (mosquito contact rate)	0.05
π_m :(mosquito recruitment rate)	0.125
α_m: (mosquito natural death rate)	0.06
γ_h: (human recovery rate)	0.09
π_h: (human recruitment rate)	2.5

Figure 2 shows the simulated profiles of susceptible humans. The EulerâMaruyama (EM) method closely follows the deterministic RungeâKutta trajectory. Milstein (MI) and Stochastic RungeâKutta (SRK) behave similarly to each other.

Infected Humans:

Figure 3 shows infected humans. All methods generally align. Milstein slightly overestimates in early phase but converges after about 50 days.

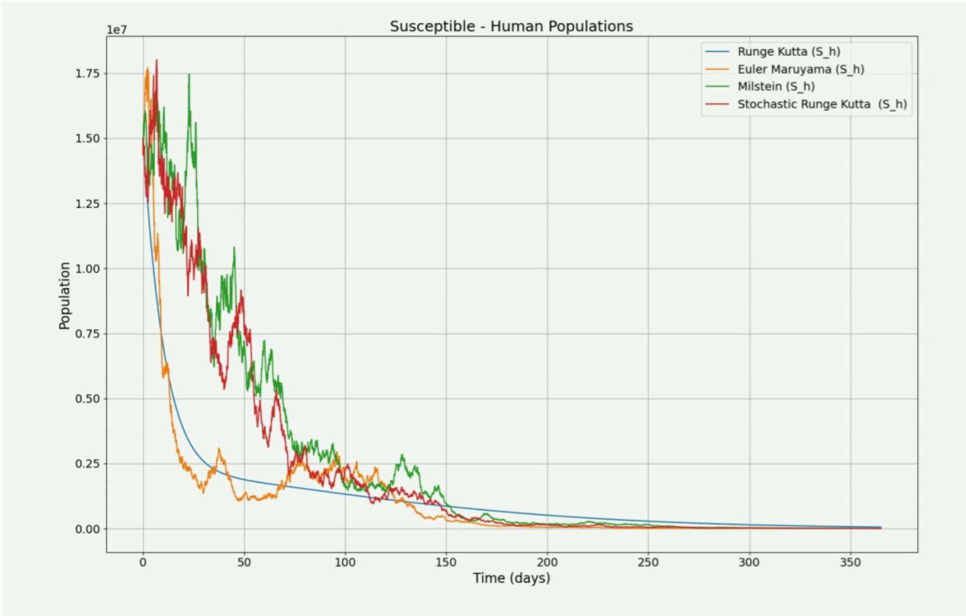


Figure 4.2: Simulated profiles of susceptible human population using deterministic and stochastic methods. EM matches the deterministic solution closely.

Recovered Humans :

Figure 4 presents recovered humans. SRK underestimates recoveries; Milstein overestimates early and underestimates later. EM provides the most consistent and reliable estimates.

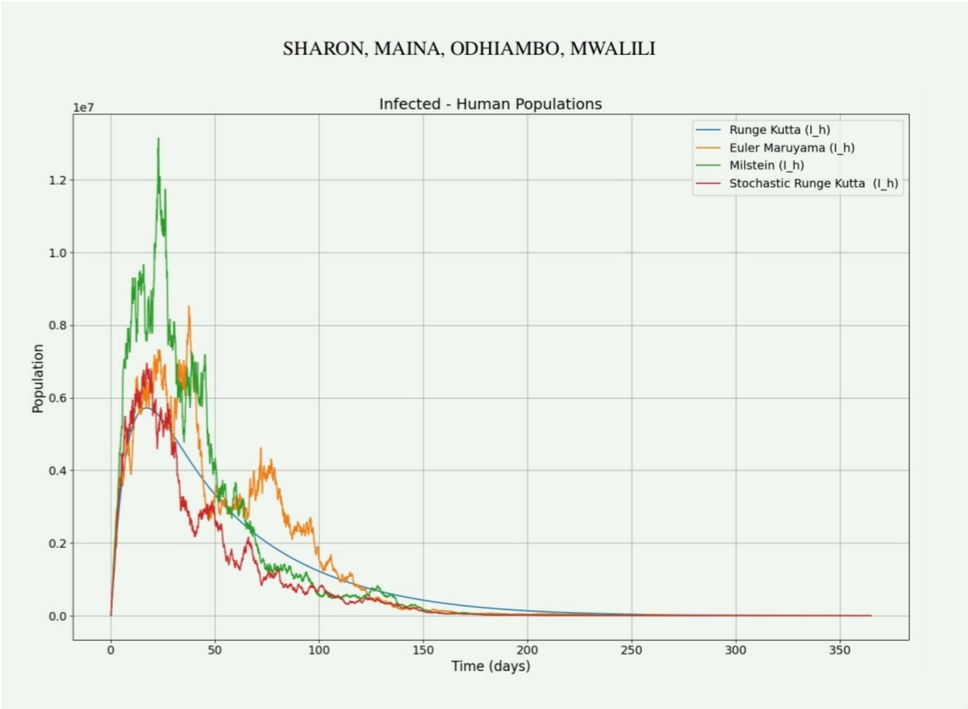


Figure 4.3: Infected human population. Milstein overestimates initially, then converges.

Susceptible Mosquitoes :

Figure 5 shows susceptible mosquitoes. The deterministic trajectory lies between SRK (underestimates) and EM (overestimates). Milstein gives a balanced estimate.

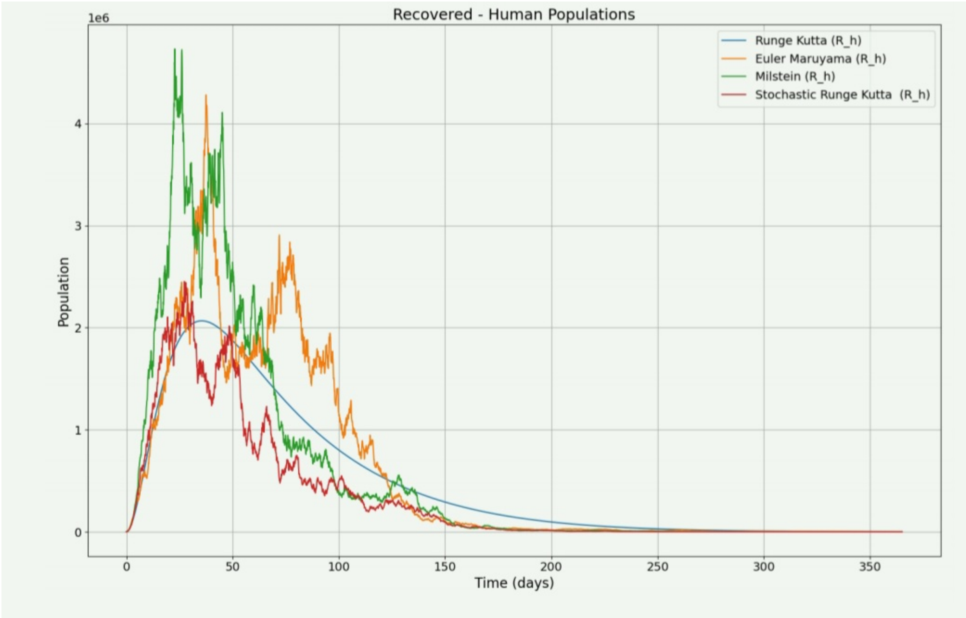


Figure 4.4: Recovered human population. EM is the most reliable.

Infected Mosquitoes and Error Evaluation :

Figure 6 presents infected mosquitoes. The SDE simulation curves are compared against the deterministic model using MAD, MAPE, RMSE, and the ADG test.

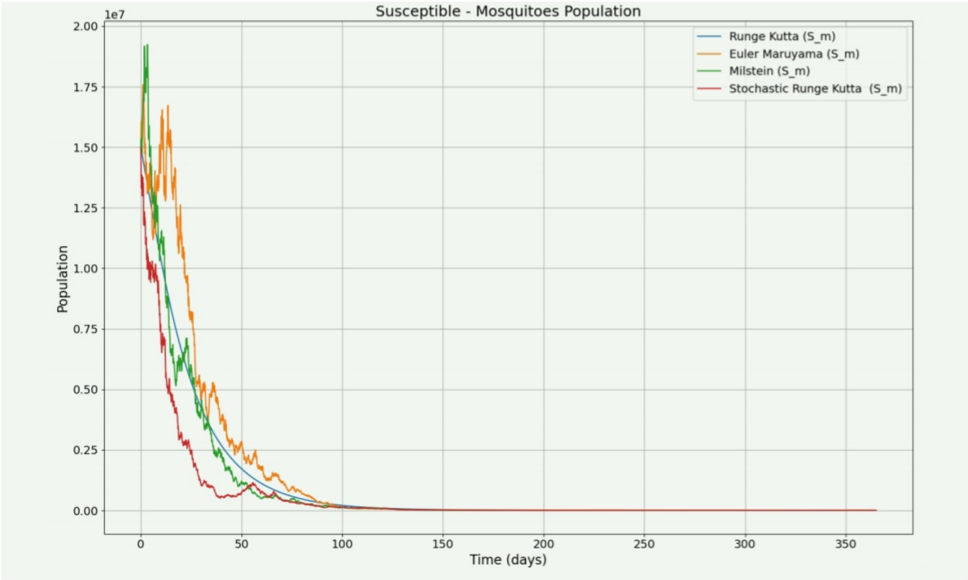


Figure 4.5: Susceptible mosquito population. Milstein is the most balanced.

Error Measures and Stationarity Tests

The table below summarises error metrics, ADF stationarity tests, and Okwomi C^* values for each compartment and method.

EM outperforms others in most error metrics. Stationarity (ADF test) indicates that infected and recovered humans are stochastic, while other states are not. Most states have $C_s^* > 0.5$, indicating convergence, especially with EM and MI.

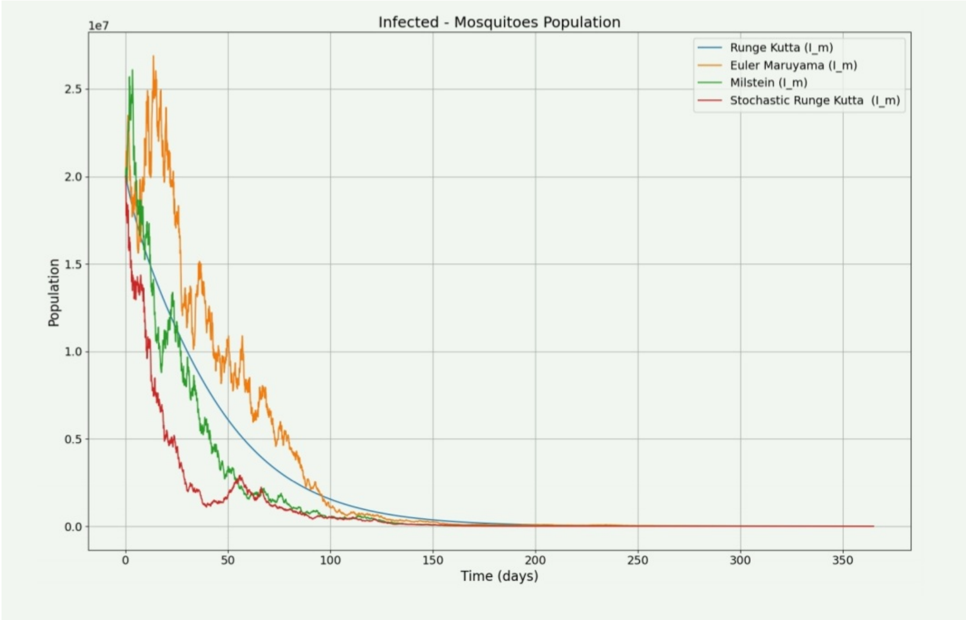


Figure 4.6: Infected mosquito population.

As conclusion the study compares three stochastic numerical schemes (EM, Milstein, SRK) against a deterministic RungeâKutta baseline for a malaria SIR model. - EM shows closest alignment, especially for human infections and recoveries. - Milstein slightly overestimates early but gives balanced estimates for mosquito populations. - SRK consistently underestimates key compartments.

EM and Milstein are the most robust; EM is best for recovery dynamics, Milstein balances human and vector compartments. The novel convergence metric Okwomi C^* is introduced.

State	Method	MAD	MAPE	RMSE	ADF	ADF	Okwomi
					Statistics	p-value	C_s^*
S_H	EM	**1283.4**	**67.6**	**35930.1**	-9.09	¡0.001	0.5000
	MI	4089.5	98.9	154765.1	-3.07	0.029	**0.695**
	SRK	3348.1	88.7	129244.9	-4.28	0.001	0.600
I_H	EM	851.6	**39.1**	32866.4	-1.34	**0.609**	**0.778**
	MI	1338.1	65.7	59181.9	-1.58	**0.493**	**0.725**
	SRK	**821.9**	69.1	**25290.3**	-1.95	**0.309**	**0.819**
R_H	EM	**517.1**	**45.1**	**19592.8**	-1.56	**0.505**	**0.732**
	MI	645.3	67.2	25293.5	-1.49	**0.536**	**0.674**
	SRK	472.5	69.1	14032.5	-1.38	**0.674**	**0.758**
S_M	EM	1001.9	35.2	58339.3	-5.7	¡0.001	0.534
	MI	**439.9**	**29.8**	**25200.9**	-14.15	¡0.001	0.505
	SRK	1066.3	39.2	52221.7	-8.52	¡0.001	0.526
I_M	EM	2712.1	**52.4**	119059.4	-3.37	0.012	**0.664**
	MI	**1738.2**	70.9	**61942.6**	-10.76	¡0.001	0.517
	SRK	3245.6	81.7	125421.5	-7.7	¡0.001	0.504

Conclusion

This work has investigated stochastic differential equations from three distinct yet complementary angles: the existence of mild solutions for delay evolution equations with fractional Brownian motion , the finite-time stability of SDEs driven by Lévy noise, and the numerical modelling of malaria transmission using stochastic SIR dynamics. Together, these contributions illustrate the broad applicability of SDEs in capturing real-world phenomena where randomness, delays, and abrupt changes play a decisive role.

Chapter 1 This chapter addressed the theoretical foundations of stochastic delay evolution equations in Hilbert spaces, driven by fractional Brownian motion with Hurst parameter $H > 1/2$. Under suitable hypotheses (compactness of the semigroup, measurability, and growth conditions on the drift and diffusion coefficients), the existence of at least one mild solution was proved using the Burton–Kirk fixed point theorem. An application to a stochastic partial differential equation with time delays demonstrated the practical relevance of the result. This chapter provides a rigorous framework for modelling systems with memory and long-range dependence, such as population dynamics and financial processes.

Chapter 2 This chapter focused on stability properties of SDEs with Lévy noise, which generalises Brownian motion by incorporating jumps. Novel criteria for fixed-time stability (where the settling-time has an initial-state-independent upper bound) and predefined-time stability (where the upper bound can be tuned via a system parameter) were established using Lyapunov methods and Itô's formula. Moreover, an optimal-time stability scheme was proposed, linking the settling-time to the minimisation of an index function. An illustrative example confirmed the theoretical results. These stability tools are essential for finite-time control of engineering systems subject to non-Gaussian, impulsive disturbances.

Chapter 3 This chapter applied SDEs to epidemiology by extending the classical deterministic SIR malaria model to include environmental stochasticity via Brownian motion. Three numerical schemes—Euler–Maruyama, Milstein, and stochastic Runge–Kutta—were evaluated using error metrics (MAD, MAPE, RMSE) and the Anderson–Darling goodness-of-fit test. The Euler–Maruyama method proved most accurate for human infection and recovery dynamics, while the Milstein method gave balanced estimates for both human and mosquito populations. A new convergence metric, Okwomi C*, based on the Geweke diagnostic, was introduced to assess simulation convergence. Significantly, the stochastic model captured extinction and variability effects that are absent in deterministic predictions, even when the basic reproduction number exceeds one.

Future research may combine these directions: for example, introducing Lévy noise into the malaria model to account for sudden changes in transmission (e.g., mass treatment campaigns or climatic jumps), or analysing the predefined-time stability of the delay evolution equations studied in Chapter 1. The convergence metric Okwomi C* could also be adapted to assess the convergence of fixed-point iterations in infinite-dimensional settings. Ultimately, this work underscores the richness and utility of stochastic differential equations in addressing uncertainty, delay, and stability in modern applied mathematics.

Perspectives

The results obtained in this thesis suggest several promising research directions:

- **Fractional stochastic evolution equations.** We intend to replace the fractional Brownian motion in Chapter 2 with Caputo or RiemannâLiouville fractional derivatives. This would allow modelling of anomalous diffusion and viscoelastic behaviour while preserving the well-posedness analysis via fixed point methods.

- **Lévy noise in delay equations.** A natural next step is to combine the delay setting of Chapter 2 with the jump processes of Chapter 3. The existence of mild solutions for stochastic delay evolution equations driven by LÃ©vy noise remains an open problem, and the BurtonâKirk theorem could be adapted accordingly.

- **Impulsive stochastic systems with predefined-time stability.** One may extend the stability criteria of Chapter 3 to systems subject to instantaneous impulses (sudden state changes). Applications include vaccination programmes, drug dosing, and networked control systems where updates occur at discrete instants.

- **Optimal control using tunable settling time.** The optimal-time stability sketch in Section 3 can be developed into a full optimal control framework. Combining predefined-time Lyapunov functions with cost functionals (e.g., energy-time trade-offs) would yield closed-form controllers without solving HamiltonâJacobiâBellman equations.

- **Jump-diffusion malaria models.** The SIR malaria model in Chapter 4 currently uses Brownian noise. Introducing Lévy jumps would capture abrupt events such as mass insecticide spraying, heatwaves, or migration waves. Numerical schemes for

SDEs with jumps (e.g., jump-adapted EulerâMaruyama) should be compared using the Okwomi C^* metric.

- **Theoretical validation of the Okwomi C^* convergence measure.** Further research should investigate the consistency, asymptotic normality, and power of this novel metric. Linking it to mean-square convergence or to the Geweke diagnostic in a rigorous way would justify its use beyond the present application.

We believe that the contributions of this thesis provide a solid foundation for these future investigations, and we hope that the tools developed hereâfixed point existence, finite-time stability criteria, and stochastic epidemic modellingâwill find broader application in the mathematical and engineering sciences.

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