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Thème :

System of Difference Equations

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Dedication

This study is dedicated to my beloved mother, who has been my source of inspiration, guidance, and strength when I thought of giving up. She continually provided her moral, spiritual, emotional, and financial support.

To my friends, who became my supporters and helped me with any problems I faced. They encouraged me to complete this important research on time and sent me inspirational messages whenever I needed them.

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Lastly, I dedicate this work to my Mighty God, who guides me, gives me strength, sharpens my mind, and provides protection and skills. All of this, I offer to You.

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Bendjelloul kamel

Abstract

The objective of this thesis is to establish the existence, uniqueness, and stability of solutions, as well as to model the results for a system of difference equations in a Banach space. Our results are recent and are based on fixed point theorems.

Key words and phrases:

Existence of solutions, uniqueness, stability, fixed point, difference equations, models.

Résumé

L'objectif de cette thèse est d'établir l'existence, l'unicité et la stabilité des solutions, ainsi que de modéliser les résultats pour un système d'équations aux différences dans un espace de Banach. Nos résultats sont récents et basés sur des théorèmes du point fixe.

Mots-clés et expressions:

Existence des solutions, unicité, stabilité, point fixe, équations aux différences, modèles.

ملخص

تهدف هذه الأطروحة إلى إثبات وجود و وحدانية واستقرار الحلول، بالإضافة إلى نمذجة النتائج لجمل من معادلات الفروق في فضاء بناخ. وتستند نتائجنا الحديثة على نظريات النقطة الصامدة.

الكلمات والعبارات المفتاحية

وجود الحلول، الوحدانية، الاستقرار، النقطة الثابتة، معادلات الفروق، النماذج.

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Introduction

We know that difference equations have been a major branch of pure and applied mathematics since their inauguration in the mid 17th century. While their history has been well studied, it is a vital field of on-going investigation, with the emergence of new connections with other parts of mathematics, fertile interplay with applied subjects, interesting reformulation of basic problems and theory in various periods, new vistas in the 20th century, and so on in this meeting we considered some of the principal parts of this story, from the launch with Newton and Leibniz up to around 1950[18].

Usually, difference equations began with Leibniz, the Bernoulli brothers and others from the 1680s, not long after Newton's fluxional equations in the 1670s.We can say applications
were made largely to geometry and mechanics; isoperimetrical problems were exercises in optimisation.Most 18th-century developments consolidated the Leibnizian tradition, extending its multi-variate form, thus leading to partial difference equations. Generalisation of isoperimetrical problems led to the calculus of variations. New figures appeared, especially Euler, Daniel Bernoulli, Lagrange and Laplace[2]. Development of the general theory of solutions included singular ones, functional solutions and those by infinite series. For example, many applications were made to mechanics, especially to astronomy and continuous media.

In the 19th century: general theory was enriched by development of the understanding of general and particular solutions, and of existence theorems. More types of equation and their solutions appeared; for example, Fourier analysis and special functions. Among new figures, Cauchy stands out. Applications were now made not only to classical mechanics but also to heat theory, optics, electricity and magnetism, especially with the impact of Maxwell. Later Poincaré introduced recurrence theorems[21],initially in connection with the three-body problem. In the 20th century: general theory was influenced by the arrival of set theory in mathematical analysis; with consequences for theorisation, including further topological aspects. New applications were made to quantum mathematics, dynamical systems and relativity theory.

In brief ,difference equations are equations that contains one or more terms involving derivatives of one variable (dependent variable) with respect to another variable (independent variable) or we can say that these are equations involving derivatives of a function or functions. They have a remarkable ability to predict the world around us. They are used to describe exponential growth and decay, population growth of species or the change in investment return over time, bank interest, even in solving radioactive decay problems, continuous compound interest problems, flow problems, cooling and heating problems, orthogonal trajectories, and also in investigating problems involving fluid mechanics, circuit design, heat transfer, population or conservation biology, seismic waves. They are used in specific field such as, in the field of medicine, where difference equations are used for modelling cancer growth or the spread of disease.

In chemistry, they are used for modelling chemical reactions and to computer radioactive half-life. In economics, they are used to find optimum investments strategies. In physics, they are used to describe the motion of waves, pendulums or chaotic systems.

We find that many statements concerning the theory of linear differential equations are also valid for the corresponding difference equations. A well-known example is the famous Poincaré theorem on the asymptotic behavior of the solutions to difference equations which was published in 1885 (see Gelfand (1967) and van Strien (1978))[12].

Also, a feature of difference equations not shared by differential equations is that they can be characterized as recursive functions. Examples of their use include modeling
population changes from one season to another ,modeling the spread of disease,modeling various business phenomena, discrete simulations applications , or giving rise to the phenomena chaos .The key is that they are discrete ,recursive relations.

This work consists of four chapters and each chapter contains more sections.
They are arranged as follows:
In **Chapter** 1, we introduce definitions, theories, and notations preliminary facts that will be used through this work.
In **Chapter** 2 , we prove the existence of solutions for difference equations in Banach space.
In **Chapter** 3 , we solve the stability problem of equilibrium point using Lyapunov and linearization of systems.
In **Chapter** 4 , we introduce applications and models of difference equations with state-dependent delays.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

1.1 Definitions of Fixed Point

Definition 1.1.1. *[14] Let X be a metric space equipped with a distance d. A map $f : X \to X$ is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that:*

$$d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2), \forall x_1, x_2 \in X.$$

The smallest λ for which the above inequality holds is the Lipschitz constant of f.

(1) If $\lambda \leq 1$ f is said to be non-expansive.

(2) If $\lambda < 1$ f is said to be a contraction.

Definition 1.1.2. *[3] Let X be a metric space,$x \in X$ is called a fixed point of a mapping:*

$$A : X \to X \ if \ Ax = x.$$

Example 1.1.1. *We have*

$$j = [a, b] \subset R$$

and

$$f : j \to j.$$

Indeed:

$$a - f(a) \leq 0$$

and

$$b - f(b) \geq 0.$$

The intermediate value theorem ensures:

$$x - f(x) = 0$$

has one solution in j and therefore f has a fixed point.

Definition 1.1.3. *[13] Given a matrix T, let*

$$\rho(T) = \max |\delta_1|, |\delta_2|,$$

Where δ_1 and δ_2 are eigenvalues of the matrix T.Then

$$\|\bar{T}\| = \sqrt{\rho(T^T T)}$$

is called the spectral norm of T and is needed a matrix norm.

Definition 1.1.4. *[13] A fixed point x^* of a map $g : \mathbb{R}^2 \to \mathbb{R}^2$ is called :*

(1) Stable if given $\epsilon > 0$ $\exists$ $\varepsilon > 0$ such that $|x - x^*| < \varepsilon$ implies $|g^m(x) - x^*| < \epsilon$ $\forall m \in \mathbb{Z}^*$.

(2) Attracting if $\forall \eta > 0$ such that $|x - x^*| < \eta$ implies that $\lim_{n\to 0} g^n(x) = x^*$ if $\eta = \infty$ we call it globally attracting.

(3) Asymptotically stable if it is both stable and attracting.It is globally asymptotically stable if it is both stable and globally attracting.

(4) Unstable if it is not stable.

1.2 Generalized Metric and Banach Spaces

Theorem 1.2.1. *[5] Let (X, d_X) and (Y, d_Y) two metric spaces and X compact. Then a follows conditions are equivalents :*

1. *$A \subset C(X, Y)$ is relatively compact i.e $\bar{A}$ compact for the topology of uniform convergence.*

2. *A is equicontinue in each point x of X ; $A(x) := \{f(x), f \in A\}$ is relatively compact.*

Theorem 1.2.2. *([20]) Let (X, d) be a complete generalized metric space with $d : X \times X \to \mathbb{R}^n$ and let $N : X \to X$ be such that:*

$$d(N(x), N(y)) \leq Md(x, y)$$

for all $x, y \in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \to 0$ as $k \to \infty$, then N has a unique fixed point $x_* \in X$. For every $x_0 \in X$ and $k \geq 1$.

Theorem 1.2.3. *[11] Let $(E, \|.\|)$ be a generalized Banach space and $N : E \to E$ is a continuous compact mapping. Moreover assume that the set:*

$$A = \{x \in E : x = \lambda N(x), for some \lambda \in (0,1)\}$$

is bounded. Then N has a fixed point.

Theorem 1.2.4. *[16] Let $\phi : [0,T] \to \mathbb{R}$ be a nonnegative differentiable function for which there exists a constant C such that:*

$$\phi'(t) \leq C\phi(t), \forall t \in [0,T].$$

Then:

$$\phi(t) \leq \phi(0) \exp(\int_0^t C(\tau)d\tau)).$$

Theorem 1.2.5. *[10] Every generalized metric space is para compact.*

Theorem 1.2.6. *[11] Let X be a generalized Banach space, C be a nonempty compact convex subset of X,*

$$G : C \to P_{cp,cv}(C)$$

be an u.s.c. multivalued map, then the operator inclusion G has at leat one fixed point, that is there exists $x \in C$ such that $x \in G(x)$.

Theorem 1.2.7. *[15] Let E be a Banach space, D a nonempty closed bounded and convex subset of E, and $N : D \to D$ a completely continuous operator. Then N has at least one fixed point.*

Proof. Let $x_0 \in D$ .For $n = 2, 3, \dots,$ define:

$$N_n := (1 - \frac{1}{n})N + \frac{1}{n}x_0.$$

Since D is convex ,we see that $N_n : D \to D$ is a contraction. Therefore each N_n has a unique fixed point $x_n \in D$,

$$x_n = N_n(x_n) = (1 - \frac{1}{n})N(x_n) + \frac{1}{n}x_0.$$

Since $N(D)$ lies in a compact subset of D,there exist a subsequence S of integers and a $u \in D$ with

$$N(x_n) \to u, as\ n \to \infty\ in\ S.$$

Thus

$$x_n = (1 - \frac{1}{n})N(x_n) + \frac{1}{n}x_0 \to u\ as\ n \to \infty\ in\ S.$$

By continuity

$$N(x_n) \to N(u)\ asn \to \infty in\ S,$$

Therefore $u = N(u)$. □

Theorem 1.2.8. *([17]) Let $B(0,1)$ be the unit ball in a Banach space X. Then $\alpha(B(0,1)) = \chi(B(0,1)) = 0$ if X is finite dimensional,*
and $\alpha(B(0,1)) = 2$, $\chi(B(0,1)) = 1$ otherwise.

1.3 Multivalued Analysis

Lemma 1.3.1. *[8] Let (X, d) be a generalized metric space. Then there exists a homeomorphism map $h : X \to \bar{X}$.*

Proof. Consider $h : X \to \bar{X}$ defined by: $h(x) = (x, \dots, x)$ for all $x \in X$
Obviously that h is bijective . To prove that h is continuous map. let $x, y \in X$ .Thus

$$d_*(h(x), h(y)) \leq \sum_{i=1}^{n} d_i(x, y).$$

For $\epsilon > 0$ we need $\delta = (\frac{\epsilon}{n}, \dots, \frac{\epsilon}{n})$, let fixed $x_0 \in X$ and $B(x_0, \delta) = \{x \in X d(x_0, x) < \delta\}$,we have:

$$d_*(h(x_0), h(x)) \leq \epsilon$$

. Let $h^{-1} : \bar{X} \to X$ is a continuous map: $h^{-1}(x, \dots, x) = x, (x, \dots, x) \in \bar{X}$ Let $(x, \dots, x), (y, \dots, y) \in \bar{X}$,then

$$d(h^{-1}(x, \dots, x), h^{-1}(y, \dots, y)) = d(x, y)$$

. And $\epsilon = (\epsilon_1, \dots, \epsilon_n) > 0$ we take $\delta = \frac{\min_{0<i<n} \epsilon_i}{n}$ And we fix $(x_0, \dots, x_0) \in \bar{X}$.

$$B((x_0, \dots, x_0), \delta) = \{(x, \dots, x) \in \bar{X} : d_*((x_0, \dots, x_0), (x, \dots, x)) < \delta\}$$

. We have

$$d_*((x_0, \dots, x_0), (x, \dots, x)) < \delta \Rightarrow \sum_{i=1}^{n} d_i(x_0, x) < \frac{\min_{0<i<n} \epsilon_i}{n}$$

Then

$$d_i(x_0, x) < \frac{\min_{0<i<n} \epsilon_i}{n}, i = 1, \dots, n \Rightarrow d(x_0, x) < \epsilon.$$

Hence h^{-1} is continuous. $\square$

Lemma 1.3.2. *[17] A subset E of a metric space (X, d) is complete if for any cauchy sequence of points $\{x_n\}$ in E there exists $x \in E$ such that:*

$$\lim_{n\to\infty} d(x_n, x) = 0$$

.

Lemma 1.3.3. *[11] Let $(X, \|.\|)$ be a generalized Banach space with $C \subset X$ a closed and convex subset of X. Assume U is an open subset of C, with $0 \in U$, and let $G : U \to C$ is a compact map. Then either,*

(a) G has a fixed point in U, or

(b) There is a point $u \in \partial U$ and $\lambda \in (0, 1)$, with $u \in \lambda G(u)$.

Proof. Let $r : C \to U$ be the standard retraction .by the Schauder theorem the compact composite $roG : U \to U$ has a fixed point $x = rG(x)$. if $G(x) \in U$ then $x = rG(x) = G(x)$ so G has a fixed point; if $G(x)$ does not belong to U, $x = rG(x) = \frac{G(x)}{\|G(x)\|}$, so $x \in \partial U$ and $\lambda = \frac{1}{\|G(x)\|} < 1$. $\square$

1.4 Completness of Metric Space

Proposition 1.4.1. *[17] Let (X, d) be a metric space.the following statements are equivalent:*

(a) Every sequence of elements of X has a convergent subsequence in X .

(b) The space X is complete and for each $\epsilon > 0$ it admits a finite covering by open balls of radius ϵ.

Chapter 2

System of Difference Equations

2.1 Main Result

In this chapter, we study the existence of solutions for differential equations of the form:

$$\begin{cases} \triangle x(k) = f(k, x(k), y(k)), \ \text{k} \in \mathbb{N}(a,b), \\ \triangle y(k) = g(k, x(k), y(k)), \ \text{k} \in \mathbb{N}(a,b), \\ x(a) = x_0, \\ y(a) = y_0, \end{cases} \tag{2.1.1}$$

Where $\mathbb{N}(a,b) = \{a, a+1, \dots, b\}$, $f, g : \mathbb{N}(a,b) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are given functions .

Let us introduce the following hypothesis:

(H_1) There exist nonnegative numbers a_i and b_i for each $i \in \{1,2\}$.

$$\begin{cases} |f(k,x,y) - f(k,\bar{x},\bar{y})| \leq a_1|x-\bar{x}| + b_1|y-\bar{y}| \\ |g(k,x,y) - g(k,\bar{x},\bar{y})| \leq a_2|x-\bar{x}| + b_2|y-\bar{y}| \end{cases}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

For our main consideration of problem (2.1.1) , a Preov fixed point is used to investigate the existence and uniqueness of solutions for system of differential equations.

Theorem 2.1.1. *Assume that (H_1) is satisfied and the matrix:*

$$A = b\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

Such that $A \in \mathbb{A}_{2*2}(\mathbb{R}_+)$. If A converges to zero. Then the problem (2.1.1) has a unique solution.

Proof. Consider the operator:

$M : C(\mathbb{N}(a,b),\mathbb{R}^n) \times C(\mathbb{N}(a,b),\mathbb{R}^n) \longrightarrow C(\mathbb{N}(a,b),\mathbb{R}^n)$ defined for:

$$(x,y) \in C(\mathbb{N}(a,b),\mathbb{R}^n) \times C(\mathbb{N}(a,b),\mathbb{R}^n)$$

We have :

$$M(x,y) = (M_1(x,y), M_2)(x,y)) \tag{2.1.2}$$

Where

$$M_1(x(k),y(k)) = x_0 + \sum_{l=a}^{k} f(l,x(l),y(l)),\ \text{k} \in \mathbb{N}(a,b)$$

And

$$M_2(x(k),y(k)) = y_0 + \sum_{l=a}^{k} g(l,x(l),y(l)),\ \text{k} \in \mathbb{N}(a,b).$$

We shall use theorem 11 to prove that M has a fixed point.Indeed,let $(x,y),(\bar{x},\bar{y}) \in C(\mathbb{N}(a,b),\mathbb{R}^n) \times C(\mathbb{N}(a,b),\mathbb{R}^n)$. Then we have for each $k \in \mathbb{N}(a,b)$

$$|M_1(x(k),y(k)) - M_1(\bar{x}(l),\bar{y}(l))| = |\sum_{l=a}^{l=k}[f(l,x(l),y(l)) - f(l,\bar{x}(l),\bar{y}(l))]|.$$

Then

$$\|M_1(x,y) - M_1(\bar{x},\bar{y})\|_\infty \leq ba_1\|x-\bar{x}\|_\infty + bb_1\|y-\bar{y}\|_\infty.$$

Similarly we have

$$\|M_2(x,y) - M_2(\bar{x},\bar{y})\|_\infty \leq ba_2\|x-\bar{x}\|_\infty + bb_2\|y-\bar{y}\|_\infty.$$

Hence

$$\|M(x,y) - M(\bar{x},\bar{y})\|_\infty = \begin{pmatrix} \|M_1(x,y) - M_1(\bar{x},\bar{y})\|_\infty \\ \|M_2(x,y) - M_2(\bar{x},\bar{y})\|_\infty \end{pmatrix} \leq b \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \|x-\bar{x}\|_\infty \\ \|y-\bar{y}\|_\infty \end{pmatrix}.$$

Therefore

$$\|M(x,y) - M(\bar{x},\bar{y})\|_\infty \leq A \begin{pmatrix} \|x-\bar{x}\|_\infty \\ \|y-\bar{y}\|_\infty \end{pmatrix}.$$

For all $(x,y),(\bar{x},\bar{y}) \in C(\mathbb{N}(a,b),\mathbb{R}^n) \times C(\mathbb{N}(a,b),\mathbb{R}^n)$ From Preov fixed point theorem, the mapping M has a unique fixed $(x,y) \in C(\mathbb{N}(a,b),\mathbb{R}^n) \times C(\mathbb{N}(a,b),\mathbb{R}^n)$ which is unique solution of problem (2.1.1).

□

Lemma 2.1.2. *Let* $\mathrm{M}_1(x(k),y(k))$ *an operator has a fixed point*

$$\mathrm{M}_1(x(k),y(k)) = x_0 + \sum_{l=a}^{k} f(l,x(l),y(l)),\ \ k \in \mathbb{N}(a,b).$$

Proof. We have $\triangle x(k) = f(k, x(k), y(k))$ and $\triangle x(k) = x(k+1) - x(k)$ so

$$x(k+1) - x(k) = f(k, x(k), y(k)),$$

If $k = a, \ldots, b$ we get:

$$x(a+1) - x(a) = f(a, x(a), y(a)),$$

$$x(a+2) = x(a+1) + f(a+1, x(a+1), y(a+1)),$$
$$x(a+3) = x(a+2) + f(a+2, x(a+2), y(a+2)), \ldots,$$
$$x(b) = x(b-1) + f(b-1, x(b-1), y(b-1)),$$
$$x(b) = x(a) + f(a, x(a), y(a)) + \ldots + f(b, x(b), y(b)),$$

Therefore :

$$M_1(x(k), y(k)) = x_0 + \sum_{l=a}^{k} f(l, x(l), y(l)),\ \text{k} \in \mathbb{N}(a, b).$$

□

Theorem 2.1.3. *Assume the following conditions:*

(H_2) *There exist nonnegative functions* $\alpha_i, \beta_i : \mathbb{N}(a) \longrightarrow \mathbb{R}_+$ *for each* $i \in \{1, 2\}$
$$\begin{cases} |f(k,x,y) - f(k,\bar{x},\bar{y})| \leq \alpha_1(k)|x - \bar{x}| + \alpha_2(k)|y - \bar{y}| \\ |g(k,x,y) - g(k,\bar{x},\bar{y})| \leq \beta_1(k)|x - \bar{x}| + \beta_2(k)|y - \bar{y}| \end{cases}$$
for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

(H_3) $h_1, h_2 : \mathbb{N}(a) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be functions such that :*

$$|h_i(k, x, y)| \leq \lambda_1(k), i = 1, 2,$$

Where λ_i *are a nonnegative functions defined on* $\mathbb{N}(a)$.

Then, for the solutions $(x(k, x_0), y(k, y_0))$ *and* $(e(k, e_0), s(k, s_0))$ *on* $\mathbb{N}(a)$ *of the initial value problem* (2.1.1) *and*

$$\begin{cases} \triangle e(k) = h_1(k, e(k), s(k)) + f(k, e(k, e_0), s(k, s_0)),\ k \in \mathbb{N}(a) = \{a, a+1, \ldots, b\}, \\ \triangle s(k) = h_2(k, e(k), s(k)) + g(k, e(k, e_0), s(k, s_0)),\ k \in \mathbb{N}(a), \\ e(a) = e_0, \\ s(a) = s_0, \end{cases} \tag{2.1.3}$$

The following inequality holds:

$$|x(k, x_0) - e(k, e_0)| \leq (|x_0 - e_0| + |y_0 - s_0| + \sum_{l=a}^{l=k} \lambda(l)) \prod_{l=a}^{l=k} (1 + \alpha(k)).\ \textit{And}$$

$$|y(k,y_0)-s(k,s_0)| \le (|y_0-s_0|+|x_0-e_0|+\sum_{l=a}^{l=k}\lambda(l))\prod_{l=a}^{l=k}(1+\alpha(k)).$$

Where

$\alpha(k)=\alpha_1(k)+\alpha_2(k)+\beta_1(k)+\beta_2(k), \lambda_k=\lambda_1(k)+\lambda_2(k),\ k\in\mathbb{N}(a).$

Proof. The problem (2.1.1) and (2.1.3) are equivalent to:

$$\begin{cases} x(k,x_0)=x_0+\sum_{l=a}^{k} f(l,x(l,x_0),y(l,y_0)),\ \text{k}\in\mathbb{N}(a) \\ y(k,y_0)=y_0+\sum_{l=a}^{k} g(l,x(l,x_0),y(l,y_0)),\ \text{k}\in\mathbb{N}(a). \end{cases}$$

And

$$\begin{cases} e(k,e_0)=e_0+\sum_{l=a}^{k}(h_1(l,e(l,e_0),s(l,s_0))+f(l,e(l,e_0),s(l,s_0))),\ \text{k}\in\mathbb{N}(a) \\ s(k,s_0)=s_0+\sum_{l=a}^{k}(h_2(l,e(l,e_0),s(l,s_0))+g(l,e(l,e_0),s(l,s_0))),\ \text{k}\in\mathbb{N}(a), \end{cases}$$

We find that:

$$\begin{cases} x(k,x_0)-e(k,e_0)=x_0-e_0+\displaystyle\sum_{l=a}^{k}(f(l,x(l,x_0),y(l,y_0))-f(l,e(l,e_0),s(l,s_0)))- \\ \sum_{l=a}^{k}(h_1(l,e(l,e_0),s(l,s_0)) \\ y(k,y_0)-s(k,s_0)=y_0-s_0+\displaystyle\sum_{l=a}^{k}(g(l,x(l,x_0),y(l,y_0))-g(l,e(l,e_0),s(l,s_0)))- \\ \sum_{l=a}^{k}(h_2(l,e(l,e_0),s(l,s_0)). \end{cases}$$

Then:

$$R(k)\le|x_0-e_0|+|y_0-s_0|+\sum_{l=a}^{k}\alpha(l)R(l)+\sum_{l=a}^{k}\lambda(l)$$

Where

$$R(k)=|x(k,x_0)-e(k,e_0)|+|y(k,y_0)-s(k,s_0)|,\ \text{k}\in\mathbb{N}(a)$$

And

$\alpha(k)=\alpha_1(k)+\alpha_2(k)+\beta_1(k)+\beta_2(k), \lambda_k=\lambda_1(k)+\lambda_2(k),\ \text{k}\in\mathbb{N}(a).$

We get

$$R(k)\le(|x_0-e_0|+|y_0-s_0|+\sum_{l=a}^{l=k}\lambda(l))\prod_{l=a}^{l=k}(1+\alpha(k))$$

Hence

$|x(k,x_0)-e(k,e_0)|\le(|x_0-e_0|+|y_0-s_0|+\sum_{l=a}^{l=k}\lambda(l))\prod_{l=a}^{l=k}(1+\alpha(k)).$

and

$$|y(k,y_0)-s(k,s_0)|\le(|y_0-s_0|+|x_0-e_0|+\sum_{l=a}^{l=k}\lambda(l))\prod_{l=a}^{l=k}(1+\alpha(k)).$$ □

Now we consider the following cauchy problem with parameter

$$\begin{cases} \triangle x(k) = f(k, x(k), y(k), \beta), \ \text{k} \in \mathbb{N}(a), \\ \triangle y(k) = g(k, x(k), y(k), \beta), \ \text{k} \in \mathbb{N}(a), \\ x(a) = x_0, \\ y(a) = y_0, \end{cases} \tag{2.1.4}$$

Where $\beta \in \mathbb{R}^m$ is a parameter such that $|\beta - \beta_0| \leq \epsilon$ and β_0 is a fixed vector in $\mathbb{R}^m$ and $f, g : \mathbb{N}(a) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ are given functions.

Theorem 2.1.4. *For fixe $\alpha_0 \in \mathbb{R}^m$ and $\delta > 0$ such that $|\alpha - \alpha_0| \leq \delta$ the functions h and u satisfies the following conditions*

(H_4) *There exist nonnegative numbers functions $\lambda_i, \gamma_i, \eta_i : \mathbb{N}(a) \longrightarrow \mathbb{R}_+$ for each $i = 1, 2$*

$$\begin{cases} |h(k, x, y, \alpha) - h(k, \bar{x}, \bar{y}, \alpha)| \leq \lambda_1(k)|x - \bar{x}| + \lambda_2(k)|y - \bar{y}| \\ |u(k, x, y, \alpha) - u(k, \bar{x}, \bar{y}, \alpha)| \leq \gamma_1(k)|x - \bar{x}| + \gamma_2(k)|y - \bar{y}| \end{cases}$$

And

$$\begin{cases} |h(k, x, y, \alpha_1) - h(k, x, y, \alpha_2)| \leq \eta_1(k)|\alpha_1 - \alpha_2| \\ |u(k, x, y, \alpha_1) - u(k, x, y, \alpha_2)| \leq \eta_2(k)|\alpha_1 - \alpha_2| \end{cases}$$

Then for the solution $(x(k, x_1, \alpha_1), y(k, y_1, \alpha_1))$ and $(p(k, p_2, \alpha_2), q(k, q_2, \alpha_2))$ of (2.1.3) the following inequality holds:

$$|x(k, x_1, \alpha_1) - p(k, p_2, \alpha_2)| \leq (|x_1 - p_2| + |y_1 - q_2| + |\alpha_1 - \alpha_2| + \sum_{l=a}^{k} \eta(l)) \prod_{l=a}^{l=k} (1 + \lambda(l)).$$

And

$$|y(k, y_1, \alpha_1) - q(k, q_2, \alpha_2)| \leq (|x_1 - p_2| + |y_1 - q_2| + |\alpha_1 - \alpha_2| + \sum_{l=a}^{k} \eta(l)) \prod_{l=a}^{l=k} (1 + \lambda(l)).$$

Where
$\lambda(k) = \lambda_1(k) + \lambda_2(k) + \gamma_1(k) + \gamma_2(k), \eta_k = \eta_1(k) + \eta_2(k), \ k \in \mathbb{N}(a).$

In ordinary differential equations the Arzela-Ascoli theorem plays an important role. In this section we give the discrete version of the Arzela-Ascoli theorem. The topology on $\mathbb{N}(0, b+1)$ will be the discrete topology. Let $(E, ||)$ be a Banach space, we denote the space of continuous functions on $\mathbb{N}(0, b+1)$ by

$$C(\mathbb{N}(a, b-1), E) = \{y : \mathbb{N}(a, b-1) \longrightarrow E, is\ continuous\}.$$

With norm

$$\|y\|_\infty = \sup_{k \in \mathbb{N}(0, b+1)} |y(k)|$$

is a banach space .Now we set and prove the discrete Arzela -Ascoli theorem.

Theorem 2.1.5. *Let A be a closed subset of $C(\mathbb{N}(a, b+1), E)$.if ψ is uniformly bounded and the set:*

$$\{y(k) : y \in \psi\}$$

is relatively compact for each $k \in \mathbb{N}(a, b+1)$.Then ψ is compact.

Proof. We need only show that every sequence in ψ has a cauchy sebsequence.Let $\psi_1 = \{l_{1,1}, l_{1,2}, \ldots\}$ be any sequence in ψ.Notice the sequence $\{l_{1,i}(0)\}$,$i = 1, 2, \ldots$ has a convergent subsequence and let $\psi_2 = \{l_{2,1}, l_{2,2}, \ldots\}$ denote this subsequence .for $\{l_{2,i}(1)\}$,$i = 1, 2, \ldots$ let $\psi_3 = \{l_{3,1}, l_{3,2}, \ldots\}$ be the subsequence of ψ_2 such that $\{l_{3,i}(1)\}$,$i = 1, 2, \ldots$ converges.since ψ_3 is a subsequence of ψ_2 then $\{l_{3,i}(0)\}$,$i = 1, 2, \ldots$ also converges.Continue this process to get a list of sequence

$$\psi_1, \psi_2, \ldots \psi_{b+2}, \psi_{b+3}$$

. in which each sequence is a subsequence of the one directly on the left of it and for each k, the sequence $\psi_k = \{l_{k,1}, l_{k,2}, \ldots\}$ has the property that $\{l_{k,i}(k-2)\}$,$i = 1, 2, \ldots$ is a convergent sequence.Thus for each $k \in \mathbb{N}(0, b+1)$, the sequence $\{l_{b+3,i}(k)\}$ is convergent .Then since $\{l_{T+3,i}(k)\}$ is Cauchy for each $k \in \mathbb{N}(0, b+1)$,and since $\mathbb{N}(0, b+1)$ is finite ,we have that there exists $n_0 \in \mathbb{N}$ independent of k such that

$$m, n \geq n_0 \Rightarrow |l_{b+3,m}(k) - l_{b+3,n}(k)| < \epsilon, k \in \mathbb{N}(0, b+1)$$

. thus ψ_{b+3} is cauchy. □

We also need the following characterization for relatively compact sets in $B \subset (\mathbb{N}, E)$, which is the discrete version of Przeradzki theorem [19].

Theorem 2.1.6. *A set $\psi \subset B \subset (\mathbb{N}, E)$ is relatively compact if the following conditions hold:*

(i) For every $k \in \mathbb{N}$ the set $\{y(k) : x \in \psi\}$ is relatively compact in E.

(ii) For every $\epsilon > 0$ there exists $N' \in \mathbb{N} \setminus \{0\}$ and $\delta > 0$ such that if $x, y \in \psi$ with $|x(N') - y(N')| \leq \delta$ then $|x(k) - y(k)| \leq \epsilon$ for all $k \in \{N', N'+1, \ldots\}$.

We shall also need the following existence principles

Theorem 2.1.7. *Let $h, g : \mathbb{N}(a, b-1) \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are continuous functions. Assume that condition*

(H5) There exist $\lambda_1, \lambda_2 \in C(\mathbb{N}(a, b-1), \mathbb{R}_+)$ such that

$$|h(k, x, y)| \leq \lambda_1(k)(|x| + |y|), k \in \mathbb{N}(a, b-1), (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and

$$|g(k, x, y)| \leq \lambda_2(k)(|x| + |y|), k \in \mathbb{N}(a, b-1), (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

holds.Then the problem (2.1.1) has at least one solution.Moreover,the solution set $S(x_0, y_0)$ is compact and the multivalued map $S : (x_0, y_0) \multimap S(x_0, y_0)$ is u.s.c.

Proof. let $M = (M_1, M_2)$ is defined in the proof of theorem (2.1.1).

Step 1:N is cotinuous.
Let $(x_n, y_n) \to (x, y)$ in $C(\mathbb{N}(a, b-1), \mathbb{R}^n) \times C(\mathbb{N}(a, b-1), \mathbb{R}^n)$ as $n \to \infty$.
Then

$$|M_1(x_n(k), y_n(k)) - M_1(x(k), y(k))|$$

.

$$= |\sum_{l=a}^{k-1} h(l, x_n(l), y_n(l)) - \sum_{l=a}^{k-1} h(l, x(l), y(l))|.$$

$$\leq |\sum_{l=a}^{b} h(l, x_n(l), y_n(l)) - \sum_{l=a}^{b} h(l, x(l), y(l))|.$$

Similarly

$$|M_2(x_n(k), y_n(k)) - M_2(x(k), y(k))|$$

$$\leq |\sum_{l=a}^{b} g(l, x_n(l), y_n(l)) - \sum_{l=a}^{b} g(l, x(l), y(l))|$$

Thus

$$\|M_1(x_n(k), y_n(k)) - M_1(x(k), y(k))\|_\infty \longrightarrow 0.$$

$$\|M_2(x_n(k), y_n(k)) - M_2(x(k), y(k))\|_\infty \longrightarrow 0.$$

Step 2: We will show that M maps bounded sets into bounded sets in $C(\mathbb{N}(a, b-1), \mathbb{R}^n) \times C(\mathbb{N}(a, b-1), \mathbb{R}^n)$
let $B_r := \{(x, y) \in C(\mathbb{N}(a, b-1), \mathbb{R}^n) \times C(\mathbb{N}(a, b-1), \mathbb{R}^n) : \|(x, y)\|_\infty \leq r\}$,
where $r = (r_1, r_2)$ and if $(x, y) \in B_r$,we obtain

$$|M_1(x(k), y(k))| = |x_0 + \sum_{l=a}^{k-1} h(l, x_n(l), y_n(l))|$$

$$\leq |x_0| + |\sum_{l=a}^{b} h(l, x(l), y(l))|.$$

$$\|M_1(x, y)\|_\infty \leq |x_0| + 2r_1 \sum_{k=a}^{b} \lambda_1(k) := l_1$$

Similarly

$$\|M_2(x, y)\|_\infty \leq |y_0| + 2r_2 \sum_{k=a}^{b} \lambda_2(k) := l_2$$

Thus $M : C(\mathbb{N}(a, b-1), \mathbb{R}^n) \to C(\mathbb{N}(a, b-1), \mathbb{R}^n)$ is completely continuous.

Step 3: it remains to show that $B = \{(x, y) \in C(\mathbb{N}(a, b-1), \mathbb{R}^n) \times C(\mathbb{N}(a, b-1), \mathbb{R}^n) : (x, y) = \lambda M(x, y)\}$ is bounded
Let $(x, y) \in B$.Then $x = \lambda M_1(x, y)$ and $y = \lambda M_2(x, y)$ for some $0 < \lambda < 1$ we have

$$|x(k)| \leq |x_0| + \sum_{l=1}^{k-1} |f(l, x(l), y(l)|$$

$$\leq |x_0| + \sum_{l=a}^{k-1} \lambda_1(l)(|x(l)| + |y(l)|)ds.$$

And

$$|y(k)| \leq |y_0| + \sum_{l=a}^{k-1} \lambda_2(l)(|x(l)| + |y(l)|)ds$$

Therefore

$$|x(k)| + |y(k)| \leq |x_0| + |y_0| + \sum_{l=a}^{k-1} \lambda(l)(|x(l)| + |y(l)|),$$

Where

$$\lambda(k) = \lambda_1(k) + \lambda_2(k), k \in \mathbb{N}(a, b-1).$$

We have:

$$|x(k)| + |y(k)| \leq (|x_0| + |y_0|)(1 + \sum_{l=a}^{k-1} \lambda(l) \prod_{\tau+1}^{k-1}(1 + \lambda(\tau))).$$

Hence

$$\|x(k)\|_\infty + \|y(k)\|_\infty \leq (|x_0| + |y_0|)(1 + \sum_{l=a}^{b} \lambda(l) \prod_{\tau+1}^{b}(1 + \lambda(\tau))).$$

this shows that B is bounded .As a consequence we deduce that M has a fixed point (x, y) which is a solution to the problem (2.1.1).

Step 4: Compatness of the solution set.For each $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n$, let $S(x_0, y_0) = \{(x, y) \in C(\mathbb{N}(a, b-1), \mathbb{R}^n) \times C(\mathbb{N}(a, b-1), \mathbb{R}^n) : (x, y) \textit{ is a solution}\}$ From *step* 3, there exists Ã such that for every $(x, y) \in S(x_0, y_0)$, $\|x(k)\|_\infty \leq$ Ã;$\|y(k)\|_\infty \leq$Ã.Since M is completely cotinuous ,$M(S(x_0, y_0))$ is relatively compact in

$$(x, y) \in C(\mathbb{N}(a, b-1), \mathbb{R}^n) \times C(\mathbb{N}(a, b-1), \mathbb{R}^n).$$

let$(x, y) \in S(x_0, y_0)$; then $(x, y) = M(x, y)$ hence $S(x_0, y_0) \subset \overline{M(S(x_0, y_0))}$. it remains to prove that $S(x_0, y_0)$ is a closed subset in $(x, y) \in C(\mathbb{N}(a, b-1), \mathbb{R}^n) \times C(\mathbb{N}(a, b-$

$1),\mathbb{R}^n)$. let $\{(x_m,y_m) : m \in \mathbb{N}\} \subset S(x_0,y_0)$ be such that $(x_m,y_m)_{m\in\mathbb{N}}$ converges to (x,y).for every $m \in \mathbb{N}$, and $k \in \mathbb{N}(a,b-1)$

$$x_m(k) = x_0 + \sum_{l=a}^{k-1} h(l, x_m(l), y_m(l)). \tag{2.1.5}$$

And

$$y_m(k) = y_0 + \sum_{l=a}^{k-1} g(l, x_m(l), y_m(l)). \tag{2.1.6}$$

Set

$$z_1(k) = x_0 + \sum_{l=a}^{k-1} h(l, x(l), y(l)). \tag{2.1.7}$$

and

$$z_2(k) = y_0 + \sum_{l=a}^{k-1} g(l, x(l), y(l)). \tag{2.1.8}$$

Since h and g are continuous functions , we can prove that

$$x(k) = x_0 + \sum_{l=a}^{k-1} h(l, x(l), y(l)), k \in \mathbb{N}(a, b-1)$$

and

$$y(k) = y_0 + \sum_{l=a}^{k-1} g(l, x(l), y(l)), k \in \mathbb{N}(a, b-1)$$

Therefore $(x,y) \in S(x_0,y_0)$ which yields that $S(x_0,y_0)$ is closed ,hence compact subset in $C(\mathbb{N}(a,b-1),\mathbb{R}^n) \times C(\mathbb{N}(a,b-1),\mathbb{R}^n)$.finally,we prove that $S(.)$ is u.s.c. by proving that the graph of S

$$\Phi_s := \{(\bar{x},\bar{y},x,y) : (x,y) \in S(\bar{x},\bar{y})\}$$

is closed .let $(\bar{x}_m,\bar{y}_m,x_m,y_m) \in \Phi_s$ be such that $(\bar{x}_m,\bar{y}_m,x_m,y_m) \to (\bar{x},\bar{y},x,y) asm \to \infty$.Since $(x_m,y_m) \in S(\bar{x}_m,\bar{y}_m)$, Then

$$x_m(k) = \bar{x_m} + \sum_{l=a}^{k-1} h(l, x_m(l), y_m(l)), k \in \mathbb{N}(a, b-1)$$

And

$$y_m(k) = \bar{y_m} + \sum_{l=a}^{k-1} g(l, x_m(l), y_m(l)), k \in \mathbb{N}(a, b-1)$$

Since $\{(\bar{x_m},\bar{y_m})\}$ is a bounded sequence ,there exists a subsequence of $\{(\bar{x_m},\bar{y_m})\}$ converging to $(\bar{x},\bar{y})$. as in Step 2,we can show that $\{(x_m,y_m) : m \in \mathbb{N}\}$ is uniformly bounded . as a sequence ,we conclude that there exists a subsequence of $\{(\bar{x_m}, \bar{y_m})\}$ converging to $(\bar{x}, \bar{y})$ in $C(\mathbb{N}(a, b-1), \mathbb{R}^n) \times C(\mathbb{N}(a, b-1), \mathbb{R}^n)$. By the continuity of h and g ,we can prove that

$$x(k) = \bar{x} + \sum_{l=a}^{k-1} h(l, x(l), y(l)), k \in \mathbb{N}(a, b-1)$$

and

$$y(k) = \bar{y} + \sum_{l=a}^{k-1} g(l, x(l), y(l)), k \in \mathbb{N}(a, b-1)$$

Thus, $(x, y) \in S(\bar{B})$.This implies that $S(.)$ is u.s.c.

□

Theorem 2.1.8. *A set $\psi \subset B \subset (\mathbb{N}, E)$ is relatively compact if the following conditions hold :*

(i) For every $l \in \mathbb{N}$ the set $\{y(l) : x \in \psi\}$ is relatively compact in E,

(ii) The functions from ψ are equiconvergent at infinity, i.e. for every $\epsilon > 0$ there exists $l(\epsilon) \in \mathbb{N}$ i.e if $|y(l) - y(\infty)| \leq \epsilon$ for all $l > l_\epsilon$ and $y \in \psi$.

Theorem 2.1.9. *Let $v, u : \mathbb{N}(a, b-1) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions satisfies*

(H6) There exist $\theta_1, \theta_2 \in C(\mathbb{N}(a, b-1), \mathbb{R}_+)$ such that

$$|u(k, x, y)| \leq \theta_1(k)(|x| + |y|), k \in \mathbb{N}(a, b-1), (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,$$

And

$$|v(k, x, y)| \leq \theta_2(k)(|x| + |y|), k \in \mathbb{N}(a, b-1), (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,$$

With

$$\sum_{l=a}^{\infty} (\theta_1(k) + \theta_2(k)) < \infty.$$

Then problem 2.1.3 has at least one solution. Moreover, the solution set $S(x_0, y_0)$ is compact and the multivalued map $S : (x_0, y_0) \multimap S(x_0, y_0)$ is u.s.c.

Proof. Consider the operator $M : BC(\mathbb{N}(a), R^n) \times C(\mathbb{N}(a), R^n) \to BC(\mathbb{N}(a), R^n)$defined for $(x, y) \in C(\mathbb{N}(a), R^n) \times C(\mathbb{N}(a), R^n)$ by

$$M(x, y) = (M_1(x, y), M_2(x, y)), (x, y) \in BC(\mathbb{N}(a), R^n) \times BC(\mathbb{N}(a), R^n) \tag{2.1.9}$$

Where

$$M_1(x(k), y(k)) = x_0 + \sum_{l=a}^{k-1} u(l, x(l), y(l)), k \in \mathbb{N}(a)$$

And

$$M_2(x(k), y(k)) = y_0 + \sum_{l=a}^{k-1} v(l, x(l), y(l)), k \in \mathbb{N}(a).$$

Step 1:$M = (M_1, M_2)$ is continuous. Let (x_m, y_m) be a sequence such that $(x_m, y_m) \longrightarrow (x, y) as\ m \to \infty$. Then

$$|M_1(x_m(k), y_m(k)) - M_1(x(k), y(k))| \leq \sum_{l=a}^{k-1} |(u(l, x_m(l), y_m(l)) - u(l, x(l), y(l)))|.$$

Similarly

$$|M_2(x_m(k), y_m(k)) - M_2(x(k), y(k))| \leq \sum_{l=a}^{k-1} |(v(l, x_m(l), y_m(l)) - v(l, x(l), y(l)))|.$$

using the condition (H_6),for every $\epsilon > 0$, there exists $h(\epsilon) \in \mathbb{N}$ such that

$$\sum_{l=h(\epsilon)}^{\infty} 2\mu(\theta_1(k) + \theta_2(k)) < \frac{\epsilon}{2}.$$

Where

$$\|x_m\| \leq \mu, \|y_m\| \leq \mu$$

for each $m \in \mathbb{N}$ Hhence

$$\begin{aligned}\|M_1(x_m, y_m) - M_1(x, y)\|_\infty \leq & \sum_{l=a}^{h(\epsilon)-1} |(u(l, x_m(l), y_m(l)) - u(l, x(l), y(l)))| \\ & + \sum_{l=h(\epsilon)}^{\infty} |(u(l, x_m(l), y_m(l)) - u(l, x(l), y(l)))| \\ & \leq \sum_{l=a}^{h(\epsilon)-1} |(u(l, x_m(l), y_m(l)) - u(l, x(l), y(l)))| \\ & +2\mu \sum_{l=h(\epsilon)}^{\infty} (\theta_1(k) + \theta_2(k)) \\ & \leq \sum_{l=a}^{h(\epsilon)-1} |(u(l, x_m(l), y_m(l)) - u(l, x(l), y(l)))| + \frac{\epsilon}{2}.\end{aligned}$$

And

$$\|M_2(x_m, y_m) - M_2(x, y)\|_\infty \leq \sum_{l=a}^{h(\epsilon)-1} |(v(l, x_m(l), y_m(l)) - v(l, x(l), y(l)))|$$

$$+\sum_{l=h(\epsilon)}^{\infty}|(v(l,x_m(l),y_m(l))-v(l,x(l),y(l)))|$$

$$\leq \sum_{l=a}^{h(\epsilon)-1}|(v(l,x_m(l),y_m(l))-v(l,x(l),y(l)))|$$

$$+2\mu\sum_{l=h(\epsilon)}^{\infty}(\theta_1(k)+\theta_2(k))$$

$$\leq \sum_{l=a}^{h(\epsilon)-1}|(v(l,x_m(l),y_m(l))-v(l,x(l),y(l)))|+\frac{\epsilon}{2}.$$

Since u, v are continuous functions ,we get

$$\|M_1(x_m,y_m)-M_1(x,y)\|_\infty \to 0 \ as\ m\to\infty$$

And

$$\|M_2(x_m,y_m)-M_2(x,y)\|_\infty \to 0 \ as\ m\to\infty$$

Step 2: We now show that $M(B_\mu)$ is equiconvergent at ∞ i.e for every $\epsilon>0$ there exists $k(\epsilon)\in\mathbb{N}$ such that

$$|M_i(x(k),y(k))-M_i(x(\infty),y(\infty))|\leq\varepsilon$$

For all $k>k_\varepsilon$ And $(x,y)\in B_\mu, i=1,2$ Where

$$B_\mu=\{(x,y)\in BC(\mathbb{N}(a),R^n)\times BC(\mathbb{N}(a),R^n):\|x\|_\infty\leq\mu,\|y\|_\infty\leq\mu\}.$$

Letting $(x,y)\in B_\mu$,then

$$|M_1(x(k),y(k))-M_1(x(\infty),y(\infty))|\leq\sum_{l=k}^{\infty}|u(l,x(l),y(l))|$$

$$\leq 2\mu\sum_{l=k}^{\infty}(\theta_1(k)+\theta_2(k)).$$

And

$$|M_2(x(k),y(k))-M_2(x(\infty),y(\infty))|\leq\sum_{l=k}^{\infty}|v(l,x(l),y(l))|$$

$$\leq 2\mu\sum_{l=k}^{\infty}(\theta_1(k)+\theta_2(k))$$

. Since $2\mu\sum_{l=k}^{\infty}(\theta_1(k)+\theta_2(k)) < \infty$, such that $\mu\sum_{l=k}^{\infty}(\theta_1(k)+\theta_2(k)) \leq \epsilon$ Then $M(B_\mu)$ is equiconvergent We conclude that M is completely continuous. Let $(x,y) \in BC(\mathbb{N}(a),R^n)$ be such that $(x,y)=M(x,y)$ We have

$$|x(k)| \leq |x_0| + \sum_{l=a}^{k-1} |u(l,x(l),y(l))|$$

$$\leq |x_0| + \sum_{l=a}^{k-1} \theta_1(l)(|x(l)|+|y(l)|)ds$$

And

$$|y(k)| \leq |y_0| + \sum_{l=a}^{k-1} \theta_2(l)(|x(l)|+|y(l)|)ds$$

Therefore

$$|x(k)|+|y(k)| \leq |x_0|+|y_0| + \sum_{l=a}^{k-1} \theta(l)(|x(l)|+|y(l)|)$$

, We have

$$|x(k)|+|y(k)| \leq (|x_0|+|y_0|)(1+)\sum_{l=a}^{k-1}\theta(k)\prod_{\tau+1}^{k-1}(1+\theta(\tau))).$$

Hence

$$\|x(k)\|_\infty + \|y(k)\|_\infty \leq (|x_0|+|y_0|)(1+\sum_{l=a}^{\infty}\theta(k)\prod_{\tau+1}^{b}(1+\theta(\tau))) = D.$$

Finally let

$$J : \{y \in BC(\mathbb{N}(a),R^n) : (\|x(k)\|_\infty, \|y(k)\|_\infty) < (D+1, D+1)\}$$

M has e fixed point $(x,y) \in J$ which is a solution of the problem (2.1.3) We can prove that the solutions $S(x_0,y_0)$ of problem (2.1.3) is compact and the multivalued operator $S : R^m R^m \to \mathbf{P}(BC(\mathbb{N}(a),R^m))$ .defined by $S(x_0,y_0) = \{(x,y) \in BC(\mathbb{N}(a),R^m))\}$ is solution of the problem (2.1.3). □

2.2 Boundary Value Problems

$$\begin{cases} \triangle^2 x(k) - u(k,x(k),y(k)) = 0, \ \mathrm{k} \in \mathbb{N}(0,b), \\ \triangle^2 y(k) - v(k,x(k),y(k)) = 0, \ \mathrm{k} \in \mathbb{N}(0,b), \\ \alpha_0 x(0) - \beta_0 \triangle x(0) = 0, \\ \xi_0 x(b+1) + \eta_0 \triangle x(b+1) = 0, \\ \bar{\alpha_0} y(0) - \bar{\beta_0} \triangle y(0) = 0, \\ \bar{\xi_0} x(b+1) + \bar{\eta_0} \triangle x(b+1) = 0, \end{cases} \tag{2.2.1}$$

Where $\beta_0, \eta_0, \bar{\beta_0}, \bar{\eta_0} \in \mathbb{R}^*, \alpha_0, \xi_0, \bar{\alpha_0}, \bar{\xi_0} \in \mathbb{R}$

Lemma 2.2.1. *A function $x \in C(\mathbb{N}(0,b),\mathbb{R}^m)$ is a solution of problem*

$$\begin{cases} \triangle^2 x(k) = -z(k), \;\; k \in \mathbb{N}(0,b), \\ \alpha_0 x(0) - \beta_0 \triangle x(0) = 0, \\ \xi_0 x(b+1) + \eta_0 \triangle x(b+1) = 0, \end{cases} \tag{2.2.2}$$

where $u \in C(\mathbb{N}(0,b),\mathbb{R}^m)$, and $\alpha_0\xi_0(b+1)+\alpha_0\eta_0+\beta_0\xi_0 \neq 0$ if and only if

$$x(k) = \sum_{i=0}^{b} \rho(k,i) z(i),$$

where

$$\rho(k,i) = \begin{cases} \dfrac{\beta_0+\alpha_0(i+1))(\eta_0+\xi_0(b+1-k))}{\alpha_0\xi_0(b+1)+\alpha_0\eta_0+\beta_0\xi_0}, \;\; i \in \{0,\dots,k-1\}, \\ \dfrac{\beta_0+\alpha_0 k)(\eta_0+\xi_0(b-i))}{\alpha_0\xi_0(b+1)+\alpha_0\eta_0+\beta_0\xi_0}, \;\; i \in \{k,\dots,b\}, \end{cases}$$

Proof.

$$\triangle^2 x(k) = z(k) \Rightarrow \triangle x(k+1) - \triangle x(k) = -z(k).$$

by summing the above equations, we get

$$\triangle x(k) = \triangle x(0) - \sum_{i=0}^{k-1} z(i). \tag{2.2.3}$$

Hence

$$x(k) = x(0) + k\triangle x(0) - \sum_{i=0}^{k-1}(k-i-1)z(i). \tag{2.2.4}$$

We have

$$x(b+1) = x(0) + (b+1)\triangle x(0) - \sum_{i=0}^{b}(k-i-1)z(i).$$

And

$$\triangle x(b+1) = \triangle x(0) - \sum_{i=0}^{b} z(i).$$

Since $\alpha_0 x(0) - \beta_0 \triangle x(0) = 0, \xi_0 x(b+1) + \eta_0 \triangle x(b+1) = 0$, Then

$$\triangle x(0) = \frac{\alpha_0}{\beta_0}, \triangle x(b+1) = \frac{\xi_0}{\eta_0}.$$

And $x(k) = x(0)(\frac{\beta_0+k\alpha_0}{\beta_0}) - \sum_{i=0}^{k-1}(k-i-1)z(i)$. □

Theorem 2.2.2. *Assume that (H_1) is satisfied and the matrix*

$$\bar{M} = \rho_* \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{M}_{2*2}(R_+),$$

Where

$$sup\{|\rho(i,j)| : (i,j) \in \mathbb{N}(0,b) \times \mathbb{N}(0,b)\}.$$

If $\bar{M} \to 0$. Then the problem (2.2.2)has unique solution.

Proof. Consider the operator $\bar{N} : C(\mathbb{N}(0,b), \mathbb{R}^m) \times C(\mathbb{N}(0,b), \mathbb{R}^m) \to C(\mathbb{N}(0,b), \mathbb{R}^m)$ defined for $(x,y) \in C(\mathbb{N}(0,b), \mathbb{R}^m) \times C(\mathbb{N}(0,b), \mathbb{R}^m)$ by

$$\bar{N}(x,y) = (\bar{N}_1(x,y), \bar{N}_2(x,y)) \tag{2.2.5}$$

Where

$$\bar{N}_1(x(k), y(k)) = \sum_{i=0}^{b} \rho(k,i) u(i, x(i), y(i)),\ \mathrm{k} \in \mathbb{N}(0,b),$$

And

$$\bar{N}_2(x(k), y(k)) = \sum_{i=0}^{b} \rho(k,i) v(i, x(i), y(i)),\ \mathrm{k} \in \mathbb{N}(0,b),$$

Thus $\bar{N}$ has a unique fixed point which is a solution of problem 2.2.1. □

Theorem 2.2.3. *Assume that*

(H7) there exist $\sigma_1, \sigma_2 \in C(\mathbb{N}(0,b), R_+)$ and $\alpha, \beta \in (0,1)$ such that

$$|u(k,x,y)| \le \sigma_1(k)(|x| + |y|)^{\alpha},\ k \in \mathbb{N}(0,b),\ (x,y) \in \mathbb{R}^m \times \mathbb{R}^m,$$

$$|v(k,x,y)| \le \sigma_2(k)(|x| + |y|)^{\beta},\ k \in \mathbb{N}(0,b),\ (x,y) \in \mathbb{R}^m \times \mathbb{R}^m,$$

if

$$\alpha_0 \xi_0 (b+1) + \alpha_0 \eta_0 + \beta_0 \xi_0 \neq 0$$

and

$$\bar{\alpha}_0 \bar{\xi}_0 (b+1) + \bar{\alpha}_0 \bar{\eta}_0 + \bar{\beta}_0 \bar{\xi}_0 \neq 0.$$

Then the problem 2.1.3 has at least one solution. Moreover, the solution set $\bar{S} = \{(x,y) \in C(\mathbb{N}(0,b), \mathbb{R}^m) \times C(\mathbb{N}(0,b), \mathbb{R}^m)\}$ (x,y) is solution of 2.2.1 is compact

2.3 Convex Case

Theorem 2.3.1. *Suppose* $F, G\mathbb{N}(0,b) \times R^m \to \boldsymbol{P}_{cp,cv}$ *such that* $(x,y) \to F(k,x,y)$ *and* $(x,y) \to G(k,x,y)$ *are u.s.c.*

(M1) there exist a continuous functions $\theta_1, \theta_2 : \mathbb{N}(0,b) \to R_+$

$$\|F(k,x,y)\|_{\boldsymbol{p}} \leq \theta_1(k)(\|x\| + \|y\|), foreach\ k \in (0,b)$$

and each $x \in \mathbb{R}^m$, *And*

$$\|G(k,x,y)\|_{\boldsymbol{p}} \leq \theta_2(k)(\|x\| + \|y\|), foreach\ k \in (0,b)$$

and each $x \in \mathbb{R}^m$,

Then problem 2.2.5 has at least one solution. Moreover, the solution set $S_{F,G}(x_0, y_0)$ *is compact and the multivalued map* $S_{F,G} : (a, \bar{b}) \multimap S_{F,G}(a,b)$ *is u.s.c.*

Proof. Set $E = C(\mathbb{N}(0,b), \mathbb{R}^m)$. Existence of solutions. Consider the operator $N : E \to \mathbf{P}(E)$ defined for $y \in E$ by

$$N(y) = \left\{\ (l_1, l_2) \in E \times E : (l_1(k), l_2(k)) = \begin{cases} x_0 + \sum_{i=0}^{k} \mu_1(i), \ \mathrm{k} \in (0,b) \\ y_0 + \sum_{i=0}^{k} \mu_2(i), \ \mathrm{k} \in (0,b) \end{cases}\right.$$

Where

$$\mu_1 \in S_{F,x,y} = \{\mu \in C(\mathbb{N}(0,b), \mathbb{R}^m) : \mu(k) \in F(k, x(k), y(k)),\ \mathrm{k} \in (0,b)\},$$

And

$$\mu_2 \in S_{G,x,y} = \{\mu \in C(\mathbb{N}(0,b), \mathbb{R}^m) : \mu(k) \in G(k, x(k), y(k)),\ \mathrm{k} \in (0,b)\}.$$

□

Chapter 3

Stability of an Equilibrium

3.1 Describing Stability

The goal of this section is to introduce basic qualitative methods to describe and classify the stability of a fixed point , or an equilibrium point, of a two-dimentional system of difference equations. Phase space diagrams are a good qualitative method for demonstrating the stability of the one or more equilibria of two-dimensional system of difference equations and the behavior of their orbits (trajectories).The only information needed ,after the equilibria are determined, is the information from chapter 1:The eigenvalues of the matrix A and the corresponding eigenvectors.Phase space diagrams will be sketched for worked examples, but a complete list of sketches can be found in [1].Note that the equilibrium point that we will focus on is the trivial, or zero,equilibrium,which every linear system possesses.

(H1) T is of the form $T = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$.For the equilibrium point $(0,0)$ in question is either a stable node or unstable node depending on the value of the eigenvalues.

(a) If $|\alpha_1| < 1$ and $|\alpha_2| < 1$,then the equilibrium point $(0,0)$ is a stable node with all trajectories approaching the origin as $n \to \infty$,with their directions determined by the eigenvectors associated with α_1 and α_2.

(b) If $|\alpha_1| > 1$ and $|\alpha_2| > 1$,then the equilibrium point $(0,0)$ is an unstable node with all trajectories tending away from the origin as $n \to \infty$,with their directions determined by the eigenvectors associated with α_1 and α_2.

(c) If $|\alpha_1| < 1$ and $|\alpha_2| > 1$,or if $|\alpha_1| > 1$ and $|\alpha_2| < 1$, the equilibrium point $(0,0)$ is a saddle node with at most two solutions approaching the origin and all others tending away from the origin as $n \to \infty$.

(d) If $\alpha_1 = \alpha_2$, then either:

(1) All trajectories approach the origin along lines passing through the origin if $\alpha_1 < 1$ or ,

(2) All trajectories tend away from the origin along lines passing through the origin if $\alpha_1 > 1$.

(e) If either $|\alpha_1|$ and or $|\alpha_2| = 1$, then the equilibrium point is a degenerate node with all points tending toward the axis of the dominate eigenvalue.

(H2) T is of form $T = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ For the equilibrium point $(0,0)$ is either a stable or unstable with only one straight line solution .

a If $|\alpha| < 1$,then the equilibrium point is stable with all trajectories approaching the origin as $n \to \infty$.

b If $|\alpha| > 1$,then the equilibrium point $(0,0)$ is unstable with all trajectories tending away from the origin as $n \to \infty$.

(H3) T is of form $T = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.Then the eigenvalues are $\gamma_{1,2} = \alpha \pm i\beta$, and there are no straight line trajectories since the general solutions for this matrix are combinations of since and cosines.

(a) If $|\gamma_1| < 1$,then the equilibrium point $(0,0)$ is a stable focus with all trajectories spiraling inward to the origin as $n \to \infty$.

(b) If $|\gamma_1| > 1$,then the equilibrium point $(0,0)$ is an unstable focus with all trajectories spiraling outward from the origin as $n \to \infty$.

(c) If $|\gamma_1| = 1$,then the equilibrium point $(0,0)$ is center with all trajectories following concentric circular paths about the origin as $n \to \infty$ (periodic orbits).

3.2 Definitions

For examining the stability of linear systems of difference equations.we note the map $g : \mathbb{R}^2 \to \mathbb{R}^2$,we refer to a point $x^* = \binom{(x_1)^*}{(x_2)^*}$ as a fixed point of g ,or an equilibrium point,if $g(x^*) = x^*$.

Definition 3.2.1. *Let U be a real vector space.Then a function $|*| : u \to \mathbb{R}$ is a vector norm if for all $x, y \in U$ and $\lambda \in \mathbb{R}$ if :*

(i) $|x| \geq 0$ *and* $|x| = 0$ *if and only if* $x = 0$.

(ii) $|\lambda x| = |\lambda||x|$

(iii) $|x + y| \leq |x| + |y|$ *(triangle inequality)*

We note the following: Let K_n be a real vector space of square matrices of size n .Then the function $\|\bar{*}\| : K_n \to \mathbb{R}$ is a matrix norm if for all $u, v \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, if:

(1) $\|\bar{u}\| \geq 0$ *and* $\|\bar{u}\| = 0$ *if and only if* $\|\bar{u}\| = 0$.

(2) $\|\lambda\bar{u}\| = |\lambda|\|\bar{u}\|$

(3) $\|\bar{u} + \bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$ *(triangle inequality)*

(4) $\|\bar{uv}\| \leq \|\bar{u}\|\|\bar{v}\|$*(submultiplivative)*

Definition 3.2.2. *Given a matrix T, let* $\rho(T) = \max|\delta_1|, |\delta_2|$,*where* δ_1 *and* δ_2 *are eigenvalues of the matrix T.Then* $\|\bar{T}\| = \sqrt{\rho(T^TT)}$ *is called the spectral norm of T and is needed a matrix norm.*

Definition 3.2.3. *A fixed point* x^* *of a map* $g : \mathbb{R}^2 \to \mathbb{R}^2$ *is called :*

(1) *Stable if given* $\epsilon > 0 \ \exists \ \varepsilon > 0$ *such that* $|x - x^*| < \varepsilon$ *implies* $|g^m(x) - x^*| < \epsilon$ $\forall m \in \mathbb{Z}^*$.

(2) *Attracting if* $\forall\eta > 0$ *such that* $|x - x^*| < \eta$ *implies that* $\lim_{n\to 0} g^n(x) = x^*$ *if* $\eta = \infty$ *we call it globally attracting.*

(3) *Asymptotically stable if it is both stable and attracting.It is globally asymptotically stable if it is both stable and globally attracting.*

(4) *Unstable if it is not stable.*

3.3 Stability of a Linear System

In order to give as much information about the fixed point $x^* = \binom{0}{0}$.

Theorem 3.3.1. *We have* $x(n + 1) = \theta x(n)$.

(i) *Asymptotically stable if* $\rho(\theta) < 1$.

(ii) *Unstable if* $\rho(\theta) > 1$.

(iii) *if* $\rho(\theta) = 1$ *the origin is unstable if the jordan form of* θ *is* $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ *And stable otherwise.*

Theorem 3.3.2. *The origin is asymptotically stable if and only if* $|tr\theta| < 1+\det(\theta) < 2$

Example 3.3.1. *let* $T = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$ *We sketch the correct canonical phase space digram,describe the stability , and find the general solution to the system.*

$$\det(\gamma\mathrm{I} - T) = [\gamma - (\frac{1}{2})]^2 + (\frac{1}{16}).$$

So $\gamma = \frac{1}{2} \pm \frac{1}{4}i$ *Then* $|\gamma| = \frac{\sqrt{5}}{4} < 1$ *by using* $\rho(T) < 1$ *that implies the origin is asymptotically stable we have an inward stable spiral. The general solution will have the form*

$$x(n) = (u_1\ u_2)|\gamma| \begin{pmatrix} cosn\omega & sinn\omega \\ -sinn\omega & cosn\omega \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

or

$$x(n) = |\gamma|^n[(\xi_1 cosn\omega + \xi_2 sinn\omega)u_1 + (-\xi_1 sinn\omega + \xi_2 cosn\omega)u_2]$$

finally

$$x_1(n) = |\frac{5}{16}|^n(\xi_1 cosn\omega)u_1 - (-\xi_1 sinn\omega)u_2$$

and

$$x_2(n) = |\frac{5}{16}|^n(\xi_2 cosn\omega)u_1 + (\xi_2 sinn\omega)u_2$$

.

3.4 Lyapunov Stability

In this section ,the notion of stability will be done by the use of a carefully chosen lyapunov function ,or by making use of laSalles'Invariance Principle.
Consider the difference equation

$$x(n+1) = g(x(n))$$

, Where $g : A \to \mathbb{R}^2, A \subset \mathbb{R}^2$, is continuous.Let x^* be a fixed point of g such that $g(x) = x^*$. For a mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, we define the variation $\triangle\phi$:

$$\triangle\phi(X) = \phi(g(x)) - \phi(x).$$

Then

$$\triangle\phi(x(n)) = \phi(x(n+1)) - \phi(x(n))$$

. Therefore, if $\triangle\phi \leq 0$,ϕ is nonincreasing along the orbit of g .

Theorem 3.4.1. *Let* $\phi : A \to \mathbb{R}, A \subset \mathbb{R}^2$ *is a lyapunov function on A if :*

(i) ϕ *is continuous on A.*

(ii) $\triangle\phi(x) \leq 0$, *whenever* x *and* $g(x) \in A$.

We note $B(x, \delta) = \{y \in \mathbb{R}^2 : |x - y| < \delta\}$ the open ball around X.Then,we say :

(1) ϕ is positive definite at x^* if $\phi(x) > 0$ and $\phi(x^*) = 0$

(2) ϕ is negative definite if $-\phi$ is positive definite

Theorem 3.4.2. *Suppose that ϕ is a positive definite lyapunov function defined on an open ball $A = B(x^*, \eta)$ around a fixed point x^* of a continuous map g on $\mathbb{R}^2$:Then ,x^* is stable if in addition ,$\triangle\phi(x) < 0$, whenever x and $g(x) \in A$, $x \neq x^*$,x^* is asymptotically stable on A ,Moreover , if $A = \mathbb{R}^2$ and $\triangle\phi(x) \to \infty$ as $|x| \to \infty$,x^* is globally asymptotically stable*

Let $\Omega(x)$ the positive limit set of $x \in \mathbb{R}^2$ and $\Theta(x)$ the positive orbit of x. A set D is said to be invariant under a map g if $g(D) \subset D$, for $x \in D, \Theta(x) \subset D$

Lemma 3.4.3. *If $\Theta(x)$ is bounded , then $\Omega(x)$ is nonempty ,compact , and invariant we define*

$$C = \{x \in \bar{A} : \triangle\phi(x) = 0\}$$

. And let D be the maximal invariant subset of C under A , and $m \in \mathbb{R}^+$,

$$\phi^{-1}(m) = \{x : \phi(x) = m, x = \mathbb{R}^2\}$$

Theorem 3.4.4. *We have ϕ is a positive definite lyapunov function defined on an open ball $A = B(x^*, \eta)$ around a fixed point x^* of a two -dimensional map g. if for $x \in A.\Theta(x)$ is bounded and contained in A ,then for some $m \in \mathbb{R}^+$, $f^n(x) \to D \cap \phi^{-1}(m)$ as $n \to \infty$.*

Theorem 3.4.5. *Let $\phi : A \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that,relative to the difference equation $x(n+1) = g(x(n))$,$\triangle\phi$ is positive definite (conversely negative definite) on a neighborhood of a fixed point x^*.if there exists a sequence $x_i \to x^*$ as $i \to \infty$ with $\phi(x_i) > 0$(conversely $\phi(x_i) < 0$),then x^* is unstable.*

3.5 Linearization of Systems

Linearization by use of a jacobian matrix is necessary to get information on the equilibrium point.Given a system of difference equations and a point Q, the jacobian is given by:

$$Dg(Q) = P = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix}_Q$$

Theorem 3.5.1. *Let $g : A \subset \mathbb{R}^2 \to \mathbb{R}^2$ a $\mathcal{C}^1$ map, where A is an open subset of $\mathbb{R}^2$, x^* is a fixed point of g, and $P = \mathrm{Dg}(x^*)$.then we have:*

i If $\rho(P) < 1$,then x^* is asymptotically stable.

ii If $\rho(P) > 1$,then x^* is unstable.

iii If $\rho(P) = 1$,then x^* may or may not be unstable.

Example 3.5.1. *There is a system :*

$$x_1(n+1) = x_2(n) - x_2(n)[x_1^2(n) + x_2^2(n)]$$

$$x_2(n+1) = x_1(n) - x_1(n)[x_1^2(n) + x_2^2(n)]$$

The equilbrium points are $(x_1^*, x_2^*) = \begin{cases} (0,0) \\ (1,-1) \\ (-1,1) \end{cases}$ *By using jacobian we find:*

$$\mathrm{Dg}(Q) = P = \begin{pmatrix} -2x_1x_2 & -3x_2^2 - x_1^2 + 1 \\ -3x_1^2 - x_2^2 + 1 & -2x_1x_2 \end{pmatrix}_Q$$

From a linearization of the system we get:

(1) $\mathrm{Dg}(0,0) = P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,$\alpha = \pm 1$ *We have a periodic solution about the equlibrium* $(0,0)$*.*

(2) $\mathrm{Dg}(1,-1) = P = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$,$(\alpha_1, \alpha_2) = (-1, 5)$ *The equilibrium point* $(1,-1)$ *is unstable .*

(3) $\mathrm{Dg}(-1,1) = P = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$,$(\alpha_1, \alpha_2) = (-1, 5)$ *The equilibrium point* $(-1,1)$ *is unstable .*

We want to comment on the stability of the system: we need the matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *.since* $b_{11} > 0$ *and* $\det B > 0$ *Let compute*

$$\phi(X) = (x_1 \ x_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 \ x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_2^2$$

So $\phi(X)$ *is clearly positive definite . We compute now*

$$\triangle\phi(X)(x_1, x_2) = ({}_2 - x_2[x_1^2 + x_2^2])^2 + (x_1 - x_1[x_1^2 + x_2^2])^2 - (x_1^2 + x_2^2)$$

By using lyapunov stabilty about the origin $(0,0)$*:*

(i) For sufficiently small values of x_1 *and* x_2 ,$\triangle\phi \leq 0$ *approach the origin asymptotically .*

(ii) For sufficiently large values of x_1 *and* x_2 *, tend away from the origin .*

Example 3.5.2. *There is a system :*

$$x(n+1) = x^2(n) + \frac{1}{4}$$

$$y(n+1) = 4x(n) - y^2(n)$$

The equilibrium points are $(x_1^*, x_2^*) = \begin{cases} (\frac{1}{2}, 1) \\ (\frac{1}{2}, -2) \end{cases}$ *By using jacobian we find:*

$$\mathrm{Dg}(Q) = P = \begin{pmatrix} 2x & 0 \\ 4 & -2y \end{pmatrix}_Q$$

From a linearization of the system we get:

(1) $\mathrm{Dg}(\frac{1}{2}, 1) = P = \begin{pmatrix} 1 & 0 \\ 4 & -2 \end{pmatrix}$,$(\alpha_1, \alpha_2) = (1, -2)$ *We have* $\alpha_1 = 1$ *and* $|\alpha_2| > 1$ *which makes this point a degenerate node.*

(2) $\mathrm{Dg}(\frac{1}{2}, -2) = P = \begin{pmatrix} 1 & 0 \\ 4 & 4 \end{pmatrix}$,$(\alpha_1, \alpha_2) = (1, 4)$ *We have* $\alpha_1 = 1$ *and* $|\alpha_2| > 1$ *which makes this point a degenerate node.*

That is inconclusive about the behavior of the system.

Example 3.5.3. *There is a system :*

$$x(n+1) = y^2(n) - \frac{1}{2}x(n)$$

$$y(n+1) = \frac{1}{4}x(n) + \frac{1}{2}y(n)$$

The equilibrium points are $(x_1^*, x_2^*) = \begin{cases} (0, 0) \\ (6, 3) \end{cases}$ *By using jacobian we find:*

$$\mathrm{Dg}(Q) = P = \begin{pmatrix} -\frac{1}{2} & 2y \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}_Q$$

From a linearization of the system we get:

(1) $\mathrm{Dg}(0, 0) = P = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$,$(\alpha_1, \alpha_2) = (-\frac{1}{2}, \frac{1}{2})$ *We have* $|\alpha_1| = |\alpha_2| < 1$ *which makes this point a stable node.*

(2) $\mathrm{Dg}(6, 3) = P = \begin{pmatrix} -\frac{1}{2} & 6 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$,$(\alpha_1, \alpha_2) = (\frac{\sqrt{7}}{2}, \frac{-\sqrt{7}}{2})$ *We have* $|\alpha_1|, |\alpha_2| > 1$ *which makes this point an unstable node.*

It is a stable node.

Example 3.5.4. *There is a system :*

$$x(n+1) = y(n)$$

$$y(n+1) = asin(x(n)) - y(n)$$

The equilibrium points are $(x_1^*, x_2^*) = (0,0)$ *By using jacobian we find:*

$$\mathrm{Dg}(Q) = P = \begin{pmatrix} 0 & 1 \\ acosx & -1 \end{pmatrix}_Q$$

From a linearization of the system we get:

(1) $a = -0.2$,$\mathrm{Dg}(0,0) = P = \begin{pmatrix} 0 & 1 \\ -0.2 & -1 \end{pmatrix}$,$(\alpha_1, \alpha_2) = (-\frac{1}{2} \pm \frac{\sqrt{2}}{2})$ *We have* $|\alpha_1|, |\alpha_2| < 1$ *which makes this point a stable node.*

(2) $a = 1$,$\mathrm{Dg}(0,0) = P = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$,$(\alpha_1, \alpha_2) = (-\frac{1}{2} \pm \frac{\sqrt{5}}{2})$ *We have* $|\alpha_1| < 1$,$|\alpha_2| > 1$ *which makes this point a saddle node.*

(3) $a = 3$,$\mathrm{Dg}(0,0) = P = \begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}$ *Which makes this point an unstable node.*

Chapter 4

Models and Applications

4.1 The Predator-Prey Model (Host Parasitoid)

This chapter gives examples of how difference equations can be applied to real world situations.

Consider a predator -prey model in which predators search over a constant area and have unlimited capacity for consuming the prey :

$$N(t+1) = rN(t)\exp(-aP(t))$$

$$P(t+1) = N(t)(1-\exp(-aP(t)))$$

Where $t \in \mathbb{Z}^*$, a is the predator's searching efficiency ($a > 0$), r is the reproductive rate of the prey,$N(t)$ is the size of the prey population, and $P(t)$ is the size of the predator population at time period t. We must to know how to solve for the fixed points (N^*, P^*):

$$N(t) = rN(t)\exp(-aP(t))$$

$$P(t) = N(t)(1-\exp(-aP(t)))$$

Then

$$(N^*, P^*) = \begin{cases} (0,0) \\ (\frac{r\ln r}{a(r-1)}, \frac{\ln r}{a}) \end{cases}$$

Now we use the jacobian matrix:

$$Dg(Q) = A = \begin{pmatrix} r\exp(-aP) & -arN\exp(-aP) \\ 1-\exp(-aP) & aN\exp(-aP) \end{pmatrix}_Q$$

For :

$$Dg(0,0) = A = \begin{pmatrix} r\exp(0) & 0 \\ 1-\exp(0) & 0 \end{pmatrix}_{(0,0)} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$$

The eigenvalues are $\alpha_1 = r, \alpha_2 = 0$ The origin is unstable if $r > 1$ and stable if $r < 1$.

And we use the jacobian matrix:

$$Dg(Q) = A = \begin{pmatrix} r\exp(-aP) & -arN\exp(-aP) \\ 1-\exp(-aP) & aN\exp(-aP) \end{pmatrix}_Q$$

For :

$$Dg(\frac{r\ln r}{a(r-1)}, \frac{\ln r}{a}) = A = \begin{pmatrix} r\exp(-a\frac{\ln r}{a}) & -ar\frac{r\ln r}{a(r-1)}\exp(-a\frac{\ln r}{a}) \\ 1-\exp(-a\frac{\ln r}{a}) & a\frac{r\ln r}{a(r-1)}\exp(-a\frac{\ln r}{a}) \end{pmatrix}_{(0,0)}$$

$$= \begin{pmatrix} 1 & -\frac{r\ln r}{(r-1)} \\ 1-\frac{1}{r} & \frac{\ln r}{(r-1)} \end{pmatrix}$$

So we use trA And $\det A$:

$$|trA| = 1 + \frac{\ln r}{(r-1)}$$

And

$$\det A = r\frac{\ln r}{(r-1)}$$

Then:

$$|trA| < \det A < 2$$

, We conclude that this point is unstable for large values of r,in particular the inequality does not hold when $(r \geq 5)$ since the middle term of the inequality exceeds the RHS values of 2 very rapidly as r increases.Very small values of r near 1 cause the LHS of the equality not to hold.

4.2 Geometric Brownian

Geometric Brownian motion (lognormal): $dX = \mu X dt + \sigma X dz$. We assume that $r > \mu$ in order to ensure convergence .
$X(t)$ follows a geometric Brownian motion and $D(X, Q)$ is linear. This is the specific setting under which Baldursson (1997) derives his oligopolistic solution. The results are identical. The equilibrium investment trigger, $X^*(Q)$, can be expressed as:

$$X^*(Q) = (\frac{\beta}{\beta-1})(\frac{r-\mu}{a})[k + (\frac{n+1}{n})\frac{bQ}{r}].$$

The equilibrium trigger is an increasing, affine function of Q, and is decreasing in n.

Example 4.2.1. *Consider the matrix*

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

It has the eigenvalue λ with multiplicity two, and the single eigenvector $v^1 = (1, 0)^t$, To achieve a basis we may define $v^2 = (0, 1)^t$ and represent arbitrary initial data in the form

$$v(0) = c_1 v^1 + c_2 v^2.$$

Direct verification shows that

$$v(t) = \{(c_1 + c_2 t)v^1 + c_2 v^2\} \exp \lambda t$$

If $\lambda \geq 0$ then the origin is unstable. If $\lambda < 0$ then, while the term $t \exp \lambda t$ increases initially, it reaches a finite maximum and $\|v(t)\|$ can be made arbitrarily small by choosing c_1 and c_2 sufficiently small; this implies not only stability but also asymptotic stability.

Conclusion and Perspectives

Here, I have reached the end of this project on the topic of systems of difference equations. It has been a wonderful and enriching learning experience for me while working on this project. Overall, this project covers: systems of difference equations, stability of equilibria, and applications.

Throughout the project, we have proved the existence and uniqueness of solutions to difference equations using several theorems. We also solved the stability problem of equilibria by using Lyapunov's theorem. Finally, we presented applications and models of difference equations.

As I can see, it is important to learn theorems related to fixed points and the generalized Banach space in order to study this topic effectively with examples.

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